# THE TATE CONSTRUCTION

Abstract. This is a note on higher algebra taught by Thomas Nikolaus and Achim Krause. The goal is to understand the Tate construction and the Tate diagonal in [NS18].

## Contents

## 1. ∞-Categories

Assume Grothendieck universe $\mathcal{U}$. We will call objects of $\mathcal{U}$ small sets. For us, categories will have large object sets and potential large morphism sets. For example we will write Set as the category of small sets. Now we give the definition of ∞-categories.

**Definition 1.1.** An ∞-category is a (large) simplicial set $\mathcal{C}$ such that for any diagram with $0 < i < n$

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathcal{C} \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

the lift exists, where $\Lambda^n_i$ denotes the $i$th inner horn of the simplices $\Delta^n$. And a functor between two ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is just a map of simplicial sets $\mathcal{C} \to \mathcal{D}$.

Recall the definition of Kan complexes.

**Definition 1.2.** A Kan complex (or an ∞-groupoid) is a (large) simplicial set $\mathcal{C}$ such that for any diagram with $0 \leq i \leq n$

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathcal{C} \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

the lift exists, where $\Lambda^n_i$ denotes the $i$th inner horn of the simplex $\Delta^n$.

Hence we also view ∞-categories as inner Kan complexes. We now explain the relation between ordinary categories and ∞-categories.

**Example 1.3.** Let $\mathcal{C}$ be an ordinary category. We define the nerve $N(\mathcal{C})$ of $\mathcal{C}$ to be the simplicial set such that $N(\mathcal{C})_n$ is the functors $[n] \to \mathcal{C}$ where $[n]$ is the poset of $\{0, 1, \dots, n\}$, i.e. $x_0 \to x_1 \to \cdots \to x_n$ with $x_i \in \mathcal{C}$. Then $N(\mathcal{C})$ is an ∞-category and the functors $N(\mathcal{C}) \to N(\mathcal{D})$ is in bijection with functors $\mathcal{C} \to \mathcal{D}$. Note that the objects of $\mathcal{C}$ is given by $N(\mathcal{C})_0$ and morphisms in $\mathcal{C}$ is given by $N(\mathcal{C})_1$.

For example, the simplicial set $\Delta^n$ is isomorphic to the nerve of the category $[n]$.

Motivated by the example above, for an ∞-category $\mathcal{C}$, we call $\mathcal{C}_0$ the objects of $\mathcal{C}$ and $\mathcal{C}_1$ the morphisms of $\mathcal{C}$. Then for any $f \in \mathcal{C}_1$, we can think about it as $f : a \to b$ where $a = \partial_1 f$ and $b = \partial_0 f$ are end points given by the face maps. And the degeneracy map from $\mathcal{C}_0$ to $\mathcal{C}_1$ gives identity maps on objects.

There is one particular relation we have on morphisms.

**Definition 1.4.** We call two different morphisms $f, g : a \to b$ equivalent if there exists $\sigma \in \mathcal{C}_2$ given by

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ & \searrow^{g} & \big\downarrow \mathrm{Id}_b \\ & & b \end{array}$$

where we denote as $f \simeq g$.

**Lemma 1.5** ([Cis19, Lemma 1.6.4.]). *The relation in Definition 1.4 is an equivalence relation.*

The main ingredient is the following lemma (Joyal's Coherence Lemma). We say a diagram $\mathrm{Sk}_1(\Delta^2) \to \mathcal{C}$ in an $\infty$-category commutes if it extends to $\Delta^2$, where $\mathrm{Sk}_1$ denotes the 1-skeleton.

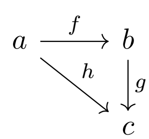
**Lemma 1.6** ([Cis19, Lemma 1.6.2.]). *Let $\mathcal{C}$ be an $\infty$-category. Let $f : \mathrm{Sk}_1(\Delta^3) \to \mathcal{C}$ be a diagram and $d^i f : \partial\Delta^2 \to \mathcal{C}$ denotes the diagram corresponds to the subcomplex of $\mathrm{Sk}_1(\Delta^3)$ ignoring the $i$ the vertice. Assume further that $d^0 f$ and $d^3 f$ commute. Then $d^1 f$ commutes if and only if $d^2 f$ commutes.*

**Remark 1.7.** If $\mathcal{C}$ is the nerve of some ordinary category, then the equivalence relation is discrete, i.e. two morphism are equivalent if and only if they are equal on the nose.

**Example 1.8.** Let $\mathcal{C}$ be a Kan complex. Hence $\mathcal{C}$ is an $\infty$-category. In particular, for a topological space $X$, $\mathrm{Sing}(X)$ is an $\infty$-category. Objects of $\mathrm{Sing}(X)$ are points of $X$. Morphisms of $\mathrm{Sing}(X)$ are paths in $X$. And the equivalence relation are given by homotopy between paths relative to end points. If one passes to the equivalence classes of $\mathrm{Sing}(X)$, then one would get the fundamental groupoid of $X$. Hence one can think $\mathrm{Sing}(X)$ as the $\infty$-category version of the fundamental groupoid of $X$.

We now discuss the composition of morphisms. As we have already seen in the previous example that the composition of paths is subtle operation: there are different ways to compose paths since we can parametrize paths differently. No matter how you parametrize them, one always runs into the issue that you can only compose them up to homotopies. Hence we give the following definition.

**Definition 1.9.** Let $\mathcal{C}$ be an $\infty$-category. Let $f : a \to b$ and $g : b \to c$ be morphisms in $\mathcal{C}$ with one common vertices. Then a composition of $f$ and $g$ is a 2-simplex $\sigma \in \mathcal{C}_2$ given by



where $h = \partial_1 \sigma$. And we write $h \simeq g \circ f$. In other words, the diagram above commutes.

**Remark 1.10.** Note that the existence of such a composition is given by the horn filling property in the definition of the $\infty$-category. In fact, by Lemma 1.6 one can see that two different choices of compostion $h \simeq g \circ f$ and $h' \simeq g \circ f$ really lead to an equivalence $h \simeq h'$.

Building on that, we give the definition of equivalence in $\infty$-categories.

**Definition 1.11.** Let $\mathcal{C}$ be an $\infty$-category. Let $f : a \to b$ be a morphism. We call $f$ a equivalence if there exists $g : b \to a$ such that $\mathrm{Id}_a \simeq g \circ f$ and $\mathrm{Id}_b \simeq f \circ g$.

In terms of $\infty$-categories, we also have the notion of functor categories.

**Definition 1.12.** Let $\mathcal{C}$ and $\mathcal{D}$ be simplical sets. We define the simplicial set of internal Hom $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D})$ with $n$-simplices $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D})_n$ given by the set of all morphisms $\mathcal{C} \times \Delta^n \to \mathcal{D}$. We call $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D})$ the simplicial mapping space between $\mathcal{C}$ and $\mathcal{D}$. When $\mathcal{C}$ and $\mathcal{D}$ are $\infty$-categories, we write $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D})$ as $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and call it the functor category.

**Remark 1.13.** If $\mathcal{C}$ and $\mathcal{D}$ are $\infty$-categories, then $\mathrm{Fun}(\mathcal{C},\mathcal{D})_0$ are just functors between $\mathcal{C}$ and $\mathcal{D}$. This justifies the name of functor categories.

It turns, in fact, under mild assumptions $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ is again an $\infty$-category.

**Proposition 1.14** ([Lur06, Proposition 1.2.7.3.]). *Let $\mathcal{C}$ be a simplical set and $\mathcal{D}$ be an $\infty$-category. Then* $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ *is an $\infty$-category.*

**Remark 1.15** ([Cis19, Corollary 3.5.12.]). Morphisms $\eta : f \to g$ in $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ is given by $\mathcal{C}\times\Delta^1 \to \mathcal{D}$ whose restrictions are $f$ and $g$. We will call them natural transformations. Moreover, $\eta$ is an equivalence in $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ if and only if $\eta_c$ is an equivalence in $\mathcal{D}$ for all objects in $c \in \mathcal{C}$.

We now talk about equivalence between $\infty$-categories.

**Definition 1.16.** A functor $f : \mathcal{C} \to \mathcal{D}$ between $\infty$-categories is called equivalent if there exists $g : \mathcal{D} \to \mathcal{C}$ and natural equivalences $f \circ g \simeq \mathrm{Id}_{\mathcal{D}}$ and $g \circ f \simeq \mathrm{Id}_{\mathcal{C}}$. We then write $\mathcal{C} \simeq \mathcal{D}$.

We want to develop the same phenomenon as the ordinary categories that equivalence is the same as fully faithful and essential surjective. In order to do that, we need the definition of mapping spaces.

**Definition 1.17.** For objects $a, b$ in an $\infty$-category $\mathcal{C}$. We define the mapping space $\mathrm{Map}_{\mathcal{C}}(a,b)$ as the pullback of the following diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(a,b) & \longrightarrow & \mathrm{Fun}(\Delta^1,\mathcal{C}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \{0\}\sqcup\{1\}} \\ \Delta^0 & \xrightarrow{(a,b)} & \mathcal{C}\times\mathcal{C} \end{array}$$

**Proposition 1.18** ([Cis19, Corollary 4.2.10., Remark 4.2.11.]). *Let $\mathcal{C}$ be an $\infty$-category. Then for any $a, b \in \mathcal{C}_0$,* $\mathrm{Map}_{\mathcal{C}}(a,b)$ *is a Kan complex.*

**Remark 1.19.** It is not obvious from Definition 1.17 that the mapping space is a "space", i.e. a Kan complex. But if one defines $\mathrm{Map}_{\mathcal{C}}(a,b)$ as in [Lur06, Proposition 1.2.2.3.] (denoted as $\mathrm{Hom}^R_{\mathcal{C}}(a,b)$) to be the simplicial set such that n-simplices are the set of all $f : \Delta^{n+1} \to \mathcal{C}$ such that $f|\Delta^{\{n+1\}} = b$ and $f|\Delta^{\{0,\dots,n\}}$ is the constant simplex at the vertex $a$, then it is obvious that $\mathrm{Map}_{\mathcal{C}}(a,b)$ is a Kan complex. In all cases, they are different simplicial model of the mapping space $\mathrm{Map}_{\mathrm{h}\mathcal{C}}(a,b)$ where $\mathrm{h}\mathcal{C}$ is the homotopy category of $\mathcal{C}$ regarded as a topological-enriched category.

**Remark 1.20.** The 0-simplices in $\mathrm{Map}_{\mathcal{C}}(a,b)$ are precisely morphisms in $\mathcal{C}$ with endpoints $a$ and $b$. And there is a 1-simplex between two difference morphisms from $a$ to $b$ if and only if the two difference morphisms are equivalent. This means that the Kan complex $\mathrm{Map}_{\mathcal{C}}(a,b)$ has $\pi_0$ as the set of equivalence classes of morphisms $a \to b$.

**Example 1.21.** By Remark 1.19, we view the mapping space as the morphism in the homotopy category of $\mathcal{C}$. Then in $\mathrm{Sing}(X)$, $\mathrm{Map}_{\mathrm{Sing}(X)}(a,b)$ is homotopy equivalent to the sapce of paths from $a$ to $b$ in $X$, and the composition map comes from path composition.

In terms of the mapping spaces, we have the following definition of fully faithful.

**Definition 1.22.** Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between $\infty$-categories. Then we call $f$ fully faithful if the induced map $\mathrm{Map}_{\mathcal{C}}(a, b) \to \mathrm{Map}_{\mathcal{D}}(f(a), f(b))$ is an homotopy equivalence of Kan complexes for each $a, b$.

Note that this is much more than saying that the $\pi_0$ is bijective. The definition of essential surjective is what one would expect.

**Definition 1.23.** Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between $\infty$-categories. Then we call $f$ essentially surjective if for every $d \in \mathcal{D}_0$, we have $c \in \mathcal{C}_0$ and an equivalence $d \simeq f(c)$.

Now we have the proposition as the ordinary category theory.

**Proposition 1.24** ([Cis19, Theorem 3.9.7.])**.** *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between $\infty$-categories. Then $f$ is equivalence if and only if $f$ is fully faithful and essentially surjective.*

Another definition one can pull over from the ordinary category theory is the full subcategory on a set of objects.

**Definition 1.25.** Let $\mathcal{C}$ be an $\infty$-category and $S \subset \mathcal{C}_0$ be a subset of objects. Then we define $\mathcal{C}_S \subset \mathcal{C}$ be the simplicial set consisting of all simplicies with vertices in $S$.

**Remark 1.26.** It is easy to check that $\mathcal{C}_S$ is indeed an $\infty$-category. If we first saturate $S$ with equivalent classes giving $S \subset \overline{S}$, then the natural map $\mathcal{C}_S \to \mathcal{C}_{\overline{S}}$ is an equivalence by Proposition 1.24.

We will now revisit the notion of compositions in the context of mapping spaces. Fix $a, b, c \in \mathcal{C}_0$ in some $\infty$-category $\mathcal{C}$. We set $\mathrm{Map}_{\mathcal{C}}(a, b, c)$ as the pullback of the following diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(a,b,c) & \longrightarrow & \mathrm{Fun}(\Delta^2, \mathcal{C}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \{0\}\sqcup\{1\}\sqcup\{2\}} \\ \Delta^0 & \xrightarrow{(a,b,c)} & \mathcal{C} \times \mathcal{C} \times \mathcal{C} \end{array}$$

Then we have the following lemma.

**Lemma 1.27** ([Cis19, 3.7.7.])**.** *$\mathrm{Map}_{\mathcal{C}}(a, b, c)$ is a Kan complex and the map*

$$\mathrm{Map}_{\mathcal{C}}(a, b, c) \xrightarrow{(\partial_2, \partial_0)} \mathrm{Map}_{\mathcal{C}}(a, b) \times \mathrm{Map}_{\mathcal{C}}(b, c)$$

*induced by $\Delta^2 \leftarrow \Lambda^2_1$ is an equivalence. In particular, by choosing a homotopy inverse $s$, we get a map of Kan complexes*

$$\circ_s : \mathrm{Map}_{\mathcal{C}}(a, b) \times \mathrm{Map}_{\mathcal{C}}(b, c) \to \mathrm{Map}_{\mathcal{C}}(a, c)$$

*composing with $\mathrm{Map}_{\mathcal{C}}(a, b, c) \xrightarrow{\partial_1} \mathrm{Map}_{\mathcal{C}}(a, c)$ induced by $\Delta^2 \leftarrow \Delta^1$.*

**Remark 1.28.** The homotopy inverse is unique up to homotopy. In fact, the space of homotopy inverse is contractible. Hence the composition is unique up to homotopy. Similarly, the associativity can be parametrized by $\mathrm{Map}_{\mathcal{C}}(a, b, c, d)$, c.f. [Cis19, 3.7.7.].

We now come to an important example of $\infty$-category. From now on, we will call Kan complexes spaces and study the $\infty$-category of spaces. This is the fundamental example of $\infty$-categories. Just like categories are enriched over sets, $\infty$-categories are enriched over Kan complexes as we have seen above. Recall that the category Kan of Kan complexes is enriched over simplicial sets.

**Construction 1.29.** For $J$ finite (nonempty) totoally ordered set. We define the simplicial enriched category $\mathfrak{C}[\Delta^J]$ with objects of $J$ and morphisms

$$\mathrm{Hom}(i,j) \coloneqq N(\{K \subset J \mid \max(K) = i \text{ and } \min(K) = j\})$$

And the composition is induced by the union $K$'s. As a poset, $\mathrm{Hom}(i,j)$ is the simplicial cube of dimension $j-i-1$, i.e. $\prod_{j-i-1} \Delta^1$. The identification is given by for instance the vertex $(1,1,0,\dots,0)$ correspondences to the subset $\{i, i+1, j\}$. Hence the simplicial mapping spaces here are all weakly contractible.

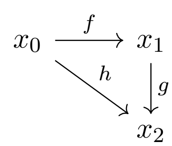
We now define the homotopy coherent nerve for simplicial enriched categories.

**Definition 1.30.** We define the simplicial homotopy coherent nerve $N_\Delta(\mathcal{C})$ of a simplicial enriched category $\mathcal{C}$ as the simplicial set

$$N_\Delta(\mathcal{C})_n \coloneqq \mathrm{Hom}(\mathfrak{C}[\Delta^n], \mathcal{C}).$$

where the Hom is taken in the category of simplicial sets. By slightly abusing notations, we will also denote it as $N$.

**Example 1.31.** In the example of $\mathcal{S} \coloneqq N(\mathrm{Kan})$, the 0-simplices are just Kan complexes and the 1-simplices are just maps of Kan complexes. As for the 2-simplices are give by the diagram




and a homotopy from $h$ to $g \circ f$. In fact, the mapping space $\mathrm{Map}_{\mathfrak{C}[\Delta^2]}(0,2)$ has 0-simplicies $\{0,2\}$ and $\{0,1,2\} = \{1,2\} \circ \{0,1\}$; and has the non-degenerate 1-simplex $\{0,2\} \subset \{0,1,2\}$. And the non-degenerate 1-simplex $\{0,2\} \to \{0,1,2\}$ encodes the homotopy $h$ to $g \circ f$ as desired.

Indeed we have the following theorem.

**Theorem 1.32.** $\mathcal{S} \coloneqq N(\mathrm{Kan})$ *is an $\infty$-category. And for any Kan complexes $x, y$, we have a homotopy equivalence of Kan complexes* $\mathrm{Map}_{\mathcal{S}}(x,y) \simeq \underline{\mathrm{Hom}}_{\mathrm{Kan}}(x,y)$, *where the latter is the simplicial mapping space as in Definition 1.12.*

In general, the following is true.

**Proposition 1.33** ([Lur06, Proposition 1.1.5.10.])**.** *If $\mathcal{C}$ is a simplicial-enriched category and mapping spaces are Kan complexes, then $N(\mathcal{C})$ is an $\infty$-category and we have a homotopy equivalence of Kan complexes* $\mathrm{Map}_{N(\mathcal{C})}(x,y) \simeq \underline{\mathrm{Hom}}_{\mathcal{C}}(x,y)$ *for any $x, y \in \mathcal{C}$.*

## 2. Limits

We will study the notion of limits in the context of $\infty$-categories. Roughly speaking, the limit of a functor is the universal cone over that functor. Throughout this section, $\mathcal{C}$ is an $\infty$-category. And we will denote $\mathrm{Fun}(I, \mathcal{C})$ as $\mathcal{C}^I$ sometimes for simplicial set $I$. In particular, we are interested in the case where $I$ is a small simplicial set. Let $F : I \to \mathcal{C}$ be a functor from $I$ to $\mathcal{C}$.

**Definition 2.1.** A cone over a functor $F : I \to \mathcal{C}$ is a pair consisting of an object $y \in \mathcal{C}_0$, together with a natural transformation $\eta : c_y \to F$ where $c_y$ is the constant functor given by $I \to \Delta^0 \xrightarrow{y} \mathcal{C}$.

**Construction 2.2.** Let $(y, \eta)$ be a cone over $F$ and $x \in \mathcal{C}_0$. We construct

$$\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{C})}(c_x, F)$$

as the composition of

$$\mathrm{Map}_{\mathcal{C}}(x, y) \xrightarrow{c} \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{C})}(c_x, c_y) \xrightarrow{\eta_*} \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{C})}(c_x, F)$$

where the first map is defined by $\mathcal{C} \to \mathcal{C}^I$ pulled back along $I \to *$. Note that the second map is defined by composition with $\eta$ which is only well-defined up to contractible choice of homotopy inverse by Lemma 1.27.

Now we give the main definition of this section.

**Definition 2.3.** A cone $(y, \eta)$ over $F$ is a limit (cone) in $\mathcal{C}$ if for all $x \in \mathcal{C}_0$, the map

$$\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{C})}(c_x, F)$$

is a homotopy equivalence. In this case, we write $y = \lim_I F$ or $y = \lim_{i \in I} F(i)$.

**Remark 2.4.** Given a functor $I \to \mathcal{C}$ of ordinary categories. Then the limits of $N(I) \to N(\mathcal{C})$ correspond precisely to ordinary limits of $I \to \mathcal{C}$.

**Example 2.5** (Terminal Object)**.** Let $I = \varnothing$, then $\mathrm{Fun}(I, \mathcal{C}) = *$. Hence $y \in \mathcal{C}$ is a limit over $*$ if $\mathrm{Map}_{\mathcal{C}}(x, y) \simeq *$ is a homotopy equivalence for any $x \in \mathcal{C}$. In this case, we write $y = *$ as the terminal object.

**Example 2.6** (Products)**.** Let $I$ be a discrete simplicial set, i.e. constant simplicial set. Then we have the functor category as $\mathrm{Fun}(I, \mathcal{C}) \simeq \prod_I \mathcal{C}$. A functor $I \to \mathcal{C}$ thus consists of a sequence of objects $\{y_i\}_{i \in I}$. And a cone is given by an object $y \in \mathcal{C}_0$ together with maps $\{\pi_i ; y \to y_i\}_{i \in I}$. Then $(y, (\pi_i))$ is the limit cone, if for each object $x \in \mathcal{C}$ we have

$$\mathrm{Map}_{\mathcal{C}}(x, y) \xrightarrow{(\pi_*)} \prod_{i \in I} \mathrm{Map}_{\mathcal{C}}(x, y_i)$$

is an equivalence. In this case we will denote $y$ as $\prod_I y_i$. For example, in $\mathcal{S}$ the products are given by products of Kan complexes. This can be easily seen by Theorem 1.32 and checking for ordinary simplicial mapping spaces which is obviously true.

Like the ordinary category theory, limits are unique.

**Lemma 2.7.** *Any two limit cones $(y, \eta)$, $(y', \eta')$ of $F$ are equivalent.*

*Proof.* Note that by the universal property, we have

$$\mathrm{Map}_{\mathcal{C}}(y, y') \simeq \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{C})}(c_y, F).$$

Let $f \leftarrow\!\shortmid \eta$, then by construction we have $\eta' \circ c_f \simeq \eta$ where where $c_f$ is the constant transformation on $f$. Similarly we get $g : y' \to y$ with $\eta \circ c_g \simeq \eta'$. Then we have $\eta \circ g \circ f \simeq \eta' \circ f \simeq \eta$. Thus by the universal property, we have $g \circ f \simeq \mathrm{Id}$. Vice versa, we have $f \circ g \simeq \mathrm{Id}$. □

**Remark 2.8.** One can show that in fact that the $\infty$-category of limit cones over $F : I \to \mathcal{C}$ is either contractible Kan complex or empty, namely the limit either uniquely exists or does not exist, c.f. [Lan21, Lemma 4.3.13.]

**Remark 2.9.** Let $I$ be a simplicial set. Then we can always form $I \subset \mathrm{Ex}^{\infty}(I)$ an $\infty$-category by gluing in fillers for inner horns that do not have fillers. Then we have $I \subset \mathrm{Ex}^{\infty}(I)$ is a anodyne map, in particular a Joyal equivalence, i.e.

$$\mathrm{Fun}(\mathrm{Ex}^{\infty}(I), \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(I, \mathcal{C})$$

for any $\infty$-category $\mathcal{C}$. Hence when taking limit, one can assume without loss of generality that $I$ is a $\infty$-category. For details, see for example [Lan21, Corollary 2.2.14.].

The first example where the notion of limit is different from the one in ordinary categories is pullback where the homotopy coherence is involved.

**Example 2.10** (Pullbacks)**.** Let $I = \Delta^1 \sqcup_{\Delta^0} \Delta^1$, given by

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ 0' & \longrightarrow & 1 \end{array}$$

Then a functor $F : I \to \mathcal{C}$ is given by

$$\begin{array}{ccc} & & b \\ & & \downarrow{\scriptstyle h} \\ c & \xrightarrow{k} & d \end{array}$$

in $\mathcal{C}$. Since $I = \Delta^1 \sqcup_{\Delta^0} \Delta^1$, we have $\mathrm{Fun}(I, \mathcal{C}) \simeq \mathcal{C}^{\Delta^1} \times_{\mathcal{C}} \mathcal{C}^{\Delta^1}$. Thus a transform $c_a \to F$ is given by the following commutative diagrams

$$\begin{array}{ccc} a & \xrightarrow{\mathrm{Id}_a} & a \\ {\scriptstyle i}\downarrow & \searrow & \downarrow{\scriptstyle f} \\ b & \xrightarrow{h} & d \end{array} \qquad \begin{array}{ccc} a & \xleftarrow{\mathrm{Id}_a} & a \\ {\scriptstyle f}\downarrow & \swarrow & \downarrow{\scriptstyle j} \\ d & \xleftarrow{k} & c \end{array}$$

for some choice of $i, j, f$. Hence a natural transformation $c_a \to F$ in $\mathrm{Fun}(I, \mathcal{C})$ up to equivalence is given by $i : a \to b$, $j : b \to c$ together with an equivalence $h \circ i \simeq k \circ j$ of maps between $a$ and $d$ in $\mathcal{C}$. This datum of equivalence is usually nontrivial, namely not equal on the nose, but unique up to homotopy. As usual we will denote the pullback as $b \times_c d$.

We now describe the limits in the $\infty$-category of spaces $\mathcal{S}$. First we need a technical lemma which is tautological in the ordinary categories setting.

**Lemma 2.11** ([Lan21, Proposition 4.3.14.]). *Let $\mathcal{C}$ be a Kan-enriched category and $F : I \to N(\mathcal{C})$ is a functor. Suppose $z \in \mathcal{S}_0$ and $x \in \mathcal{C}_0$. Then we have natural equivalences*

$$\mathrm{Map}_{\mathcal{S}}(z, \mathrm{Map}_{N(\mathcal{C})^I}(c_x, F)) \simeq \mathrm{Map}_{\mathcal{S}^I}(c_z, \mathrm{Map}_{N(\mathcal{C})}(x, F(-)))$$

*where* $\mathrm{Map}_{N(\mathcal{C})}(x, F(-)) := \mathrm{Map}_{N(\mathcal{C})}(x, -) \circ F$. *Moreover, if $\mathcal{C} = \mathrm{Kan}$, then the mapping spaces above are also equivalent to* $\mathrm{Map}_{\mathcal{S}^I}(c_{z\times x}, F)$.

**Remark 2.12.** Recall from Lemma 1.27 that in $\infty$-categories, compositions are somehow subtle. However, in Lemma 2.11, one can make $\mathrm{Map}_{N(\mathcal{C})}(x, -)$ as a functor easily since it is a homotopy coherent nerve of a Kan-enriched category. Hence one can simply form $\underline{\mathrm{Hom}}_{\mathcal{C}}(x, -)$ as a functor on $\mathcal{C}$ and take the homotopy coherent nerve.

The first consequence is the following which says that limits in $\mathcal{S}$ can be detected simply on $\pi_0$.

**Proposition 2.13.** *A cone $(y, \eta)$ over $F$ in $\mathcal{S}$ is a limit if and only if for each $x \in \mathcal{S}_0$ the map*

$$[x, y]_{\mathcal{S}} = \pi_0(\mathrm{Map}_{\mathcal{S}}(x, y)) \to \pi_0(\mathrm{Map}_{\mathcal{S}^I}(c_x, F)) = [c_x, F]_{\mathcal{S}^I}$$

*is an isomorphism, where $[-,-]$ denotes the homotopy class.*

*Proof.* Assume $\pi_0(\mathrm{Map}_{\mathcal{S}}(x, y)) \simeq \pi_0(\mathrm{Map}_{\mathcal{S}^I}(c_x, F))$ for every $x$. First note that we have the following diagram

$$\begin{array}{ccc} [z, \mathrm{Map}_{\mathcal{S}}(x, y)]_{\mathcal{S}} & \xrightarrow{\simeq} & [z \times x, y]_{\mathcal{S}} \\ \downarrow & & \downarrow{\scriptstyle\simeq} \\ [z, \mathrm{Map}_{\mathcal{S}^I}(c_x, F)]_{\mathcal{S}} & \xrightarrow{\simeq} & [c_{z\times x}, F]_{\mathcal{S}^I} \end{array}$$

where the isomorphism on the first row is by the usual product internal Hom adjointness; the isomorphism on the second row is by Lemma 2.11; and the isomorphism on the second column is by assumption. Hence this forces the first column to be isomorphism. Since $z$ is arbitrary, we win. ☐

**Example 2.14.** The pullback in $\mathcal{S}$ exists. Moreover, the pullback of a diagram $F : b \to c \leftarrow d$ in $\mathcal{S}$ can be computed as the ordinary limit $b \times_c c^{\Delta^1} \times_c d$ of simplicial sets

$$\begin{array}{ccccccc} b & & & c^{\Delta^1} & & & d \\ & \searrow & \swarrow{\scriptstyle \mathrm{ev}_1} & & \searrow{\scriptstyle \mathrm{ev}_0} & \swarrow & \\ & c & & & & c & \end{array} .$$

To see this, first note that by Proposition 2.13 it suffices to check that any object $x$ in $\mathcal{S}$, we have $[x, b \times_c c^{\Delta^1} \times_c d]_{\mathcal{S}} \simeq [c_x, F]_{\mathcal{S}^I}$. Note that by Example 2.10 $[c_x, F]_{\mathcal{S}^I}$ is given by the datum of the following diagram

$$\begin{array}{ccc} x & \longrightarrow & d \\ \downarrow & & \downarrow \\ b & \longrightarrow & c \end{array}$$

with a homotopy $x \times \Delta^1 \to c$ with restrictions $x \times \{0\} \to c = x \to d \to c$ and $x \times \{1\} \to c = x \to b \to c$. Hence by adjointness of products and simplicial mapping spaces, this is equivalent of giving a map from $a$ to the diagram of simplicial sets above. One interesting application is that the pullback of $* \to c \leftarrow *$ is the loop space of $c$.

In fact, all the small limits exist in the $\infty$-category of spaces.

**Proposition 2.15.** *Fix any functor $F : I \to \mathcal{S}$. Then the mapping space $\mathrm{Map}_{\mathcal{S}^I}(c_{\mathrm{pt}}, F)$ is the limit of $F$. In particular, $\mathcal{S}$ has all small limits.*

*Proof.* We first check the unviersal property, i.e. we need to show that

$$\mathrm{Map}_{\mathcal{S}}(x, \mathrm{Map}_{\mathcal{S}^I}(c_{\mathrm{pt}}, F)) \to \mathrm{Map}_{\mathcal{S}^I}(c_x, F)$$

is an equivalence. However, this is again by Lemma 2.11 since $x \simeq x \times \mathrm{pt}$. Now applying the equivalence to $x = \mathrm{Map}_{\mathcal{S}^I}(c_{\mathrm{pt}}, F)$, we have a natural transformation $\eta : c_x \to F$ corresponding to the identity on $\mathrm{Map}_{\mathcal{S}^I}(c_{\mathrm{pt}}, F)$. Hence we have the limit cone $(\mathrm{Map}_{\mathcal{S}^I}(c_{\mathrm{pt}}, F), \eta)$. □

**Example 2.16.** Let $I$ be an simplicial set and $F = c_x$ for some space $x \in \mathcal{S}_0$. Then we have

$$\lim_I F \simeq \mathrm{Map}_{\mathcal{S}^I}(c_{\mathrm{pt}}, c_x) \simeq \underline{\mathrm{Hom}}(I, \mathrm{Map}_{\mathcal{S}^I}(c_{\mathrm{pt}}, c_x)) \simeq \underline{\mathrm{Hom}}(I, x)$$

where the second equivalence is using the pullback definition of the mapping space. For example, if $I$ itself is a Kan complex, then this coincides with the usual mapping space in Kan. This is more than limits in ordinary category theory. For example, if $I$ is the sphere, then one get all the higher homotopy groups.

**Example 2.17** (Sequential Limits)**.** Let $\overline{I}$ be the nerve of the totally ordered set $(\mathbb{N}, \leq)$ and $I = \Delta^1 \sqcup_{\Delta^0} \Delta^1 \sqcup_{\Delta^0} \Delta^1 \sqcup_{\Delta^0} \dots$ given by $0 \to 1 \to 2 \to 3 \to \dots$. Note that in $I$ there is no map from $i$ to $i+2$. In other words $I \subset \overline{I}$ is the saturation of $I$ into an $\infty$-category as in Remark 2.9. In $\mathcal{S}$, a functor $I^{\mathrm{op}} \to \mathcal{S}$ is given by a sequence of spaces

$$\dots a_3 \to a_2 \to a_1 \to a_0$$

Similarly as in Example 2.14, the limit in $\mathcal{S}$ of the above diagram is given by the ordinary limit of the following simplicial sets

$$\dots a_1 \xleftarrow{\mathrm{ev}_1} a_1^{\Delta^1} \xrightarrow{\mathrm{ev}_0} a_1 \to a_0 \xleftarrow{\mathrm{ev}_1} a_0^{\Delta^1} \xrightarrow{\mathrm{ev}_0} a_0.$$

After introducing colimits, we will see that the colimit of some functor $F : I \to \mathcal{S}$ is the mapping telescope of $\{F(n)\}_n$.

Now we will end this section by explaining how to think about limits in arbitrary $\infty$-categories in terms of limits in $\mathcal{S}$. Again we will adhere some functorality property of the mapping space functor.

**Theorem 2.18.** *Assume that $\mathcal{C}$ is a Kan-enriched category and $F : I \to N(\mathcal{C})$ is a functor. Then a cone $(y, \eta)$ over $F$ is a limit cone if and only if for any $x \in \mathcal{C}$ the induced cone $(\mathrm{Map}_{N(\mathcal{C})}(x, y), \mathrm{Map}_{N(\mathcal{C})}(x, \eta))$ is a limit cone of $\mathcal{S}$. In fact, we have*

$$\mathrm{Map}_{N(\mathcal{C})}(x, \lim_I F) \simeq \lim_I \mathrm{Map}_{N(\mathcal{C})}(x, F(-))$$

*where the latter is the limit taking in $\mathcal{S}$ with respect to the functor $\mathrm{Map}_{N(\mathcal{C})}(x, -) \circ F : I \to \mathcal{S}$.*

*Proof.* Note that $(y,\eta)$ being a limit cone over $F$ in $N(\mathcal{C})$ is saying that

$$\mathrm{Map}_{N(\mathcal{C})}(x,y) \simeq \mathrm{Map}_{N(\mathcal{C})^I}(c_x,F)$$

is an equivalence which is the same of saying that

$$\mathrm{Map}_{\mathcal{S}}(z,\mathrm{Map}_{N(\mathcal{C})}(x,y)) \simeq \mathrm{Map}_{\mathcal{S}}(z,\mathrm{Map}_{N(\mathcal{C})^I}(c_x,F))$$

is an equivalence for any $z \in \mathcal{S}_0$. Using Lemma 2.11, we can identify the latter with $\mathrm{Map}_{\mathcal{S}^I}(c_z,\mathrm{Map}_{N(\mathcal{C})}(x,F(-)))$. But this is precisely the definition of $\mathrm{Map}_{N(\mathcal{C})}(x,y)$ being the limit cone of $\mathrm{Map}_{N(\mathcal{C})}(x,F(-))$. □

**Remark 2.19.** In fact, every $\infty$-category can be realized as the nerve of some Kan-enriched category, c.f. [Lan21, Corollary 3.2.26.]. Hence Theorem 2.18 describes limits in arbitrary $\infty$-categories.

We will end this section remarking on the relation between the definition of limit introduced here and the one in [Lur06].

**Remark 2.20.** In Contruction 2.2, a cone over a functor $F: I \to \mathcal{C}$ is a functor

$$I \times \Delta^1 / I \times \{0\} \coloneqq (I \times \Delta^1) \sqcup_{K \times \{0\}} \Delta^0 \to \mathcal{C},$$

i.e. constant at one end. This is a "smaller" model $I^{\triangleleft} \coloneqq \Delta^0 \star I$ called the left cone of $I$ which is a quotient of $I \times \Delta^1/I \times \{0\}$. Indeed, $\mathrm{Fun}(I \times \Delta^1/I \times \{0\}, \mathcal{C}) \simeq \mathrm{Fun}(I^{\triangleleft}, \mathcal{C})$ is an equivalence for any $\infty$-category $\mathcal{C}$. This means that $\mathrm{Map}_{\mathcal{C}^I}(c_x,F)$ is equivalent to the pullback of

$$\mathcal{C}^{I^{\triangleleft}} \to \mathcal{C} \times \mathcal{C}^I \xleftarrow{(x,F)} \Delta^0.$$

In fact a even smaller model $\mathcal{C}_{/F}$ is used in [Lur06], i.e. $\mathrm{Map}_{\mathcal{C}^I}(c_x,F)$ is equivalent to the pullback of

$$\mathcal{C}_{/F} \to \mathcal{C} \times \mathcal{C}^I \xleftarrow{(x,F)} \Delta^0,$$

c.f. [Lan21, Lemma 4.3.9.].

## 3. Colimits

In this section we will discuss colimits in $\infty$-categories. As in the ordinary category theory, one can have everything done by the opposite categories. For $\infty$-categories, we have the following construction.

**Construction 3.1.** Let $\mathrm{Op} : \Delta \to \Delta$ denotes the functor giving the opposite linear ordering. More explicitly, for each object $[n] \in \Delta$, we have $\mathrm{Op}([n]) = [n]^{\mathrm{op}}$ sending $i \mapsto n - i$. And for each morphism $\alpha : [m] \to [n]$ in $\Delta$, we set $\mathrm{Op}(\alpha) : [m] \to [n]$ given by $\mathrm{Op}(\alpha)(i) = n - \alpha(m - i)$. Then for any simplicial set $K : \Delta^{\mathrm{op}} \to \mathrm{Set}$, we define the opposite simplicial set $K^{\mathrm{op}}$ as

$$K^{\mathrm{op}} : \Delta^{\mathrm{op}} \xrightarrow{\mathrm{Op}} \Delta^{\mathrm{op}} \to \mathrm{Set}.$$

It is easy to see that if $\mathcal{C}$ is an $\infty$-category, then $\mathcal{C}^{\mathrm{op}}$ is also an $\infty$-category.

**Definition 3.2.** The colimit of a functor $F : I \to \mathcal{C}$ is the limit over of $F^{\mathrm{op}} : I^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$.

Explicitly, this means the universal cone under a functor in the following sense

**Definition 3.3.** A cone under a functor $F : I \to \mathcal{C}$ is an object $y \in \mathcal{C}_0$ together with a natural transformation $\eta : F \to c_y$. And we call a cone $(y, \eta)$ a colimit cone if for any $z \in \mathcal{C}_0$, the natural map

$$\mathrm{Map}_{\mathcal{C}}(y, z) \xrightarrow{\eta^*} \mathrm{Map}_{\mathcal{C}^I}(F, c_z)$$

is a homotopy equivalence where $\eta^*$ is given by precomposite with $\eta$.

Most of the properties of limits formally dualize to the properties of colimits. But it is still not clear that what the colimits in the $\infty$-category of spaces look like. First of all, we need a proposition analogus to Proposition 2.13. The proof is the same as before.

**Proposition 3.4.** *A cone $(y, \eta)$ under $F$ in $\mathcal{S}$ is a colimit if and only if for each $z \in \mathcal{S}$ the map*

$$[y, z]_{\mathcal{S}} = \pi_0(\mathrm{Map}_{\mathcal{S}}(y, z)) \to \pi_0(\mathrm{Map}_{\mathcal{S}^I}(F, c_z)) = [F, c_z]_{\mathcal{S}^I}$$

*is a natural isomorphism in $z$.*

We now give a couple of basic examples.

**Example 3.5** (Initial Object)**.** Let $I = \varnothing$, then $\mathrm{Fun}(\mathcal{C}, I) = *$. Hence $y \in \mathcal{C}$ is a colimit under $*$ if $\mathrm{Map}_{\mathcal{C}}(y, x) \simeq *$ is a homotopy equivalence for any $x \in \mathcal{C}$. In this case, we write $y = \varnothing$ as the initial object. In $\mathcal{S}$, the initial object is the empty space.

**Example 3.6** (Coproducts)**.** Let $I$ be a discrete simplicial set. Let $F : I \to \mathcal{C}$ be a functor given by a family of objects $\{y_i\}_{i \in I}$. Then a cone under $F$ is given by an object $y \in \mathcal{C}_0$ together with maps $\{\iota_i ; y_i \to y\}_{i \in I}$. Then $(y, (\iota_i))$ is the colimit cone, if for each object $x \in \mathcal{C}$ we have

$$\mathrm{Map}_{\mathcal{C}}(y, z) \xrightarrow{(\iota_*)} \prod_{i \in I} \mathrm{Map}_{\mathcal{C}}(y_i, z)$$

is an equivalence. We will denote $y$ as $\coprod_I y_i$. In $\mathcal{S}$ the coproducts are just given by disjoint of Kan complexes.

**Example 3.7** (Pushouts)**.** Let $I = \Delta^1 \sqcup_{\Delta^0} \Delta^1$, given by

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & & \\ 1' & & \end{array}$$

Then a functor $F : I \to \mathcal{C}$ is given by

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \\ c & & \end{array}$$

in $\mathcal{C}$. We will denote the colimit of $F$ as $b \sqcup_a c$ if exists. For any $z \in \mathcal{C}_0$, the universal property is given by

$$\mathrm{Map}_{\mathcal{C}}(b \sqcup_a c, z) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{C}}(b, z) \times_{\mathrm{Map}_{\mathcal{C}}(a,z)} \mathrm{Map}_{\mathcal{C}}(c, z)$$

where the latter pullback is taken in $\infty$-category $\mathcal{S}$. Argue similarly as in Example 2.14, we can see that the pushout in $\mathcal{S}$ is given by the Kan complex associated to ordinary colimit $b \sqcup_a (a \times \Delta^1) \sqcup_a c$ of simplicial sets

$$\begin{array}{ccccccc} b & & & a \times \Delta^1 & & & c \\ & \nwarrow & \nearrow^{\mathrm{Id}\times\{0\}} & & \nwarrow^{\mathrm{Id}\times\{1\}} & \nearrow & \\ & & a & & a & & . \end{array}$$

which is the known as the double mapping cylinder. As before, even the given simplicial set might not be a Kan complex, but one can always glue all the fillers to get a universal Kan complex.

We will next see that $\mathcal{S}$ has all small colimits. In fact, we will show a general fact that if an $\infty$-category has all coproducts and pushouts, then it has all small colimits. We first show this philosophy in some easy examples.

**Example 3.8** (Coequalizers)**.** A coequalizer is given by the colimit of the diagram $I = \Delta^1 \sqcup_{\partial\Delta^1} \Delta^1$, i.e. the diagram $0 \rightrightarrows 1$ . Assuming an $\infty$-category $\mathcal{C}$ has all pushouts and coproducts. We claim that the coequalizer of $F :\ a \underset{g}{\overset{f}{\rightrightarrows}} b$ is the pushout of

$$\begin{array}{ccc} a \sqcup a & \xrightarrow{(f,g)} & b \\ \downarrow{\scriptstyle \nabla} & & \\ a & & \end{array}$$

where the vertical map is the codiagonal. By universal property, it suffices to show that for any $z \in \mathcal{C}_0$, we can realize $\mathrm{Map}_{\mathcal{C}}(F, c_z)$ as the pullback of

$$\begin{array}{ccc} & & \mathrm{Map}_{\mathcal{C}}(b, z) \\ & & \downarrow{\scriptstyle (f_*, g_*)} \\ \mathrm{Map}_{\mathcal{C}}(a, z) & \xrightarrow{\Delta} & \mathrm{Map}_{\mathcal{C}}(a, z) \times \mathrm{Map}_{\mathcal{C}}(a, z). \end{array}$$

By Example 2.14, the pullback is given by $h \in \mathrm{Map}(a, z)_0$, $k \in \mathrm{Map}(b, z)_0$ and a homotopy equivalence $(h, h) \simeq (k \circ f, k \circ g)$. But this is exactly the definition of $\mathrm{Map}_{\mathcal{C}}(F, c_z)$. Conversely, every pushout can also be realized as coequalizers. Namely, the pushout of any diagram $c \xleftarrow{f} a \xrightarrow{g} b$ can be given by the coequalizer $a \overset{f}{\underset{g}{\rightrightarrows}} b \sqcup c$. This can be checked using a similar argument.

**Example 3.9** (Sequential Colimits)**.** Let $\overline{I}$ be the nerve of the totally ordered set $(\mathbb{N}, \leq)$ and $I = \Delta^1 \sqcup_{\Delta^0} \Delta^1 \sqcup_{\Delta^0} \Delta^1 \sqcup_{\Delta^0} \ldots$ given by $0 \to 1 \to 2 \to 3 \to \ldots$. A functor $I \to \mathcal{C}$ is given by a sequence of objects

$$y_0 \xrightarrow{f_0} y_1 \xrightarrow{f_1} y_2 \xrightarrow{f_2} y_3 \to \ldots.$$

We will write the colimit (if exists) as $y_\infty := \operatorname{colim} y_i$. Again we can form $y_\infty$ in terms of coequalizers and coproducts, namely

$$\coprod y_i \overset{\mathrm{Id}}{\underset{(f_i)}{\rightrightarrows}} \coprod y_i \longrightarrow y_\infty$$

is a coequalizer diagram. Again, this can be checked using mapping spaces. Note $y_\infty$ is also known as the mapping telescope.

In order to state the general result, we first introduce the following notion.

**Definition 3.10.** A simplicial set $I$ is called finite if it has finitely many non-degenerate simplices. And a colimit is called finite if the index simplicial set is finite.

We now state the theorem.

**Theorem 3.11.** *For an $\infty$-category $\mathcal{C}$, the following are equivalent:*

*(1) $\mathcal{C}$ admits an initial object and all pushouts;*

*(2) $\mathcal{C}$ admits all finite coproducts and all coequalizers;*

*(3) $\mathcal{C}$ admits all finite colimits.*

*Proof.* We refer to [Lur06, Proposition 4.4.2.6.] or [Lan21, Proposition 4.3.28.] for a more detailed argument. We first show the equivalence between (1) and (2). Assume (1). Note that the coproduct $a \sqcup b$ of $a$ and $b$ can be realized as the pushout of the diagram $a \leftarrow \varnothing \to b$. Inductively, one has all finite coproducts. And we have already seen in Example 3.8, that coequalizers can be obtained from finite coproducts and pushouts. Conversely, assume (2). Note that the initial object is the coproduct of the empty set. And Example 3.8 again tells us how to translate coequalizers into get pushouts assuming finite coproducts exist.

Obviously, (3) implies (2). Hence it suffices to show (2) implies (3). Let $I$ be a finite simplicial set and $F : I \to \mathcal{C}$ be a functor. Since $I$ is finite, it has a finite skeleton filtration $I_0 \subset I_1 \subset \cdots \subset I_n = I$ with $n$ being the dimension of $I$ and $I_n$ being the pushout of simplicial sets

$$\begin{array}{ccc} \coprod_J \partial\Delta^n & \longrightarrow & I_{n-1} \\ \downarrow & & \downarrow \\ \coprod_J \Delta^n & \longrightarrow & I_n \end{array}$$

for some index $J$. When $n = 0$, $I$ is discrete hence the colimit exists by assumption. By induction, we may assume that the colimits $\operatorname{colim} F|_{\coprod_J \partial\Delta^n}$ and $\operatorname{colim} F|_{I_{n-1}}$ exist. However, note that

$$\operatorname{colim} F|_{\coprod_J \Delta^n} \simeq \coprod_J \operatorname{colim} F|_{\Delta^n} \simeq \coprod_J F|_{\Delta^n}(n).$$

The last equivalence is because $\Delta^n$ is the nerve of a poset with terminal object $n$. In particular, $\operatorname{colim} F|_{\coprod_J \Delta^n}$ exists. Then the colimit of $F$ can be realized as the pushout

$$\operatorname{colim} F \simeq \operatorname{colim} F|_{\coprod_J \Delta^n} \sqcup_{\operatorname{colim} F|_{\coprod_J \partial\Delta^n}} \operatorname{colim} F|_{I_{n-1}}.$$

□

Given Theorem 3.11, one concludes that $\mathcal{S}$ has all finite colimits. But one needs more to see it admits all small colimits.

**Theorem 3.12.** *If an $\infty$-category $\mathcal{C}$ admits all coproducts. Then the following are equivalent:*

*(1) $\mathcal{C}$ admits all colimits;*

*(2) $\mathcal{C}$ admits all pushouts;*

*(3) $\mathcal{C}$ admits all coequalizers;*

*(4) $\mathcal{C}$ admits all geometric realizations, i.e. colimits indexed by $N(\Delta^{\mathrm{op}})$.*

*Proof.* We have already seen that (2) and (3) are equivalent and obviously (1) implies the rest. We will first show that (2) and (3) implies (1). Assume we have a functor $F : I \to \mathcal{C}$. Again we write $I$ as the union of its skeleta $I_0 \subset I_1 \subset \ldots$ with possible infinitely many steps. By Theorem 3.11, we know that all colimits $\operatorname{colim} F|_{I_n}$ exist. Then the colimit of $F$ can be realized as the sequential limit of $\operatorname{colim} F|_{I_n}$'s

$$\operatorname{colim} F \simeq \operatorname*{colim}_{n\in\mathbb{N}} F|_{I_n}$$

by checking the universal property. Then we win by writing the sequential colimit as a coequalizer as in Example 3.9.

It remains to show (4) implies (2). For any diagram $F : I \to \mathcal{C}$ given by $c \leftarrow a \to b$, we claim that the pushout of $F$ can be realized as the geometric realization of the following simplicial diagram (with obvious degeneracy maps) of the two sided bar construction

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} b \sqcup a \sqcup a \sqcup c \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} b \sqcup a \sqcup c \rightrightarrows b \sqcup c$$

Indeed, we define a map $u : I \to \Delta^{\mathrm{op}}$ given by $[0] \xleftarrow{d_0} [1] \xrightarrow{d_1} [0]$. Then we left Kan extend along $u$ to a functor $u_!F : \Delta^{\mathrm{op}} \to \mathcal{C}$. Denote the canonical map $\Delta^{\mathrm{op}} \to *$ by $h$. Then by Remark 6.30, the colimit of $F$ is given by left Kan extend along $I \to \Delta^{\mathrm{op}} \to *$, namely $(h \circ u)_!F$. On the other hand, the colimit of $u_!F : \Delta^{\mathrm{op}} \to \mathcal{C}$ is given by $h_!(u_!F)$ which is the same as $(h \circ u)_!F$. Hence it suffices to consider the functor $u_!F : \Delta^{\mathrm{op}} \to \mathcal{C}$. The pointwise formula for left Kan extension says that in degree $n$, the simplicial diagram $u_!F$ is given by

$$\operatorname*{colim}_{(u(i)\to[n]) \text{ in } \Delta^{\mathrm{op}}} F(i)$$

Now the index category decomposes into disjoint union of categories, namely one for each surjection $[n] \to [1]$ in $\Delta$ (corresponding to $a$) and two for each $[n] \to [0]$ in $\Delta$ (corresponding to $b$ and $c$ respectively). This gives the desired construction. ☐

As a direct consequence, we have the following.

**Corollary 3.13.** *The $\infty$-category $\mathcal{S}$ has all small colimits.*

**Remark 3.14.** Working with the opposite category, we also have $\mathcal{S}$ has all small limits.

We now define two more important $\infty$-categories.

**Definition 3.15.** The $\infty$-category $\mathcal{S}_*$ of pointed spaces is the nerve $N(\mathrm{Kan}_*)$ where $\mathrm{Kan}_*$ is the simplicial enriched category of pointed Kan complexes.

**Definition 3.16.** The $\infty$-category $\mathrm{Cat}_\infty$ of $\infty$-categories is the nerve $N(\mathrm{Cat}_\infty^\Delta)$ where $\mathrm{Cat}_\infty^\Delta$ is the simplicial enriched category where objects are $\infty$-categories and morphisms between two $\infty$-categories $\mathcal{C}$, $\mathcal{D}$ are the maximal Kan complex $\underline{\mathrm{Hom}}_{\mathrm{Cat}_\infty^\Delta}(\mathcal{C}, \mathcal{D})$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

It turns out that both $\mathcal{S}_*$ and $\mathrm{Cat}_\infty$ have small limits and colimits.

**Proposition 3.17.** *The $\infty$-category $\mathcal{S}_*$ has all small limits and colimits.*

**Remark 3.18.** The strategry of the proof of Proposition 3.17 is exactly the same as Corollary 3.13, namely by proving the existence of products (resp. coproducts) and pullbacks (pushouts). One just need to be slightly careful with the coproducts in $\mathcal{S}_*$ since it is given by wedge products instead of disjoint unions as in the category of pointed topological spaces.

Following the same philosophy, we have the following theorem. See [Lur06, 3.3.3.] for a different argument.

**Theorem 3.19** ([Lan21, Theorem 4.3.37.])**.** *The $\infty$-category $\mathrm{Cat}_\infty$ has all small limits and colimits.*

## 4. Derived Categories as ∞-Categories

We will introduce the derived ∞-category of an abelian category. This will be an enhancement of the ordinary derived category. We will first introduce an ∞-category with morphisms being chain homotopy classes. Then we will use the Dwyer-Kan localization to turn it in the derived ∞-category.

Throughout this section, let $\mathcal{A}$ be an abelian category. We will construct the ∞-category $\mathcal{D}(\mathcal{A})$ with objects being chain complexes in $\mathcal{A}$ and equivalent classes of morphisms being the usual $R\mathrm{Hom}$. We start the following definition.

**Definition 4.1.** Let $\mathrm{Ch}(\mathcal{A})$ be the category of chain complexes of $\mathcal{A}$ enriched over $\mathrm{Ch}(\mathbb{Z})$.

**Remark 4.2.** $\mathrm{Ch}(\mathbb{Z})$-enriched categories are usually called differential graded categories or simply dg categories. Recall that in the case of $\mathrm{Ch}(\mathcal{A})$, for $x, y \in \mathrm{Ch}(\mathcal{A})$ we have

$$\mathrm{Hom}_*(x,y) \coloneqq \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_n(x,y)$$

where $\mathrm{Hom}_n(x,y) \coloneqq \prod_{p\in\mathbb{Z}}(x^p, y^{p+n})$. And the differential is given by $d(f) = d_y \circ f - (-1)^n f \circ d_x$. See [Kel06] for more details.

In order to turn a $\mathrm{Ch}(\mathbb{Z})$-enriched category into a simplicial enriched category, we first recall the following equivalence known as the Dold-Kan correspondence.

**Theorem 4.3** (Dold-Kan Correspondence)**.** *There is an equivalence of categories given by the adjoint pair* $(\Gamma, N)$*:*

$$\Gamma : \mathrm{Ch}(\mathbb{Z})_{\geq 0} \xrightarrow{\simeq} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Ab}) : N_*.$$

**Remark 4.4.** The adjoint pair $(\Gamma, N_*)$ is defined as follows: $\Gamma$ sends $c \in \mathrm{Ch}(\mathbb{Z})_{\geq 0}$ to the simplicial abelian group $\Gamma(c)$ where $\Gamma(c)_n = \bigoplus_{[n]\twoheadrightarrow[k]} c_k$ and for $\alpha : [m] \to [n]$ in $\Delta$ the corresponding map $\Gamma(c)_n \to \Gamma(c)_m$

$$\alpha^* : \bigoplus_{[n]\twoheadrightarrow[k]} c_k \to \bigoplus_{[m]\twoheadrightarrow[j]} c_j$$

on the summand indexed by some $\sigma : [n] \to [k]$ is given by

$$c_k \xrightarrow{d^{j-k}} c_j \hookrightarrow \bigoplus_{[m]\twoheadrightarrow[j]} c_j$$

where $[m] \twoheadrightarrow [j] \hookrightarrow [k]$ is a factorization of $[m] \to [n] \to [k]$ and $d^{j-k}$ is the composition of differentials. Hence unless $k = j - 1$, the morphism will be zero.

On the other hand, $N_*$ is normalized Moore complex functor, namely for any simplicial group $s$, we let $N_*(c)$ denote the chain complex with

$$N_n(c) = \bigcap_{1\leq i\leq n} \ker(d_i^n)$$

with differential given by $d_0^n$.

See [Lur17, Lemma 1.2.3.13.] for a proof that this is an equivalence. Moreover, both $\Gamma$ and $N_*$ are lax monoidal with respect to the usual tensor product of chain of complexes and the pointwise tensor product of simplicial abelian groups. However, $\Gamma$ is not symmetric while $N_*$ is symmetric. One way to think about it is the Alexander-Whitney map, c.f. [nA, Monoidal Dold-Kan Correspondence].

We can now associate any chain complexes to a simplicial set.

**Construction 4.5.** Let $K$ denote the composition of the functors:

$$K : \mathrm{Ch}(\mathbb{Z}) \xrightarrow{\tau_{\geq 0}} \mathrm{Ch}(\mathbb{Z})_{\geq 0} \xrightarrow{\Gamma} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Ab}) \xrightarrow{\mathrm{can}} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set}).$$

where $\tau_{\geq 0}$ is the canonical truncation functor, c.f. [Sta, 0118] and can is the forgetful functor. Note that all functors are lax monoidal. Hence this gives a functor from dg categories to simplicial-enriched categories. And the lax monoidal structure of $K$ gives the compatibiligy of compositions.

**Remark 4.6.** The functor $K$ enjoys with two nice properties:

(1) The essential image of $K$ lands not just inside simplicial sets but actually Kan complexes with a canonical base point (given by 0). In fact, any simplicial abelian group is a Kan complex, i.e. $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Ab}) \xrightarrow{\mathrm{can}} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$ factors through $\mathrm{Kan}_*$, c.f. [nA, Theorem 3.1.].

(2) If $c \in \mathrm{Ch}(\mathbb{Z})_{\geq 0}$, then the homotopy groups $\pi_n(K(c), 0)$ of the pointed Kan complex is the homology groups $H_i(c)$. Since $N_* \circ \Gamma(c) \simeq c$, it suffices to show that for any simplicial abelian group $x$, we have $H_n(N_*(x)) \simeq \pi_n(x, 0)$. Indeed, every element $\sigma \in \ker(N_n(x) \to N_{n-1}(x))$ can be identified with a map $\sigma : \Delta^n \to x$ with $\sigma|_{\partial\Delta^n} = 0$ which corresponds to an element $[\sigma] \in \pi_n(x, 0)$. Moreover, by [Ker, Remark 3.2.2.21.], $[\sigma] = 0$ in $\pi_n(x, 0)$ if and only if there exists an $(n+1)$-simplex $\tau$ such that $d_0^{n+1}(\tau) = \sigma$ and $d_i^{n+1}(\tau)$ is the constant map for $1 \leq i \leq n+1$. However, this is exactly the condition of $\sigma \in \mathrm{im}(N_{n+1}(x) \to N_n(x))$.

We now form the definition of the dg nerve.

**Definition 4.7.** For a dg category $\mathcal{C}$, we let

$$N_{\mathrm{dg}}(\mathcal{C}) := N_\Delta(\mathcal{C}_\Delta)$$

where $\mathcal{C}_\Delta$ is the simplicial-enriched category obtained from $\mathcal{C}$ by applying $K$ to Hom. Note that by Remark 4.6 (1), this functor really gives an $\infty$-category. When there is no confusion, we will also denote it as $N$.

**Definition 4.8.** For an abelian category $\mathcal{A}$, we define the $\infty$-category $\mathcal{K}(\mathcal{A})$ as

$$\mathcal{K}(\mathcal{A}) := N(\mathrm{Ch}(\mathcal{A})).$$

**Remark 4.9.** One should think about $\mathcal{K}(\mathcal{A})$ with objects and morphisms in $\mathrm{Ch}(\mathcal{A})$ and equivalent relation being chain homotopies by Remark 4.6 (2).

We will see that $\mathcal{K}(\mathcal{A})$ has all finite limits and colimits without any assumption on $\mathcal{A}$.

**Proposition 4.10.** *The $\infty$-category $\mathcal{K}(\mathcal{A})$ admits all finite limits and colimits.*

*Proof.* By Theorem 3.11, it suffices to show the existence of the initial object, the terminal object and all pullbacks and pushouts. Note that 0 is both the initial and the terminal object. Obviously we have $\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(0,x) = K(\mathrm{Hom}(0,x)) \simeq *$ and $\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(0,x) \simeq *$ being contractible for any $x \in \mathcal{K}(\mathcal{A})_0$.

Before we show the existence of pullbacks and pushouts, we first take a look into products and coproducts. For $c, d \in \mathcal{K}(\mathcal{A})$, we claim that $c \oplus d$ is both the product and the coproduct. For it being a coproduct We need to show that for any $x \in \mathcal{K}(\mathcal{A})_0$

$$\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(c \oplus d, x) \simeq \mathrm{Map}_{\mathcal{K}(\mathcal{A})}(c, x) \times \mathrm{Map}_{\mathcal{K}(\mathcal{A})}(d, x)$$

is an equivalence. This is true for $\mathrm{Hom}_*$ hence it boils down to the fact that the functor $K$ preserves products. Similarly, one can check that $c \oplus d$ is the product.

Now we show the existence of pushouts. Let $y \xleftarrow{f} x \xrightarrow{g} y'$ be a pushout diagram. Let $c(f,g)$ denotes the complex $c(f,g)_n = y_n \oplus y'_n \oplus x_{n-1}$ and differentials given by $d = (d_y + f, d_{y'} - g, d_x)$. One can check that there is a chain homotopy between $x \xrightarrow{f} y \hookrightarrow c(f,g)$ and $x \xrightarrow{g} y' \hookrightarrow c(f,g)$. This leads to pushout square

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & \ulcorner & \downarrow \\ y' & \longrightarrow & c(f,g). \end{array}$$

Similarly, we also have description for pullbacks. □

**Remark 4.11.** When $y' = 0$, we call the pushout of $y \xleftarrow{f} x \xrightarrow{g} 0$ the cofibre of $x \to y$. And the pullback of $x \xrightarrow{g} y \xleftarrow{f} 0$ the fibre of $x \to y$. In particular, this gives a cofibre functor. If one consider $\mathcal{E} \subset \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{K}(\mathcal{A}))$ to be the full subcategory of cofibre square, then the forgetful functor $\theta : \mathcal{E} \to \mathrm{Fun}(\Delta^1, \mathcal{K}(\mathcal{A}))$ is a trivial Kan fibration by [Lur06, Proposition 4.3.2.15.]. Hence $\theta$ admits a section $\mathrm{Fun}(\Delta^1, \mathcal{K}(\mathcal{A})) \to \mathcal{E}$. And we will call the composition cofibre functor

$$\mathrm{cofib} : \mathrm{Fun}(\Delta^1, \mathcal{K}(\mathcal{A})) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{K}(\mathcal{A})) \to \mathcal{K}(\mathcal{A})$$

where the last map is given by evaluation at the final object of $\Delta^1 \times \Delta^1$. Similarly, one can define the fibre functor.

Note that we have not identified morphisms which are quasi-isomorphic. For example in $\mathcal{K}(\mathbb{Z})$, there is no inverse map of $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$. To fix this, we need to invert all quasi-isomorphisms. In order to do this, we introduce the notion of Dwyer-Kan localization.

**Definition 4.12** (Dwyer-Kan Localization)**.** Let $\mathcal{C}$ be an $\infty$-category and $W \subset \mathcal{C}_1$ be a subset of morphisms. A functor $f : \mathcal{C} \to \mathcal{C}'$ between $\infty$-categories is called a Dwyer-Kan localization at $W$ if $f$ takes $W$ to equivalences in $\mathcal{C}'$; and for any $\infty$-category $\mathcal{D}$, the functor

$$\mathrm{Fun}(\mathcal{C}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is fully faithful with essential image the full subcategory $\mathrm{Fun}^W(\mathcal{C}, \mathcal{D})$ on those functors $\mathcal{C} \to \mathcal{D}$ which takes $W$ to equivalence.

**Remark 4.13.** It is not hard to see that the Dwyer-Kan localization is unique up to equivalence if exists, c.f.[Lan21, Lemma 2.4.4.]. Hence we will write $\mathcal{C}[W^{-1}]$ for such $\mathcal{C}'$.

**Example 4.14.** Let $\mathcal{C} = \Delta^1$ and $W$ be the nontrivial morphism $0 \to 1$. Then $\mathcal{C}[W^{-1}] \simeq \Delta^0$. In order to check the universal property, we need to show that for any $\infty$-category $\mathcal{D}$ the functor

$$\mathrm{Fun}(\Delta^0, \mathcal{D}) \to \mathrm{Fun}^W(\Delta^1, \mathcal{D})$$

is an equivalence. Note that for any object $a \xrightarrow{\simeq} a'$ in $\mathrm{Fun}^W(\Delta^1, \mathcal{D})$, it is equivalence to $a \xrightarrow{\mathrm{Id}} a$. Indeed, by the definition of equivalence of $a \xrightarrow{\simeq} a'$, there is an inverse map $a' \to a$ such that the composition $a \to a' \to a$ is equivalent to Id. This shows essential surjectivity.

As for fully faithfulness, note that the mapping space $\mathrm{Map}_{\mathrm{Fun}^W(\Delta^1,\mathcal{D})}(f, g)$ of $f : a \xrightarrow{\simeq} a'$ and $g : b \xrightarrow{\simeq} b'$ is the pullback of

$$\begin{array}{ccc} & & \mathrm{Map}_{\mathcal{D}}(a', b') \\ & & \downarrow f^* \\ \mathrm{Map}_{\mathcal{D}}(a, b) & \xrightarrow{g_*} & \mathrm{Map}_{\mathcal{D}}(a, b') \end{array}.$$

But since $f, g$ are equivalences, so does $f^*$ and $g_*$. Hence one can identify all the mapping spaces which shows the fully faithfulness.

Now the question will be the existence of such localization. This comes to the following proposition.

**Proposition 4.15.** *Let $\mathcal{C}$ be an $\infty$-category and $W \subset \mathcal{C}_1$ be a subset of morphisms. The Dwyer-Kan localization $\mathcal{C}[W^{-1}]$ exists and the functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ is essentially surjective.*

*Proof.* Inspired by Example 4.14, we take $\mathcal{C}[W^{-1}]$ to be the pushout of the following diagram in $\mathrm{Cat}_\infty$

$$\begin{array}{ccc} \bigsqcup_W \Delta^1 & \longrightarrow & \mathcal{C} \\ \downarrow & \ulcorner & \downarrow \\ \bigsqcup_W \Delta^0 & \longrightarrow & \mathcal{C}[W^{-1}] \end{array}.$$

Then we claim that for any $\infty$-category $\mathcal{D}$, the corresponding diagram of functor categories

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) & \longrightarrow & \mathrm{Fun}(\bigsqcup_W \Delta^0), \mathcal{D}) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Fun}(\mathcal{C}, \mathcal{D}) & \longrightarrow & \mathrm{Fun}(\bigsqcup_W \Delta^1, \mathcal{D}) \end{array}$$

is a pullback diagram. Assuming the claim, since we have already seen in Example 4.14 that

$$\mathrm{Fun}(\bigsqcup_W \Delta^0, \mathcal{D}) \simeq \prod_W \mathrm{Fun}(\Delta^0, \mathcal{D}) \to \prod_W \mathrm{Fun}(\Delta^1, \mathcal{D}) \simeq \mathrm{Fun}(\bigsqcup_W \Delta^1, \mathcal{D})$$

is full faithful, it is easy to see that the pullback along $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is also fully faithful. Moreover, by tracing the diagram, one can see that $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D})$ is exactly $\mathrm{Fun}^W(\mathcal{C}, \mathcal{D})$. Hence it suffices to show the claim, which is given by the Lemma 4.17.

∎

**Remark 4.16.** Note that the universal property of pushouts a priori only gives a pullback diagram for $\mathrm{Map}_{\mathrm{Cat}_\infty}(-,-)$ which by definition is the maximal Kan subcomplex of $\mathrm{Fun}(-,-)$. Thus the claim does not follow immediately from the universal property.

**Lemma 4.17.** *Given a pushout out diagram in* $\mathrm{Cat}_\infty$

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}' \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{D} & \longrightarrow & \mathcal{D}' \end{array},$$

*the corresponding diagram of functor categories*

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{D}',\mathcal{E}) & \longrightarrow & \mathrm{Fun}(\mathcal{C}',\mathcal{E}) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Fun}(\mathcal{D},\mathcal{E}) & \longrightarrow & \mathrm{Fun}(\mathcal{C},\mathcal{E}) \end{array}$$

*is a pullback diagram for any $\infty$-category $\mathcal{E}$.*

*Proof.* We prove by checking the corresponding statement for $\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{B},-)$ for any $\infty$-category $\mathcal{B}$. Observe that

$$\mathrm{Fun}(\mathcal{B},\mathrm{Fun}(\mathcal{D}',\mathcal{E})) \simeq \mathrm{Fun}(\mathcal{D}',\mathrm{Fun}(\mathcal{B},\mathcal{E}))$$

since both sides can be identified with $\mathrm{Fun}(\mathcal{B}\times\mathcal{D}',\mathcal{E})$ using the usual adjointness of products and internal Homs for simplicial sets. However, by definition, $\mathrm{Map}_{\mathrm{Cat}_\infty}(-,-)$ is the maximal Kan subcomplex of $\mathrm{Fun}(-,-)$, hence one gets an identification

$$\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{B},\mathrm{Fun}(\mathcal{D}',\mathcal{E})) \simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{D}',\mathrm{Fun}(\mathcal{B},\mathcal{E})).$$

After applying this observation, it reduces to show that

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{D}',\mathrm{Fun}(\mathcal{B},\mathcal{E})) & \longrightarrow & \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}',\mathrm{Fun}(\mathcal{B},\mathcal{E})) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{D},\mathrm{Fun}(\mathcal{B},\mathcal{E})) & \longrightarrow & \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C},\mathrm{Fun}(\mathcal{B},\mathcal{E})) \end{array}$$

which is clearly true by using the universal property of pushouts on $\mathrm{Map}_{\mathrm{Cat}_\infty}(-,\mathrm{Fun}(\mathcal{B},\mathcal{E}))$. □

**Remark 4.18.** A "fancy" way to interpret Lemma 4.17 is that $\mathrm{Cat}_\infty$ is actually a $(\infty,2)$-category, i.e. there are not only mapping spaces but also mapping categories.

With the notion of Dwyer-Kan localization, we can finally form the definition of the derived $\infty$-category.

**Definition 4.19.** Let $W \subset \mathcal{K}(\mathcal{A})_1$ be the subset of quasi-isomorphisms. Then we define derived $\infty$-category $\mathcal{D}(\mathcal{A})$ to be

$$\mathcal{D}(\mathcal{A}) \coloneqq \mathcal{K}(\mathcal{A})[W^{-1}].$$

However, one drawback of such approach is that the mapping spaces in $\mathcal{D}(\mathcal{A})$ is rather implicit. We will see how to compute these mapping spaces in the next section.

## 5. Slices and Filtered Colimits

In order to compute the mapping spaces of the derived $\infty$-category $\mathcal{D}(\mathcal{A})$ of some abelian category $\mathcal{A}$, we realizes them as special colimits as mapping spaces in $\mathcal{K}(\mathcal{A})$ and in many cases, they are equal on the nose. In fact, we will see many cases that colimits and limits in $\mathcal{D}(\mathcal{A})$ are simply lifting a diagram to $\mathcal{K}(\mathcal{A})$ and can be computed within.

In general, mapping spaces of a Dwyer-Kan localization is somehow subtle to understand. Many homopoty theories are developed to attack this question such as model categories and calculus of fractions. However, it is rather easy in $\mathcal{D}(\mathcal{A})$. To start with, we need some definitions.

**Definition 5.1.** Let $\mathcal{C}$ be an $\infty$-category and $x \in \mathcal{C}_0$. We define the slice category $\mathcal{C}_{/x}$ over $x$ as the pullback in $\mathrm{Set}_\Delta$

$$\begin{array}{ccc} \mathcal{C}_{/x} & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{C}) \\ \downarrow & \lrcorner & \downarrow d_0 \\ \Delta^0 & \xrightarrow{\;x\;} & \mathcal{C} \end{array}$$

and the slice category $\mathcal{C}_{x/}$ under $x$ as the pullback

$$\begin{array}{ccc} \mathcal{C}_{x/} & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{C}) \\ \downarrow & \lrcorner & \downarrow d_1 \\ \Delta^0 & \xrightarrow{\;x\;} & \mathcal{C} \end{array} .$$

**Lemma 5.2** ([Lur06, Corollary 2.1.2.2., Lemma 5.5.5.12.])**.** *Let $\mathcal{C}$ be an $\infty$-category and $x \in \mathcal{C}_0$. Then the slice categories $\mathcal{C}_{/x}$ and $\mathcal{C}_{x/}$ are both $\infty$-categories. Moreover, the mapping spaces of $f : y \to x$ and $g : z \to x$ in $\mathcal{C}_{/x}$ are given by pullback in $\mathcal{S}$*

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}_{/x}}(f, g) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(y, z) \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{\;f\;} & \mathrm{Map}_{\mathcal{C}}(y, x) \end{array}$$

**Remark 5.3.** Slice category admits limits/colimits if the original one does, c.f. [Lan21, Proposition 4.3.33].

**Notation 5.4.** Let $c \in \mathcal{K}(\mathcal{A})_0$. We define $\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c}$ to be the full subcategory of $\mathcal{K}(\mathcal{A})_{/c}$ with objects $(c' \to c)$ in $\mathcal{K}(\mathcal{A})_{/c}$ being quasi-isomorphisms. Dually, we define $\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{c/}$ with objects $(c' \to c)$ in $\mathcal{K}(\mathcal{A})_{c/}$ being quasi-isomorphisms.

Our goal is to use the following proposition to understand the mapping spaces in derived $\infty$-categories. We will prove it in the following section.

**Proposition 5.5.** *Let $c, d \in \mathcal{D}(\mathcal{A})_0$. Then we have the following equivalences*

$$\mathrm{Map}_{\mathcal{D}(\mathcal{A})}(c, d) \simeq \operatorname*{colim}_{(\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c})^{\mathrm{op}}} \mathrm{Map}_{\mathcal{K}(\mathcal{A})}(-, d) \simeq \operatorname*{colim}_{\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{d/}} \mathrm{Map}_{\mathcal{K}(\mathcal{A})}(c, -)$$

*where we view* $\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(-,d)$ *as a functor from* $(\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c})^{\mathrm{op}}$ *to* $\mathcal{S}$ *and similarly* $\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(c,-):\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{d/}\to\mathcal{S}$.

**Remark 5.6.** Again one can view $\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(-,d)$ (resp. $\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(c,-)$) as the homotopy coherent nerve of the underlying simplicial Hom functor to make $\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(-,d)$ into a functor. In general, one needs to use Lurie's straightening-unstraightening equivalence to makes sense of such functors, c.f.[Lur06, 3.2.] or [Lan21, 3.3.]. We briefly explain the idea here. Lurie's straightnening-unstraightnening equivalence states that for every $\infty$-category $\mathcal{C}$, there is an equivalence of $\infty$-categories

$$\mathrm{CoCart}(\mathcal{C})\simeq\mathrm{Fun}(\mathcal{C},\mathrm{Cat}_\infty)$$

where $\mathrm{CoCart}(\mathcal{C})$ is the subcategory of the slice category $(\mathrm{Cat}_\infty)_{/\mathcal{C}}$ whose objects are cocartesian fibrations $\mathcal{D}\to\mathcal{C}$ and morphisms are those preserves cocartesian morphisms. Applying to the inclusion $\mathcal{S}\to\mathrm{Cat}_\infty$, this gives

$$\mathrm{LFib}(\mathcal{C})\simeq\mathrm{Fun}(\mathcal{C},\mathcal{S})$$

where $\mathrm{LFib}(\mathcal{C})$ is the full subcategory of $(\mathrm{Cat}_\infty)_{/\mathcal{C}}$ whose objects are left fibrations over $\mathcal{C}$. Hence the left fibration $\mathcal{C}_{x/}\to\mathcal{C}$ (by [Lur06, Corollary 2.1.2.2.]) whose fibres are $\mathrm{Map}_{\mathcal{C}}(x,y)$ gives a functor $\mathcal{C}\to\mathcal{S}$ which is $\mathrm{Map}_{\mathcal{C}}(x,-)$.

**Remark 5.7.** Even though proving Proposition 5.5 needs some machinaries, it is not hard to see that there is a map

$$\operatorname*{colim}_{(\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c})^{\mathrm{op}}}\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(-,d)\to\mathrm{Map}_{\mathcal{D}(\mathcal{A})}(c,d).$$

Since quasi-isomorphisms become equivalences in $\mathcal{D}(\mathcal{A})$, we have a map

$$\operatorname*{colim}_{(\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c})^{\mathrm{op}}}\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(-,d)\to\operatorname*{colim}_{(\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c})^{\mathrm{op}}}\mathrm{Map}_{\mathcal{D}(\mathcal{A})}(-,d)$$

where the latter is a constant diagram by design. Since the index category obviously contains $c\xrightarrow{\mathrm{Id}}c$ the colimit is $\mathrm{Map}_{\mathcal{D}(\mathcal{A})}(c,d)$.

We now make the following definitions.

**Definition 5.8.** We say $c\in\mathcal{K}(\mathcal{A})_0$ is $\mathcal{K}$-projective if $\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(c,-)$ sends quasi-isomorphisms to homotopy equivalences. Similarly, We say $c\in\mathcal{K}(\mathcal{A})_0$ is $\mathcal{K}$-injective if $\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(-,c)$ sends quasi-isomorphisms to homotopy equivalences.

As a direct corollary of Proposition 5.5, we have the following.

**Corollary 5.9.** *If* $c\in\mathcal{K}(\mathcal{A})_0$ *is* $\mathcal{K}$*-projective or* $d\in\mathcal{K}(\mathcal{A})_0$ *is* $\mathcal{K}$*-injective, then*

$$\mathrm{Map}_{\mathcal{D}(\mathcal{A})}(c,d)\simeq\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(c,d)$$

*is an homotopy equivalence. In particular, given* $c\in\mathcal{K}(\mathcal{A})$ *and a quasi-isomorphism* $c'\to c$ *with* $c'$ *begin* $\mathcal{K}$*-projective. Then*

$$\mathrm{Map}_{\mathcal{D}(\mathcal{A})}(c,d)\simeq\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(c',d).$$

*Analogously, we have a corresponding statement for* $\mathcal{K}$*-injective.*

**Remark 5.10.** Recall the fundamental lemma of homological algebra states that bounded below, levelwise projective implies $\mathcal{K}$-projective; analogously bounded above levelwise injective implies $\mathcal{K}$-injective.

**Remark 5.11.** Corollary 5.9 recovers the usual computation for $R\mathrm{Hom}$ even though we have not proved any result on enough projective/injective objects.

We now want to understand limits and colimis in $\mathcal{D}(\mathcal{A})$. However, observe from Corollary 5.9 that if every object in $\mathcal{K}(\mathcal{A})$ is quasi-isomorphic to a $\mathcal{K}$-projective, then every limit cone in $\mathcal{K}(\mathcal{A})$ goes to a limit cone in $\mathcal{D}(\mathcal{A})$. Similarly, enough injective assures colimits are preserved. However, finite limits and colimits exist in $\mathcal{D}(\mathcal{A})$ and are preserved from $\mathcal{K}(\mathcal{A})$ without any of these results.

In order to understand limits and colimits, we first introduce the notion of (co)filtered in the context of $\infty$-category.

**Definition 5.12.** An $\infty$-category $\mathcal{C}$ is called filtered if for any finite simplicial set $K$ and any map $K \to \mathcal{C}$ extends over $(K \times \Delta^1) \sqcup_{K\times\{1\}} \Delta^0$. Dually, we call $\mathcal{C}$ cofilterd if $\mathcal{C}^{\mathrm{op}}$ if filtered. More explicitly, any map $K \to \mathcal{C}$ extends over $(K \times \Delta^1) \sqcup_{K\times\{0\}} \Delta^0$. In particular, filtered colimits are defined to be colimits indexed by a filtered $\infty$-category and dually for cofiltered limits.

**Remark 5.13.** By writing a finite simplicial set $K$ with dimension $n$ as the pushout of simplicial sets

$$\begin{array}{ccc} \coprod_J \partial\Delta^n & \longrightarrow & K' \\ \downarrow & & \downarrow \\ \coprod_J \Delta^n & \longrightarrow & K \end{array}$$

one may see that an $\infty$-category $\mathcal{C}$ is filtered if and only if it has the (right) extension property with respect to the inclusion $\partial\Delta^n \subset \Lambda^{n+1}_{n+1}$. See [Lur06, Lemma 5.3.1.14.] for a detailed proof.

**Remark 5.14.** Recall that a 1-category $\mathcal{C}$ is called filtered if

(1) for any $a, b \in \mathcal{C}$ there exists $c$ and morphisms $a \to c$, $b \to c$;

(2) for any two parallel morphisms $a \rightrightarrows b$ there exists a morphism $b \to c$ such that the two compositions are equal.

Then we claim that $\mathcal{C}$ is filtered if and only if $N(\mathcal{C})$ is filtered in the $\infty$-categorical sense.

Assume first that $N(\mathcal{C})$ is filtered. To check (1), take $K$ to be $\partial\Delta^1$ with two vertices given by $a, b$. As for (2), take $K$ to be the simplicial set obtained from $\partial\Delta^2$ by collapsing the initial face $\Delta^1$ to a point. Conversely, assume that $\mathcal{C}$ is filtered. Note that by Remark 5.13, it suffices to check the lifting property with respect to $\partial\Delta^n \subset \Lambda^{n+1}_{n+1}$. Since $\mathcal{C}$ is a 1-category, it suffices to consider the case when $n = 1, 2$ by [Cis19, Proposition 1.4.11.]. However, when $n = 1$ and $n = 2$, these two lifting properties correspond to condition (1) and (2) respectively. See [Lur06, Proposition 5.3.1.15.] for a more general statement.

**Remark 5.15.** Note that by Remark 2.20, an $\infty$-category $\mathcal{C}$ being filtered is equivalent of saying that for any finite diagram there exists a cone under such diagram, which is a weaker condition than asking for a universal cone, i.e. the colimit. In particular, if an $\infty$-category has finite colimits, then it is filtered.

The crucial property of filtered colimits is presented by the following lemma.

**Lemma 5.16.** *If $I$ is a filtered $\infty$-category and $J$ is a finite simplicial set, then the diagram*

$$\begin{array}{ccc} \mathrm{Fun}(I \times J, \mathcal{S}) & \xrightarrow{\lim_J} & \mathrm{Fun}(I, \mathcal{S}) \\ \downarrow{\scriptstyle \mathrm{colim}_I} & & \downarrow{\scriptstyle \mathrm{colim}_I} \\ \mathrm{Fun}(J, \mathcal{S}) & \xrightarrow{\lim_J} & \mathcal{S} \end{array}$$

*commutes.*

*Proof.* We only give a sketch here. As discussed before, one can first reduce to the case where $J$ is a pullback diagram (the case where $J$ is empty is trivial). The statement is equivalent of checking the commutativity for homotopy pullbacks and filtered homotoply colimits in the model category Kan. It reduces to a general fact in a combinatorial model category.

On one hand, in a combinatorial model category, filtered colimits are already homotopy colimits by [nA, filtered homotopy colimit, Proposition 2.1.] and preserve weak equivalences. One the other hand, every diagram $a \to b \leftarrow c$ is weak equivalent to diagram [DS95, Proposition 10.6.] $a' \to b' \leftarrow c'$ with $b'$ fibrant and morphisms $a' \to b'$, $c' \to b'$ fibrant and the homotopy pullback of $a \to b \leftarrow c$ is computed by the usual pullback of $a' \to b' \leftarrow c'$, c.f. [Lur06, A.2.4]. Then the statement follows from the commutativity for pullbacks and filtered colimits in an ordinary category. □

**Lemma 5.17.** *If $x \in \mathcal{S}_0$ is obtained from a finite simplicial sets, then*

$$\mathrm{Map}_{\mathcal{S}}(x, -) : \mathcal{S} \to \mathcal{S}$$

*commutes with filtered colimits.*

*Proof.* Fix a filtered diagram $F : I \to \mathcal{S}$. We need to check that for $s \in \mathcal{S}_0$,

$$\mathrm{Map}_{\mathcal{S}}(x, \operatorname*{colim}_I F) \simeq \operatorname*{colim}_I \mathrm{Map}_{\mathcal{S}}(x, F(-))$$

is an equivalence. We use [nA, filtered homotopy colimit, Proposition 2.1.] again that filtered colimits in $\mathcal{S}$ are actually taken in Kan. By Theorem 1.32, $\mathrm{Map}_{\mathcal{S}}(-,-) \simeq \underline{\mathrm{Hom}}_{\mathrm{Kan}}(-,-)$, hence the statement reduces to show that in the simplicial enriched category Kan, the functor $\underline{\mathrm{Hom}}_{\mathrm{Kan}}(x,-)$ commutes with filtered colimits. Since colimits in Kan are computed levelwise and $\underline{\mathrm{Hom}}_{\mathrm{Kan}}(x,-)_n = \mathrm{Hom}_{\mathrm{Kan}}(x \times \Delta^n, -)$ and $x \times \Delta^n$ is again a finite Kan complex, we reduce to show that $\mathrm{Hom}_{\mathrm{Kan}}(x,-)$ commutes with filtered colimits.

By [Ker, Proposition 3.6.1.7.], there is some simplicial set $y$ which is a finite coproduct of standard simplices that surjects onto $x$. Moreover, since $y \times_x y$ is also a finite simplicial set, one can apply [Ker, Proposition 3.6.1.7.] again to find a simplicial set $z$ which is a finite coproduct of standard simplices that surjects onto $y \times_x y$. Then $x$ can be realized as the coequalizer of $z$ and $y$. Hence one can realizes $\mathrm{Map}_{\mathcal{S}}(x,-)$ as finite products and equalizers of $\mathrm{Map}_{\mathcal{S}}(\Delta^n,-)$. By Lemma 5.16, $\mathrm{colim}_I \mathrm{Map}(x, F(-))$ is equivalent to finite products and equalizers of $\mathrm{colim}_I \mathrm{Map}(\Delta^n, F(-))$. Hence, it suffices to show the statement for $\Delta^n$ which is obvious since colimits are computed levelwise in Kan. □

**Remark 5.18.** This shows that finite Kan complexes are compact objects in $\mathcal{S}$.

**Corollary 5.19.** *If $K \in \mathcal{S}_0$ is obtained from a finite simplicial sets, then*

$$[K, -]_{\mathcal{S}} = \pi_0(\mathrm{Map}_{\mathcal{S}}(K, -)) : \mathcal{S} \to N(\mathrm{Set})$$

*commutes with filtered colimits.*

*Proof.* Note that $\pi_0 : \mathcal{S} \to N(\mathrm{Set})$ commutes with arbitrary colimits. Indeed $\pi_0$ is the left adjoint to the canonical embedding of discrete sets $N(\mathrm{Set}) \to \mathcal{S}$, hence preserves colimits. One can also check that $\pi_0$ preserves all the coproducts and pushouts. □

Now we get back to the discussion of derived $\infty$-categories with the results above.

**Lemma 5.20.** *The sliced categories $(\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c})^{\mathrm{op}}$ and $\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{d/}$ defined in Proposition 5.5 are filtered.*

*Proof.* We will show a stronger statement that $\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{d/}$ (and dually $(\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c})^{\mathrm{op}}$) have all finite colimits. Since $\mathcal{K}(\mathcal{A})_{d/}$ has all finite colimits, we will check that $\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{d/} \subset \mathcal{K}(\mathcal{A})_{d/}$ is closed under finite colimits. Consider the following composition of functors

$$\mathcal{K}(\mathcal{A})_{d/} \to \mathrm{Fun}(\Delta^1, \mathcal{K}(\mathcal{A})) \xrightarrow{\mathrm{cofib}} \mathcal{K}(\mathcal{A})$$

where the first is the canonical map and the second is the cofibre functor as in Remark 4.11. On the level of objects, it sends $(c \to c')$ to $\mathrm{cofib}(c \to c')$. By slightly abusing notations we also denote the composition as cofib. Note that we can characterize $\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{d/}$ by the cofibre functor, namely $(c \to c')$ is a quasi-isomorphism if and only if $\mathrm{cofib}(c \to c')$ is acyclic, i.e. zero homology groups. Moreover, since cofibre is a colimit, cofib preserves colimits. Hence it suffices to check that the acyclic objects in $\mathcal{K}(\mathcal{A})$ is closed under finite colimits. Then it amounts to check it for finite coproducts and pushouts which can be done explicitly by the formula of (homotopy) pushout. The proof is similar for $(\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c})^{\mathrm{op}}$. □

We now come to the main proposition.

**Proposition 5.21.** *The derived $\infty$-category $\mathcal{D}(\mathcal{A})$ admits all finite limits and colimits. Moreover, $\mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ preserves all finite limits and colimits.*

*Proof.* Clearly, 0 is both $\mathcal{K}$-projective and $\mathcal{K}$-injective. Hence by Corollary 5.9, 0 is still the initial and the terminal object. As before, it suffices to show the statement for finite products and finite coproducts, pushouts and pullbacks. We will just show it for pushouts and the rest will be similar. Given a pushout diagram

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & \ulcorner & \downarrow \\ c & \longrightarrow & d \end{array}$$

in $\mathcal{K}(\mathcal{A})$, it is the same as giving a pullback in $\mathcal{S}$

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{K}(\mathcal{A})}(d, z) & \longrightarrow & \mathrm{Map}_{\mathcal{K}(\mathcal{A})}(c, z) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Map}_{\mathcal{K}(\mathcal{A})}(b, z) & \longrightarrow & \mathrm{Map}_{\mathcal{K}(\mathcal{A})}(a, z) \end{array}$$

for any $z \in \mathcal{K}(\mathcal{A})$. On one hand, the corresponding diagram in $\mathcal{D}(\mathcal{A})$ arises as colimits indexed by $\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{z/}$ by Proposition 5.5. On the other hand, $\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{z/}$ is filtered by Lemma 5.20. Hence we win by Lemma 5.16.

Finally, we check for the existence of pushouts in $\mathcal{D}(\mathcal{A})$. Given a pushout diagram $c \leftarrow a \rightarrow b$ in $\mathcal{D}(\mathcal{A})$, we have

$$\pi_0(\mathrm{Map}_{\mathcal{D}(\mathcal{A})}(a, c)) \simeq \operatorname*{colim}_{\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{c/}} \pi_0(\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(a, -))$$

by Proposition 5.5 and Corollary 5.19. However, this is filtered of colimit of sets. This means that up to homotopy equivalence, there exists quasi-isomorphism $c \to c'$ such that the composition $a \to c \to c'$ comes from $\mathcal{K}(\mathcal{A})$. Argue similarly for $a \to b$, we have the following diagram

$$\begin{array}{ccccc} c & \longleftarrow & a & \longrightarrow & b \\ \downarrow{\scriptstyle \mathrm{qi}} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{qi}} \\ c' & \longleftarrow & a & \longrightarrow & b' \end{array}.$$

Now we get an equivalent diagram $c' \leftarrow a \rightarrow b'$ that comes from $\mathcal{K}(\mathcal{A})$. □

Note that the Dwyer-Kan localization (or Verdier quotient) here is somehow tame. In general, it could be very wild. We state the following two propositions without proof.

**Proposition 5.22.** *Take* $\mathrm{Set}_\Delta := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$ *the 1-category of simplicial sets and consider its nerve* $N(\mathrm{Set}_\Delta)$. *Take* $W$ *as the weak equivalences of simplicial sets, then one have*

$$N(\mathrm{Set}_\Delta)[W^{-1}] \simeq \mathcal{S}.$$

**Proposition 5.23** ([Lan21, Theorem 2.4.10.])**.** *Take* $\mathrm{Ch}(\mathcal{A})$ *as the 1-category of chain complexes and consider its nerve* $N(\mathrm{Ch}(\mathcal{A}))$. *Take* $W$ *as quasi-isomorphisms. Then one have*

$$N(\mathrm{Ch}(\mathcal{A}))[W^{-1}] \simeq \mathcal{D}(\mathcal{A}).$$

*Moreover, take* $V$ *to be the chain homotopy equivalences, then*

$$N(\mathrm{Ch}(\mathcal{A}))[V^{-1}] \simeq \mathcal{K}(\mathcal{A}).$$

**Remark 5.24.** The two examples above show that even thought, one start with some 1-category with discrete mapping spaces, after apply Dwyer-Kan localization, the mapping spaces could become very different. Moreover, limits and colimits behave entirely different.

## 6. Derived Functors

We will discuss derived functors in this section. Again $\mathcal{A}$ will be an abelian category and $\mathcal{D}(\mathcal{A})$ will be its derived category.

**Construction 6.1.** Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories, i.e. $F(0) = 0$ and $F(a \oplus a') \simeq F(a) \oplus F(a')$ for any $a, a' \in \mathcal{A}$. We get a induced functor of dg categories

$$\mathrm{Ch}(F) : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B}).$$

Or in other words, it is an enriched functor over $\mathrm{Ch}(\mathbb{Z})$. Then one gets an induced functor

$$\mathcal{K}(F) : \mathcal{K}(\mathcal{A}) = N(\mathrm{Ch}(\mathcal{A})) \to N(\mathrm{Ch}(\mathcal{B})) = \mathcal{K}(\mathcal{B})$$

which commutes finite limits and colimits. Note that $\mathcal{K}(F)$ certainly preserves 0 which is the terminal object. And since pushouts are given by double mapping cones (as in 4.10) which is a direct sum of objects, it is preserved by $\mathcal{K}(F)$ by the additivity of $F$. Hence it preserves finite colimits. One can argue dually for limits. In other words, the functor $\mathcal{K}(F)$ is exact.

**Definition 6.2.** A functor between two $\infty$-categories is called exact if it preserves limits and colimtis.

We would like to get an induced functor $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ such that

$$\begin{array}{ccc} \mathcal{K}(\mathcal{A}) & \xrightarrow{\mathcal{K}(F)} & \mathcal{K}(\mathcal{B}) \\ \downarrow{\scriptstyle p_{\mathcal{A}}} & & \downarrow{\scriptstyle p_{\mathcal{B}}} \\ \mathcal{D}(\mathcal{A}) & \xrightarrow{\mathcal{D}(F)} & \mathcal{D}(\mathcal{B}) \end{array}$$

is a commutative diagram in $\mathrm{Cat}_\infty$. To construct such a $\mathcal{D}(F)$, one would incline to use the universal property of Dwyer-Kan localization, namely the functor $\mathcal{K}(F)$ preserves quasi-isomorphisms. However, this rarely happens.

**Lemma 6.3.** *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Then the following are equivalent:*

*(1) There exists a functor $\mathcal{D}(F) : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ making the following square*

$$\begin{array}{ccc} \mathcal{K}(\mathcal{A}) & \xrightarrow{\mathcal{K}(F)} & \mathcal{K}(\mathcal{B}) \\ \downarrow{\scriptstyle p_{\mathcal{A}}} & & \downarrow{\scriptstyle p_{\mathcal{B}}} \\ \mathcal{D}(\mathcal{A}) & \xrightarrow{\mathcal{D}(F)} & \mathcal{D}(\mathcal{B}) \end{array}$$

*commutative in* $\mathrm{Cat}_\infty$*;*

*(2) The functor $\mathcal{K}(F)$ preserves quasi-isomorphisms;*

*(3) The functor $F$ is exact.*

*Proof.* Assume (3), then $\mathrm{Ch}(F)$ preserves quasi-acyclic objects, hence quasi-isomorphisms. Thus $\mathcal{K}(F)$ preserves quasi-isomorphisms which is (2). Assume (2), then the universal property of Dwyer-Kan localization implies (1). (1) implies (2) is obvious. As for (2) implies (1), we show for example

$F$ preserves kernel. Let $a = a_1 \to a_0$ and replace $a_0$ be the image if necessary we may assume the morphism is surjective. Then $a$ is quasi-isomorphic to $\ker(a_1 \to a_0)$. Since quasi-isomorphism is preserved, we know that $F(\ker(a_1 \to a_0)) \simeq \ker(F(a_1) \to F(a_0))$. Similarly for cokernel. □

What one should hope for is the second best, i.e. there exists a natural transformation $\eta : \mathcal{D}(F) \circ p_{\mathcal{A}} \to p_{\mathcal{B}} \circ \mathcal{K}(F)$. Obviously, one can take $\mathcal{D}(F)$ to be zero. Hence one should ask for the "largest one", i.e. a universal one.

**Definition 6.4.** A left derived functor $LF$ of $F : \mathcal{A} \to \mathcal{B}$ is given by a functor $LF : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ together with a natural transformation

$$\eta : LF \circ p_{\mathcal{A}} \to p_{\mathcal{B}} \circ \mathcal{K}(F)$$

such that for any other functor $H : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$, the mapping space

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{D}(\mathcal{A}),\mathcal{D}(\mathcal{B}))}(H, LF) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Fun}(\mathcal{K}(\mathcal{A}),\mathcal{D}(\mathcal{B}))}(H \circ p_{\mathcal{A}}, p_{\mathcal{B}} \circ K(F))$$

is an equivalence by precomposing with $p_{\mathcal{A}}$ and applying $\eta$.

**Remark 6.5.** If a left derived functor exists, then it is unique up to a trivial Kan complex by the universal property. However, the left derived functor might not exist in our generality. We will see that it exists if, for example, there is enough projectives.

One immediate consequence is the following.

**Proposition 6.6.** *If $LF$ exists, then it preserves finite limits and colimits.*

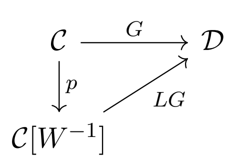
*Proof.* We first show that $LF$ preserves direct sums. By Theorem 6.10 and the fact that $\mathcal{K}(F)$ preserving direct sumes, it suffices to show that for any $a, b \in \mathcal{K}(\mathcal{A})$ the canonical map

$$\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/a} \times \mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/b} \to \mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/a\oplus b}$$

is cofinal. But for any $c \to a \oplus b$ one can just take $\mathrm{im}(c \to a) \oplus \mathrm{im}(c \to b)$. Hence $LF$ preserves direct sums. However, all pullbacks and pushouts in $\mathcal{D}(\mathcal{A})$ are commuted by mapping cones by Proposition 5.21 hence is also preserved by $LF$. □

We want to give a slightly more general definition of derived functors in the context of Dwyer-Kan localization.

**Definition 6.7.** Given an $\infty$-category $\mathcal{C}$ with a class of morphisms $W$ and a functor $G : \mathcal{C} \to \mathcal{D}$ between infinite categories. Then the left derived functor $LG$ is given by a diagram




with a natural transformation $\eta : LG \to G$ such that

$$\eta^* : \mathrm{Map}_{\mathrm{Fun}^W(\mathcal{C},\mathcal{D})}(H, LG) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(H, G)$$

is an equivalence for any $H \in \mathrm{Fun}^W(\mathcal{C}, \mathcal{D})$. Here we have slightly abused notations by identifying $\mathrm{Fun}^W(\mathcal{C}, \mathcal{D})$ and $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D})$. We say that $LG$ is the absolute left derived functor if for any functor $T : \mathcal{D} \to \mathcal{E}$ of $\infty$-categories the induced triangle

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} & \xrightarrow{T} & \mathcal{E} \\ \downarrow{\scriptstyle p} & & & \nearrow{\scriptstyle T\circ LG} & \\ \mathcal{C}[W^{-1}] & & & & \end{array}$$

exhibits $(T \circ LG, T \circ \eta)$ as the left derived functor of $T \circ G$.

**Remark 6.8.** It is useful to talk about the notion of absolute left derived functors. It is in someway saying that the left derived functor is stable base change. For example, Grothendieck spectral sequences crucially uses the notion of absolute. And we will see that there are cases where a left derived functor is not absolute.

**Remark 6.9.** We will see later that providing a left derived functor is some best approximation of providing a section to $\mathrm{Fun}^W(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. And this is the principle of right Kan extensions, providing a right adjunction to the inclusion.

We now give the main theorem.

**Theorem 6.10.** *For any functor $G : \mathcal{K}(\mathcal{A}) \to \mathcal{D}$ to any $\infty$-category $\mathcal{D}$, we have that*

$$LG(c) \simeq \lim_{\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c}} G$$

*provided that the limit exists for each $c \in \mathcal{K}(\mathcal{A})$. Here we view $G$ as a functor $\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c} \xrightarrow{d_1} \mathcal{K}(\mathcal{A}) \to \mathcal{D}$.*

**Example 6.11.** We have the following examples.

(1) Let $c = 0$, then by Theorem 6.10 we have

$$LG(0) \simeq \lim_{\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/0}} G \simeq G(0).$$

since 0 is the initial object in the quasi-acyclic category $\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/0}$.

(2) For any $c \in \mathcal{K}_0$ with a projective resolution, i.e. $p : \hat{c} \xrightarrow{\simeq} c$ is quasi-isomorphic with $\hat{c}$ being $\mathcal{K}$-projective, then we claim that $(p : \hat{c} \xrightarrow{\simeq} c)$ is initial in $\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c}$. To see this, we note that for any other object $d$ quasi-isomorphic to $c$, we have a pullback diagram in $\mathcal{S}$ by Lemma 5.2

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{K}(\mathcal{A})_{/c}}((\hat{c} \to c), (d \to c)) & \longrightarrow & \mathrm{Map}_{\mathcal{K}(\mathcal{A})}(\hat{c}, d) \\ \downarrow & \lrcorner & \downarrow \\ \Delta^0 & \xrightarrow{p} & \mathrm{Map}_{\mathcal{K}(\mathcal{A})}(\hat{c}, c) \end{array}$$

Since $\hat{c}$ is $\mathcal{K}$-projective, $\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(\hat{c}, -)$ preserves quasi-isomorphism. Hence the left vertical map is an equivalence. Therefore, the right vertical map is also an equivalence showing that

$\mathrm{Map}_{\mathcal{K}(\mathcal{A})_{/c}}((\hat{c} \to c), (d \to c))$ is contractible. Thus we win. This shows that $LG(c) = G(\hat{c})$ for any $\mathcal{K}$-projective resolution $\hat{c} \xrightarrow{\simeq} c$ of $c$ for any functor $G$.

Note that if we assume that every object $c$ there exists a $\mathcal{K}$-projective resolution $\hat{c}$ then

$$LG(c) = G(\hat{c})$$

for any functor $G$. In particular, this is compatible with postcomposition with functors $T : \mathcal{D} \to \mathcal{E}$, i.e. in this case every left derived functor is absolute.

We now turn to an example of the original setup.

**Example 6.12.** Assume that $F : \mathcal{A} \to \mathcal{B}$ is an additive functor between small abelian categories. Assume also that $\mathcal{B}$ satisfies (AB4*), i.e. it has all infinite products and they are exact, c.f. [Sta, Definition 079B]. Then the large $\infty$-category $\mathcal{D}(\mathcal{B})$ has all (small) limits. To see this we use the fact that a large $\infty$-category $\mathcal{C}$ has all small colimits when $\mathcal{C}$-indexed colimits in the $\infty$-category of large spces commutes with all small products not just finite ones. Then one can mimic the proof of Proposition 5.21 and deduce that $\mathcal{D}(\mathcal{B})$ has all limits and comes from $\mathcal{K}(\mathcal{B})$. In particular, it has all the cofiltered limits. Hence $LF$ exists by Theorem 6.10 and $LF$ preserves finite colimits and limits.

There is a dual notion of right derived functor.

**Definition 6.13.** Given an $\infty$-category $\mathcal{C}$ with a class of morphisms $W$ and a functor $G : \mathcal{C} \to \mathcal{D}$ between infinite categories. Then the right derived functor $LG$ is given by a diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} \\ \downarrow{\scriptstyle p} & \nearrow_{RG} & \\ \mathcal{C}[W^{-1}] & & \end{array}$$

with a natural transformation $\eta : G \to RG$ such that

$$\eta_* : \mathrm{Map}_{\mathrm{Fun}^W(\mathcal{C},\mathcal{D})}(RG, H) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(G, H)$$

is an equivalence for any $H \in \mathrm{Fun}^W(\mathcal{C}, \mathcal{D})$. Similarly, we say that $RG$ is the absolute right derived functor if for any functor $T : \mathcal{D} \to \mathcal{E}$ of $\infty$-categories the induced triangle

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} & \xrightarrow{T} & \mathcal{E} \\ \downarrow{\scriptstyle p} & & & \nearrow_{T \circ RG} & \\ \mathcal{C}[W^{-1}] & & & & \end{array}$$

exhibits $(T \circ RG, T \circ \eta)$ as the right derived functor of $T \circ G$.

Then the dual of the formula Theorem 6.10 applying to $\mathcal{K}(\mathcal{A})^{\mathrm{op}} \simeq \mathcal{K}(\mathcal{A}^{\mathrm{op}})$ gives:

**Theorem 6.14.** *For any functor $G : \mathcal{K}(\mathcal{A}) \to \mathcal{D}$ to any $\infty$-category $\mathcal{D}$, we have that*

$$RG(c) \simeq \operatorname*{colim}_{(\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{c/})^{\mathrm{op}}} G$$

*provided that the colimit exists for each $c \in \mathcal{K}(\mathcal{A})$.*

Again we provide some examples.

**Example 6.15.** For any $a, b \in \mathcal{K}(\mathcal{A})_0$, consider the functors (again after a choice) of mapping spaces

$$\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(a, -) : \mathcal{K}(\mathcal{A}) \to \mathcal{S},$$
$$\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(-, b) : \mathcal{K}(\mathcal{A})^{\mathrm{op}} \to \mathcal{S}.$$

We have

$$\mathrm{Map}_{\mathcal{D}(\mathcal{A})}(a, -) \simeq R\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(a, -),\ \mathrm{Map}_{\mathcal{D}(\mathcal{A})}(-, b) \simeq R\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(-, b).$$

Combing with Theorem 6.14, one get Proposition 5.5. We will see next time that this is a general phenomenon of a Dwyer-Kan localization. We will develop the machinary in the next section to derive a nonadditive functor from start. In this way, one can provide the derived functor

$$\mathrm{Map}_{\mathcal{K}(\mathcal{A})}(-, -) : \mathcal{K}(\mathcal{A}) \times \mathcal{K}(\mathcal{A})^{\mathrm{op}} \to \mathcal{S}.$$

We now give a proof (sketch) of Theorem 6.10 and dually for 6.14.

*Proof of Theorem 6.10.* Assume that $\lim_{\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c}} G$ exists for any $c$. We claim that there is a functor

$$LG : \mathcal{K}(\mathcal{A}) \to \mathcal{D}$$

given by $c \mapsto \lim_{\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c}} G$. This can be formed into a functor by considering a family of diagram parametrized by $\mathcal{K}(\mathcal{A})$. More specificly, each fibre $\Delta^0 \xrightarrow{c} \mathcal{K}(\mathcal{A})$ gives a diagram

$$\begin{array}{ccc} \mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c} & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{K}(\mathcal{A})^{\mathrm{qi}}) \\ \downarrow{\scriptstyle G} & \lrcorner & \downarrow{\scriptstyle (d_0, G)} \\ \mathcal{D} & \longrightarrow & \mathcal{K}(\mathcal{A}) \times \mathcal{D} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_2} \\ \Delta^0 & \xrightarrow{\quad c \quad} & \mathcal{K}(\mathcal{A}) \end{array}.$$

Then the dual version of [Lur06, Proposition 4.2.2.7.] provides a map $\mathcal{K}(\mathcal{A}) \diamond_{\mathcal{K}(\mathcal{A})} \mathrm{Fun}(\Delta^1, \mathcal{K}(\mathcal{A})^{\mathrm{qi}}) \to \mathcal{D} \times \mathcal{K}(\mathcal{A})$. Then we define $LG$ to be the composition

$$\mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{A}) \diamond_{\mathcal{K}(\mathcal{A})} \mathrm{Fun}(\Delta^1, \mathcal{K}(\mathcal{A})^{\mathrm{qi}}) \to \mathcal{D} \times \mathcal{K}(\mathcal{A}) \to \mathcal{D}$$

which sends $c \mapsto \lim_{\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c}} G$. This gives a natural transformation $\eta : LG \to G$ by projecting onto $\mathrm{Id} : c \to c$.

Next, note that $LG$ preserves weak equivalences. This is because if $p : c \xrightarrow{\simeq} c'$ is a quasi-isomorphim, then for any $(d \to c') \in \mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c'}$ the slice category over it admits a terminal object $c \xrightarrow{\mathrm{Id}} c$. Hence by [Lan21, Proposition 5.2.6.], $p^* : \mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c} \to \mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/c'}$ admits a right adjoint hence coinitial by [Lan21, Remark 5.2.8.]. Hence they give equivalent limit.

Observe also that if $G$ preserves quasi-isomorphisms, then $\eta : LG \to G$ is an equivalence since $LG$ is given by a constant diagram by design. Here we used Remark 1.15 we can check a natural transformation is an equivalence by pointwise.

Finally, we claim that the $LG$ constructed above is the left derived functor. We need to check that

$$\eta^* : \mathrm{Map}_{\mathrm{Fun}^W(\mathcal{C},\mathcal{D})}(H, LG) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(H, G)$$

is an equivalence for any $H \in \mathrm{Fun}^W(\mathcal{C},\mathcal{D})$. Consider the functor

$$L : \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(H, G) \to \mathrm{Map}_{\mathrm{Fun}^W(\mathcal{C},\mathcal{D})}(LH, LG) \simeq \mathrm{Map}_{\mathrm{Fun}^W(\mathcal{C},\mathcal{D})}(H, LG)$$

where the last equivalence is given by the above observation. Now it suffices to check that $L$ is actually an inverse to $\eta^*$. It boils down to check that $\eta^* \circ L$ and $L \circ \eta^*$ are homotopic to each other on $LG$. But this is easy to see using the explicit formula and limits commute □

We then summarize the above phenomenon into perspective using adjunctions and Kan extensions. As one can see directly from the definition the functor $L$ is the right adjoint to the inclusion

$$\iota : \mathrm{Fun}^W(\mathcal{C},\mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C},\mathcal{D})$$

namely for any $H \in \mathrm{Fun}^W(\mathcal{C},\mathcal{D})$ and $G \in \mathrm{Fun}(\mathcal{C},\mathcal{D})$ to find a $L(G)$ such that

$$\mathrm{Map}_{\mathrm{Fun}^W(\mathcal{C},\mathcal{D})}(H, L(G)) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(\iota(H), G).$$

We now give some definitions to make things precise.

**Definition 6.16.** Let $\mathcal{C}$ and $\mathcal{D}$ be $\infty$-categories with functors

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D} \ .$$

A natural transformation $\epsilon : L \circ R \to \mathrm{Id}_{\mathcal{D}}$ is called counit of an adjunction if for any given pair of objects $c \in \mathcal{C}$, $d \in \mathcal{D}$ the induced map

$$\mathrm{Map}_{\mathcal{C}}(c, R(d)) \xrightarrow{L} \mathrm{Map}_{\mathcal{D}}(L(c), L \circ R(d)) \xrightarrow{\epsilon_*} \mathrm{Map}_{\mathcal{D}}(L(c), d)$$

is a homotopy equivalence.

Dually, a natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \to R \circ L$ is called unit of an adjunction if for any given pair of objects $c \in \mathcal{C}$, $d \in \mathcal{D}$ the induced map

$$\mathrm{Map}_{\mathcal{D}}(L(c), d) \xrightarrow{R} \mathrm{Map}_{\mathcal{C}}(R \circ L(c), R(d)) \xrightarrow{\eta^*} \mathrm{Map}_{\mathcal{C}}(c, R(d))$$

is a homotopy equivalence.

If either a counit or a unit exists, then we call $L$ is a left adjunction to $R$, and denote it as $L \dashv R$.

We will see that a counit exists is equivalent to a unit exists. To do this, we introduce the following terminology.

**Definition 6.17.** Given $L, R$ as in Definition 6.16 and natural transformations $\epsilon : L \circ R \to \mathrm{Id}_{\mathcal{D}}$, $\eta : \mathrm{Id}_{\mathcal{C}} \to R \circ L$, we say that the Zig-Zag identities hold if the following compositions

$$L = L \circ \mathrm{Id}_{\mathcal{C}} \xrightarrow{\eta} L \circ R \circ L \xrightarrow{\epsilon} \mathrm{Id}_{\mathcal{D}} \circ L = L,$$

$$R = \mathrm{Id}_{\mathcal{C}} \circ R \xrightarrow{\eta} R \circ L \circ R \xrightarrow{\epsilon} R \circ \mathrm{Id}_{\mathcal{D}} = R$$

are equivalent to $\mathrm{Id}_L \in \mathrm{Fun}(\mathcal{C},\mathcal{D})$ and $\mathrm{Id}_R \in \mathrm{Fun}(\mathcal{D},\mathcal{C})$ respectively.

The main ingredients is the following.

**Proposition 6.18.** *Given $L, R$ as in Definition 6.16 and natural transformations $\epsilon : L \circ R \to \mathrm{Id}_{\mathcal{D}}$, $\eta : \mathrm{Id}_{\mathcal{C}} R \circ L$, then $\epsilon$ and $\eta$ satisfies the Zig-Zag identities, then there are counit and unit of an adjunction. Conversely, given a unit of an adjunction, then there is a unique counit such that the Zig-Zag identities hold.*

*Proof.* We will only prove the first part. Then second part is formal (using the $\infty$-categorical Yoneda lemma). Assume that $\eta$ and $\epsilon$ satisfies the Zig-Zag identities, then we claim that

$$\mathrm{Map}_{\mathcal{C}}(c, R(d)) \xrightarrow{L} \mathrm{Map}_{\mathcal{D}}(L(c), L \circ R(d)) \xrightarrow{\epsilon_*} \mathrm{Map}_{\mathcal{D}}(L(c), d)$$

and

$$\mathrm{Map}_{\mathcal{D}}(L(c), d) \xrightarrow{R} \mathrm{Map}_{\mathcal{C}}(R \circ L(c), R(d)) \xrightarrow{\eta^*} \mathrm{Map}_{\mathcal{C}}(c, R(d))$$

are inverse to each other. However, this is exact the condition of being a Zig-Zag identity. □

**Remark 6.19.** Note that even in the case where $F : \mathcal{C} \to \mathcal{D}$ is an equivalence, it is not entirely obvious that it is the left and right adjoint to its inverse. Note that one really has to choose a natural transformation that satisfies the Zig-Zag identity. See [RV22, Lemma 2.1.11.].

**Remark 6.20.** One can also characterize adjunctions by cartesian and cocartesian fibration. Namely, an adjuntion is a bicartesian fibration $\mathcal{E} \to \Delta^1$, i.e. a functor which is both a cartesian and a cocartesian fibration by [Lur06, Proposition 5.2.2.8.]. For example, if one wants to prove that $f : \mathcal{C} \to \mathcal{D}$ has a right adjoint, then by Lurie's straightening-unstraightening equivalence it suffices to show the corresponding cocartesian fibration $\mathcal{E} \to \Delta^1$ is a cartesian. This happens for example when the right exist locally, c.f. Proposition 6.24.

We now state some easy consequences of adjunctions.

**Lemma 6.21** ([Lur06, Proposition 5.2.1.3.])**.** *Given a functor $L : \mathcal{C} \to \mathcal{D}$ of $\infty$-catgories, then a right adjoint $(R, \eta)$ is unique (up to a trivial Kan complex) if exists. And similarly for left adjoint.*

**Lemma 6.22** ([Lur06, Proposition 5.2.6.2.])**.** *The composition of left adjoints is again the left adjoint with right adjoint given by the composition of right adjoints. More coherently*

$$\mathrm{Cat}_{\infty}^{L} \xrightarrow{\simeq} (\mathrm{Cat}_{\infty}^{R})^{\mathrm{op}}$$

*where $\mathrm{Cat}_{\infty}^{L}$ is the subcategory of $\mathrm{Cat}_{\infty}$ consists of only left adjoint functors and similarly for $\mathrm{Cat}_{\infty}^{R}$; and the map is given by sending a left adjoint to its associated right adjoint.*

We now observe the relation between adjoint functors with limits and colimits.

**Remark 6.23.** Let $I$ be a small $\infty$-category and consider the constant functor $\Delta : \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$. Assume that $\Delta$ admits a left adjoint functor $L$. Then for any functor $F : I \to \mathcal{C}$ we get from the unit a map

$$F \to \Delta \circ L(F)$$

such that the induced map

$$\mathrm{Map}_{\mathcal{C}}(L(F), y) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{C}^I}(F, \Delta(y))$$

is an equivalence. This shows that $L(F) \simeq \mathrm{colim}_I F$. Dually for limits. In fact, the opposite is also true.

**Proposition 6.24** ([Lan21, Proposition 5.1.10.])**.** *Let $I$ be a small $\infty$-category. If $\mathcal{C}$ has all $I$-shaped colimits then $\Delta$ admits a left adjoint. More generally, assume that $L : \mathcal{C} \to \mathcal{D}$ has pointwise right adjoint, namely for any object $d \in \mathcal{D}_0$ there is an object $R(d) \in \mathcal{C}_0$ with a map $L \circ R(d) \to d$ such that*

$$\mathrm{Map}_{\mathcal{C}}(c, R(d)) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{D}}(L(c), d)$$

*is an equivalence. Then $L$ admits a right adjoint $R$ given pointwise by $R(d)$.*

We first give two basic examples.

**Example 6.25.** The homotopy class $\pi_0 : \mathcal{S} \to N(\mathrm{Set})$ is left adjoint to the inclusion functor $N(\mathrm{Set}) \to \mathcal{S}$ of discrete spaces.

**Example 6.26.** The forgetful functor $\mathcal{C}_{/x} \to \mathcal{C}$ from the slice category of an $\infty$-category $\mathcal{C}$ is a left adjoint if $\mathcal{C}$ has products. The right adjoint is given by $c \to c \times x$.

Similarly as in the ordinary categories case, one can check the following proposition by checking the corresponding statement in mapping spaces.

**Proposition 6.27.** *Left adjoint functors preserves colimits and right adjoint functors preserve limits.*

We finally come to the point of the relation between adjoint functors and derived functors.

**Definition 6.28.** Given a functor $p : \mathcal{C} \to \mathcal{C}'$ and any functor $F : \mathcal{C} \to \mathcal{D}$ between $\infty$-categories. Then a triangle

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle p}\downarrow & \nearrow_{p_*(F)} & \\ \mathcal{C}' & & \end{array}$$

together with a natural transformation $\eta : p_*(F) \circ p \to F$ is said to exhibts $p_*(F)$ as the right Kan extension $F$ along $p$ if it is terminal, i.e.

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}',\mathcal{D})}(H, p_*(F)) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(H \circ p, F)$$

is an equivalence for any $H$. Similarly, one can define the left Kan extension and we will denote it as $p_!(F)$.

**Remark 6.29.** The limit over $F : I \to \mathcal{C}$ is actually the right Kan extension of $I \to *$ where $*$ is the terminal simplicial set, c.f. [nA, Kan extension, Proposition 2.2] or [Cis19, Theorem Proposition 6.4.9.].

**Remark 6.30.** Analogously to the case of limit, the colimit of a functor $F : I \to \mathcal{C}$ is given by left Kan extension along $I \to *$. See [nA, Kan extension, Proposition 2.2.] or combine [Cis19, Proposition 6.1.14.] and the proof of [Cis19, Proposition 6.4.9.].

By Proposition 6.24, we have the following corollary.

**Corollary 6.31.** *If the right Kan extension $p_*(F)$ exists for any $F : \mathcal{C} \to \mathcal{D}$ then the restriction functor*

$$\mathrm{Fun}(\mathcal{C}', \mathcal{D}) \underset{p_*}{\overset{p^*}{\rightleftarrows}} \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \ .$$

*exhibits a right adjoint $p_*$. Similarly, if the left Kan extension $p_!(F)$ exists for any $F : \mathcal{C} \to \mathcal{D}$ then the restriction functor*

$$\mathrm{Fun}(\mathcal{C}', \mathcal{D}) \underset{p^*}{\overset{p_!}{\rightleftarrows}} \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \ .$$

*exhibits a left adjoint $p_!$.*

## 7. Nonabelian Derived Functors

We will discuss nonabelian derived functors and nonabelian derived $\infty$-category, a.k.a the animation in this section. This was originally used by Quillen to derive a non additive functor such as cotangent complexes.

We start with the following observation.

**Proposition 7.1.** *Assume that $\mathcal{K}(\mathcal{A})$ has enough $\mathcal{K}$-projectives. Let $\mathcal{K}(\mathcal{A})^p$ denotes the full subcategory of $\mathcal{K}$-projective objects. Then we have*

1. *The composition $\mathcal{K}(\mathcal{A})^p \overset{i}{\hookrightarrow} \mathcal{K}(\mathcal{A}) \xrightarrow{p_{\mathcal{A}}} \mathcal{D}(\mathcal{A})$ is an equivalence and $i$ is left adjoint to the projection $p_{\mathcal{A}}$.*
2. *For any $F : \mathcal{K}(\mathcal{A}) \to \mathcal{D}$, the left derived functor $LF : \mathcal{D}(\mathcal{A}) \to \mathcal{D}$ is given by the restriction along $i$ after identify $\mathcal{D}(\mathcal{A})$ with $\mathcal{K}(\mathcal{A})^p$.*
3. *In particular, $LF : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ preserves connective part.*

*Proof.* (1) and (2) is by Example 6.11. (3) is by the fact that when restricting to the connective part $\mathcal{K}$-projective is equivalent of being levelwise projective. □

**Remark 7.2.** If the Dwyer-Kan admits a fully faithful right adjoint (resp. left adjoint) is called a Bousfield localization (resp. coBousfield localization).

**Remark 7.3.** If $\mathcal{A}$ has enough projectives, then in $\mathcal{K}(\mathcal{A})$ being projective is the same as being contractible and levelwise projective. It is equivalent of being split exact in the sense [Wei94, Exercise 2.2.1.] and levelwise projective. But it is not enough being acyclic and levelwise projective, e.g.

$$\cdots \to \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \to \dots$$

is not projective since $\mathbb{Z}/2$ is not a direct summand of $\mathbb{Z}/2$ which is not split exact. Hence being projective in the category of chain of complexes is the same as being trivial cofibrant in the model category of the chain of complexes, c.f. [DS95, Lemma 7.10.]. Moreover by the same criterion, in $\mathcal{K}(\mathcal{A})_{\geq 0}$ being split exact is automatic with acyclic and levelwise projective objects.

We will mainly try to answer the following question: How can one universally characterize the functor $\mathcal{K}(F)$, especially in the case where $F$ is not additive

**Remark 7.4.** The above question is not as easy as it seems when $F$ is not additive. Namely given a pointed functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories, if there exists a functor $\mathcal{K}(\mathcal{A})_{\geq 0} \to \mathcal{D}(\mathcal{B})$ given on chain complexes representing objects by apply $F$ levelwise, then $F$ must already be additive. Indeed, take any split short exact sequence

$$0 \to a \to a \oplus b \to b \to 0$$

in $\mathcal{K}(\mathcal{A})_{\geq 0}$, it will become a fibre sequence in $\mathcal{D}(\mathcal{B})$ by the splitting condition. But the coproduct in $\mathcal{D}(\mathcal{B})$ is the same as the coproduct in $\mathcal{K}(\mathcal{B})$ hence $F(a \oplus b)$ must be the coproduct hence direct sum of $F(a)$ and $F(b)$. Hence $F$ is additive.

In order to overcome the issue of $F$ being not additive, we need to interlude the Yoneda lemma and size issues.

**Convention 7.5.** We fix countabel many Grothendieck universes, namely

$$\{\text{small sets}\} \subset \{\text{large sets}\} \subset \{\text{very large sets}\} \subset \dots$$

which are models of ZFC. We also choose them in a way that the set of all small sets is a large set and the set of all larges sets is a very large sets, etc. Then we extend to $\infty$-categories,

$$\{\text{small } \infty\text{-categories}\} \subset \{\text{large } \infty\text{-categories}\} \subset \{\text{very large } \infty\text{-categories}\} \subset \dots$$

where small $\infty$-categories are the ones where their underlying simiplicial sets are small, and similarly for large $\infty$-categories and very large $\infty$-categories, etc. We will mainly focus on the first three. Now we have two different $\infty$-category of spaces, namely

(1) $\mathcal{S}$ is the large $\infty$-category of small spaces (Kan complexes);

(2) $\hat{\mathcal{S}}$ is the very large $\infty$-category of large spaces (Kan complexes).

And we have a fully faithful inclusion $\mathcal{S} \subset \hat{\mathcal{S}}$. If $\mathcal{C}$ is a large $\infty$-category then its mapping spaces belong to $\hat{\mathcal{S}}$. Similarly, we have $\mathrm{Cat}_\infty^{\mathrm{small}}$ as the $\infty$-category small $\infty$-categories and $\mathrm{Cat}_\infty$ by the $\infty$-category of large $\infty$-categories. The notation is inconsistent with the previous ones. In this manner, everything exists, but one just needs to keep track of the sizes.

**Example 7.6.** For example, if $\mathcal{C} \in \mathrm{Cat}_\infty$ is large, then it Dwyer-Kan localization $\mathcal{C}[W^{-1}] \in \mathrm{Cat}_\infty$ is also large. In the case where $\mathcal{A}$ is a large abelian category, the mapping spaces of $\mathcal{D}(\mathcal{A})$ is given by a colimit indexed over $\mathcal{K}(\mathcal{A})^{\mathrm{qi}}_{/-}$ which are also large.

**Definition 7.7.** We say a large $\infty$-category $\mathcal{C}$ is locally small if for any pair of objects $x, y \in \mathcal{C}_0$ the mapping space $\mathrm{Map}_{\mathcal{C}}(x, y) \in \hat{\mathcal{S}}$ is equivalent to an object in $\mathcal{S}$, i.e. essentially small. In this case,

$$\mathrm{Map}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \hat{\mathcal{S}}$$

factors through $\mathcal{S} \subset \hat{\mathcal{S}}$.

We now come to the Yoneda lemma.

**Construction 7.8.** For any large $\infty$-category $\mathcal{C}$, we have a functor to the category of large presheaves

$$j : \mathcal{C} \to \hat{\mathcal{P}}(\mathcal{C}) \coloneqq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \hat{\mathcal{S}})$$

given by $c \in \underline{c} \coloneqq \mathrm{Map}_{\mathcal{C}}(-, c)$. Again we use Remark 5.6 to make the mapping space into a functor. And we call $j$ the Yoneda embedding. Note that $\hat{\mathcal{P}}(\mathcal{S})$ is a very large $\infty$-category. However, if $\mathcal{C}$ is locally small, then we have

$$j : \mathcal{C} \to \mathcal{P}(\mathcal{S}) \coloneqq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

lands in the category of small presheaves.

Yoneda lemma for $\infty$-categories justifies the name of the above contruction being an embedding.

**Theorem 7.9.** *Let $\mathcal{C}$ be a large $\infty$-category, then we have the following for the Yoneda functor*

$$j : \mathcal{C} \to \hat{\mathcal{P}}(\mathcal{C}).$$

*(1) $j$ is fully faithful.*

*(2) For any $F : \mathcal{C}^{\mathrm{op}} \to \hat{\mathcal{S}}$ and any $x \in \mathcal{S}$, there is a natural equivalence*

$$\mathrm{Map}_{\hat{\mathcal{P}}(\mathcal{C})}(\underline{x}, F) = F(x).$$

*(3) Every object $F \in \hat{\mathcal{S}}$ is a (large) colimit of objects of the form $\underline{x}$ for $x \in \mathcal{C}$, i.e. the representable objects.*

**Remark 7.10.** We first give a remark explaining why keeping track of the size issue is important here. In the statment of (3), for a given functor $F : \mathcal{C}^{\mathrm{op}} \to \hat{\mathcal{S}}$, we consider the pullback diagram (in the very very large $\infty$-category of very large $\infty$-categories)

$$\begin{array}{ccc} \mathcal{C}_{/F} & \longrightarrow & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \hat{\mathcal{S}})_{/F} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{C} & \xrightarrow{j} & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \hat{\mathcal{S}}) \end{array} .$$

Even though one starts with some $\mathcal{C}$ being locally small, $\mathcal{C}_{/F}$ is large. Note that an object in $\mathcal{C}_{/F}$ is given by $\underline{x} \to F$ for $x \in \mathcal{C}_0$. But to define such $\underline{x}$ and $F$, one needs to specify values one all the objects in $\mathcal{C}$. In this manner, one can see that mapping spaces are also large.

*Sketch of Theorem 7.9.* Yoneda embedding is tautological in the view of straightening-unstraightening equivalence. The straightening-unstraightening equivalence states that

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \hat{\mathcal{S}}) \simeq \mathrm{RFib}(\mathcal{C})$$

the presheaves category is equivalent to the right fibrations over $\mathcal{C}$, where the latter is a full subcategory $\mathrm{Cat}_{\infty/\mathcal{C}}$. The equivalence sends $\underline{x}$ to the canonical right fibration $\mathcal{C}_{/x} \to \mathcal{C}$. And for any functor $F$ in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \hat{\mathcal{S}})$, the corresponding right fibration $p : \mathcal{E} \to \mathcal{C}$ is characterized by $F(x) \simeq \mathcal{E}_x$ where $\mathcal{E}_x$ is the fibre over $\Delta^0 \xrightarrow{x} \mathcal{C}$.

However, it is not hard to see that $\mathrm{Map}_{\mathrm{Cat}_{\infty/\mathcal{C}}}(\mathcal{C}_{/x}, \mathcal{E})$ is the following pullback in $\hat{\mathcal{S}}$

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Cat}_{\infty/\mathcal{C}}}(\mathcal{C}_{/x}, \mathcal{E}) & \longrightarrow & \mathrm{Fun}(\mathcal{C}_{/x}, \mathcal{E}) \\ \downarrow & \lrcorner & \downarrow p_* \\ \Delta^0 & \xrightarrow{\mathrm{can}} & \mathrm{Fun}(\mathcal{C}_{/x}, \mathcal{C}) \end{array}$$

In fact, by Lemma 5.2, it is true if one replace $\mathrm{Fun}(-,-)$ by $\mathrm{Map}_{\mathrm{Cat}_\infty}(-,-)$. But $\mathrm{Map}_{\mathrm{Cat}_\infty}(-,-)$ is the maximal Kan subcomplex of $\mathrm{Fun}(-,-)$. Hence the corresponding diagram is also a pullback since, $p$ is a right fibration and hence $p_*$ is conservative. Apply the same pullback diagram, one can see that

$$\mathrm{Map}_{\mathrm{Cat}_{\infty/\mathcal{C}}}(\Delta^0, \mathcal{E}) \simeq \mathcal{E}_x$$

viewing $\Delta^0$ as $\Delta^0 \xrightarrow{x} \mathcal{C}$. However, since $\Delta^0 \xrightarrow{\mathrm{Id}_x} \mathcal{C}_{/x}$ is right-anodyne, the induced map

$$\mathrm{Map}_{\mathrm{Cat}_{\infty/\mathcal{C}}}(\mathcal{C}_{/x}, \mathcal{E}) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Cat}_{\infty/\mathcal{C}}}(\Delta^0, \mathcal{E}) \simeq \mathcal{E}_x$$

is a trivial fibration. This proves (2). (1) and (3) then follows from the corresponding 1-categorical argument. □

Note that Theorem 7.9 immediately implies that if $\mathcal{C}$ is small, then for any $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is a small colimit of representable objects. Provided the Yoneda embedding, the following proposition is proved by the same argument of the 1-categorical version.

**Proposition 7.11** ([Cis19, Theorem 6.3.4.]). *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor from a small $\infty$-category to a possible large $\infty$-category. Assume that $\mathcal{D}$ admits all small limits, then*

*(1) There is an essentially unique colimit preserving functor from $\mathcal{P}(\mathcal{C}) \to \mathcal{D}$ extending $F$, i.e. a diagram*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow{\scriptstyle j} & \nearrow_{j_!(F)} & \\ \mathcal{P}(\mathcal{C}) & & \end{array}$$

*with a invertible natural transformation $F \xrightarrow{\simeq} j_!(F) \circ j$. In other words, this is a left Kan extension of $F$ along $j$.*

*(2) If $\mathcal{D}$ is locally small, then this is left adjoint to the restricted Yoneda embedding*

$$\mathcal{D} \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{F^*} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}).$$

One immediate consequence is the following.

**Corollary 7.12.** *Let $\mathcal{C}$ be a small $\infty$-category and $\mathcal{D}$ be a possible large $\infty$-category. Then the small colimits preserving functors from $\mathcal{P}(\mathcal{C})$ to $\mathcal{D}$ is given by the equivalence*

$$j^* : \mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

*where the inverse is given by the left Kan extension $j_!$. In other words, $\mathcal{P}(\mathcal{C})$ is the universal $\infty$-category obtained from $\mathcal{C}$ by freely adjoining small colimits.*

We can also adjoin specific class of colimits.

**Construction 7.13** ([Lur06, Definition 5.3.5.1.]). Let $K$ be any class (or large set) of small colimits, e.g.

(1) $K$ is the class of all small colimits;

(2) $K$ is the class of all finite colimits;

(3) $K$ is the class of geometric realizations, i.e. $\Delta^{\mathrm{op}}$-indexed colimits.

We form $\mathcal{P}_K(\mathcal{C})$ as the smallest full subcategory of $\mathcal{P}(\mathcal{C})$ which contains all representable objects and is closed under $K$-indexed colimtis.

**Proposition 7.14.** *For any large $\infty$-category $\mathcal{D}$ which admits $K$-indexed colimits, the restriction along $j : \mathcal{C} \to \mathcal{P}_K(\mathcal{C})$ induces an equivalence*

$$j^* : \mathrm{Fun}^{K-\mathrm{colim}}(\mathcal{P}_K(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

*where* $\mathrm{Fun}^{K\text{-colim}}(\mathcal{P}_K(\mathcal{C}), \mathcal{D})$ *is the functors that preserve* $K$*-indexed colimits. The inverse is given by the left Kan extension* $j_!$

**Example 7.15.** Let $\mathcal{C}$ be an $\infty$-category and $K$ is the class of small filtered colimits. Then we define

$$\mathrm{Ind}(\mathcal{C}) := \mathcal{P}_K(\mathcal{C}).$$

Then the objects in $\mathrm{Ind}(\mathcal{C})$ can be represented by functors $F : I \to \mathcal{C}$ where $I$ is filtered and we write it as $\text{"}\operatorname{colim}_I\text{"}F$. For any pair of objects $F, G$ of $\mathrm{Ind}(\mathcal{C})$, we have

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\text{"}\operatorname*{colim}_I\text{"}F(i), \text{"}\operatorname*{colim}_J\text{"}G(j)) \simeq \lim_I \operatorname*{colim}_J \mathrm{Map}_{\mathcal{C}}(F(i), G(j))$$

is an equivalence. In particular, if $\mathcal{C}$ is the nerve of a 1-category, then so is $\mathrm{Ind}(\mathcal{C})$.

**Remark 7.16** ([Lur06, Proposition 5.3.3.3.])**.** If $\mathcal{C}$ has finite limits, then $\mathrm{Ind}(\mathcal{C})$ can be identified with those presheaves preserving finite limits.

We now come back to our discussion of nonabelian derived functors. The following proposition is due to Dold-Kan, Quillen, etc.

**Proposition 7.17** ([Lur17, Proposition 1.3.3.14.])**.** *Assume that* $\mathcal{A}$ *is an abelian category has enough compact projective objects. Then we have a sequence of equivalences*

$$\mathcal{D}(\mathcal{A})_{\geq 0} \simeq \mathcal{P}_\Sigma(N(\mathcal{A}^{cp})) \simeq \mathrm{Fun}^{\Pi}(N(\mathcal{A}^{cp})^{\mathrm{op}}, \mathcal{S})$$

*where* $\mathcal{A}^{cp}$ *is the full subcategory of compact projective objects;* $\Sigma$ *is the class of geometric realizations and filtered colimits; and* $\Pi$ *is the functors that preserve finite products.*

**Remark 7.18.** We give an intuitive way of think Proposition 7.17. By viewing $\mathcal{D}(\mathcal{A})_{\geq 0}$ as $\mathcal{K}(\mathcal{A}^p)_{\geq 0}$, then the classical Dold-Kan correspondence shows that one can view objects in $\mathcal{K}(\mathcal{A}^p)_{\geq 0}$ as simplicial objects in $\mathcal{A}^p$. Then simplicial enrichment shows that objects in $\mathcal{K}(\mathcal{A}^p)_{\geq 0}$ are geometric realizations of $\mathcal{A}^p$. On the other hand, $\mathcal{A}^p$ is generated by $\mathcal{A}^{cp}$ by filtered colimits. The last step is generally true. From now on, we will call $\Sigma$ the class of sifted colimits. By [Lur06, Lemma 5.5.8.14.], sifted colimits are exactly those commutes with finite products in $\mathcal{P}(\mathcal{C})$.

We have an immediate corollary.

**Corollary 7.19.** *A functor* $F : \mathcal{D}_{\geq 0}(\mathcal{A}) \to \mathcal{E}$ *that preserves sifted colimits is uniquely determined by its restriction to* $\mathcal{A}^{cp}$*.*

**Example 7.20** (Nonabelian Derived Functors)**.** Let $\mathcal{A} \to \mathcal{B}$ be a (not necessarily additive) functor between abelian categories. Then it induces a functor $F : \mathcal{A}^{cp} \to \mathcal{D}(\mathcal{B})_{\geq 0}$. Kan extension then gives its nonabelian derived functor $LF : \mathcal{D}(\mathcal{A})_{\geq 0} \to \mathcal{D}(\mathcal{B})_{\geq 0}$ that preserves sifted colimits by Corollary 7.19. And one can compute $LF(a)$ by first resolving $a$ by a simplicial object of $\mathcal{A}^p$, then writing the simplicial object as a filtered colimit of simplicial object in $\mathcal{A}^{cp}$ and finally applying $F$ on that.

**Remark 7.21.** In the previous section, we get the left derived functor from right Kan extend along the Dwyer-Kan localization $\mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$. However, in this section we get the left derived functor (it is a left derived functor if one start with an additive functor) from left Kan extend along the animation $\mathcal{A}^{cp} \to \mathcal{P}_\Sigma(N(\mathcal{A}^{cp})) \simeq \mathcal{D}(\mathcal{A})_{\geq 0}$.

Using the above observation, we can now generalize the construction of derived category in nonabelian settings.

**Definition 7.22.** An object $c \in \mathcal{C}_0$ of an $\infty$-category is called projective if $\mathrm{Map}_{\mathcal{C}}(c, -)$ commutes with geometric realizations.

**Remark 7.23.** Recall that if $\mathcal{C}$ is a 1-category with finite colimits, then an object $c$ being projective means $\mathrm{Hom}_{\mathcal{C}}(c, -)$ preserves effective epimorphisms, i.e. colimits along $\Delta^{\mathrm{op}}_{\leq 1}$ (coequalizers $a \rightrightarrows b$ with a common section $b \to a$.). If $\mathcal{C}$ is the category of sets or an abelian category, then every epimorphism is effective. Hence in these nice categories, being projective is the same as the usual notion of being projective. Moreover, $c$ is projective in the $\infty$-category $N(\mathcal{C})$ if and only if $c$ is projective in $\mathcal{C}$. This is essentially [Lur17, Lemma 1.3.3.10.], namely the colimit of some simplicial diagram $\Delta^{\mathrm{op}} \to \mathcal{C}$ has the same colimit of its left Kan extension of the restriction $\Delta^{\mathrm{op}}_{\leq 1} \to \mathcal{C}$ since $\mathcal{C}$ is a 1-category. But left Kan extension preserves colimits by Remark 6.30. This is in the same spirit as Theorem 3.12.

**Definition 7.24** ([Lur06, Proposition 5.5.8.10.])**.** Let $\mathcal{C}$ be an ordinary category which admits small colimits and is generated under small colimits by $\mathcal{C}^{cp}$. Then the animation $\mathrm{Ani}(\mathcal{C})$ of $\mathcal{C}$ is defined as

$$\mathrm{Ani}(\mathcal{C}) \coloneqq \mathcal{P}_{\Sigma}(N(\mathcal{C}^{cp})) \simeq \mathrm{Fun}^{\Pi}(N(\mathcal{C}^{cp})^{\mathrm{op}}, \mathcal{S}).$$

**Example 7.25.** If $\mathcal{C} = \mathrm{Set}$, then the animation $\mathrm{Ani}(\mathrm{Set})$ is just the $\infty$-category of spaces $\mathcal{S}$. Note that the compact projective objects of Set are exactly finite sets. Clearly we have

$$\mathrm{Fun}^{\Pi}(N(\mathrm{Fin})^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}(\Delta^0, \mathcal{S}) \simeq \mathcal{S}.$$

More explicitly, take $N(\mathrm{Fin})$ as a full subcategory of $\mathcal{S}$. Then $\mathrm{Ind}(N(\mathrm{Fin})) \simeq N(\mathrm{Set})$. Then [Ker, Variant 10.1.1.2.] shows that the geometric realization of a simplicial diagram $S = S_\bullet$ in $N(\mathrm{Set})$

$$N(\Delta)^{\mathrm{op}} \xrightarrow{S} N(\mathrm{Set}) \subset \mathcal{S}$$

is the Kan complex $\mathrm{Sing}(|S|)$ where we regard $S$ as a simplicial set. This shows that $\mathcal{S}$ is generated by Fin under sifted colimits.

## 8. Spectra

In this section, we will introduce the concept of a spectra which is essential to the study of homotopy theory. Recall that $\mathcal{S}_*$ is the $\infty$-category of pointed Kan complexes which can be realized either by $N(\mathrm{Kan}_*)$ or by $\mathcal{S}_{\Delta^0/}$. Note that the terminal object $\Delta^0$ is now the 0 object, i.e. both the initial and the terminal object. Following this spirit, we will denote $\Delta^0$ as $*$ in $\mathcal{S}_*$ thinking it as the point space. Note also that if $X$, $Y$ are pointed spaces, then $\mathrm{Map}_{\mathcal{S}_*}(X,Y)$ is a pointed space with base point $X \to *$.

**Construction 8.1.** Let $X$ be a pointed space in $\mathcal{S}_*$, we define the suspension $\Sigma X$ of $X$ as the pushout in $\mathcal{S}$

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & \Sigma X \end{array} .$$

and we define the loop space $\Omega X$ of $X$ as the pullback in $\mathcal{S}$

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & X \end{array} .$$

Let $\mathcal{E} \subset \mathrm{Fun}(\Delta^0 \times \Delta^0, \mathcal{S}_*)$ be the full subcategory of pairs $(X, \Sigma X)$, then the forgetful functor $\theta : \mathcal{E} \to \mathrm{Fun}(\Delta^0, \mathcal{S}_*)$ is a trivial Kan fibration by [Lur06, Proposition 4.3.2.15.]. Hence $\theta$ admits a section $\mathrm{Fun}(\Delta^0, \mathcal{S}_*) \to \mathcal{E}$. And we will call the composition the suspension functor

$$\Sigma : \mathcal{S}_* \simeq \mathrm{Fun}(\Delta^0, \mathcal{S}_*) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{S}_*) \to \mathcal{S}_*.$$

Similarly, one can define the loop space functor.

**Lemma 8.2.** *The suspension functor $\Sigma$ is left adjoint to the loop space functor $\Omega$.*

*Proof.* Let $I$ denote the simplicial set $\Delta^1 \sqcup_{\Delta^0} \Delta^1$ given by $0 \to 1 \leftarrow 0'$ and $J$ denote the simplicial set $\Delta^1 \sqcup_{\Delta^0} \Delta^1$ given by $1 \leftarrow 0 \to 1'$. Let $i : I \to I^{\triangleleft}$ and $j : J \to J^{\triangleright}$ be the canonical inclusions. Note that $I^{\triangleleft} \simeq J^{\triangleright}$ are both $\Delta^1 \times \Delta^1$, namely the square. Since $\mathcal{S}_*$ has all pullbacks and pushouts, by Lemma 8.3, we have adjunctions

$$\mathrm{Fun}(J, \mathcal{S}_*) \underset{j^*}{\overset{j_!}{\rightleftarrows}} \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{S}_*) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Fun}(I, \mathcal{S}_*) .$$

All the functor categories are fibred over $\mathcal{S}_* \times \mathcal{S}_*$ by evaluating at the intermediate vertices of the commutative square. Pulling back along $(*,*) : \Delta^0 \to \mathcal{S}_* \times \mathcal{S}_*$ gives full subcategories of the functor categories with the intermediate vertices being $*$. Denote the corresponding functor categories as $\mathrm{Fun}(J, \mathcal{S}_*)_{/(*,*)}$, $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{S}_*)_{/(*,*)}$ and $\mathrm{Fun}(I, \mathcal{S}_*)_{/(*,*)}$ respectively. Clearly all $j_!$, $j^*$, $i^*$, $i_*$ preserve $*$ since it is both initial and terminal. Hence restricting on the full subcategories give adjunctions

$$\mathrm{Fun}(J, \mathcal{S}_*)_{/(*,*)} \underset{j^*}{\overset{j_!}{\rightleftarrows}} \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{S}_*)_{/(*,*)} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Fun}(I, \mathcal{S}_*)_{/(*,*)} .$$

But both $\mathrm{Fun}(J, \mathcal{S}_*)_{/(*,*)}$ and $\mathrm{Fun}(I, \mathcal{S}_*)_{/(*,*)}$ are clearly equivalent to $\mathcal{S}_*$ by evaluation functors. Hence the composition gives the desired adjunction between $\Sigma$ and $\Omega$. $\square$

The following lemma is used in Lemma 8.2.

**Lemma 8.3** ([RV22, Corollary 4.3.5.]). *If an $\infty$-category $\mathcal{C}$ admits all $I$-shaped limits, then the limit functor $\mathrm{Fun}(I, \mathcal{C}) \to \mathcal{C}$ is given by*

$$\mathrm{Fun}(I, \mathcal{C}) \xrightarrow{i_*} \mathrm{Fun}(I^{\triangleleft}, \mathcal{C}) \xrightarrow{\mathrm{eva}} \mathcal{C}$$

*where $i_*$ is the right adjoint of the restriction functor $\mathrm{Fun}(I^{\triangleleft}, \mathcal{C}) \xrightarrow{i^*} \mathrm{Fun}(I, \mathcal{C})$ and eva is the evaluation at 0. Similarly, the colimit functor is given by composing the left adjoint $i_!$ of the restriction functor and evaluation at $\infty$.*

Recall the definition of the homotopy group.

**Definition 8.4.** Let $X \in \mathcal{S}_*$, we define the $n$th homotopy group of $X$ as

$$\pi_n(X) := [(\Delta^n, \partial\Delta^n), (X, *)] = \pi_0(\mathrm{Map}_{\mathcal{S}_*}(S^n, X))$$

where $S^n$ is the pointed $n$-sphere. See [Ker, 3.2.2.] for details.

Lemma 8.2 immediately implies the following corollary.

**Corollary 8.5.** *For any $X \in \mathcal{S}_*$ and $n \geq 0$, we have*

$$\pi_n(\Omega X) \simeq \pi_{n+1}(X).$$

The following result will also be used latter.

**Lemma 8.6.** *Let $X \to Y \to Z$ be a fibre sequence in $\mathcal{S}_*$. Then it induces a long exact sequence of homotopy groups (sets)*

$$\cdots \to \pi_1(X) \to \pi_1(Y) \to \pi_1(Z) \to \pi_0(X) \to \pi_0(Y) \to \pi_0(Z) \to 0.$$

*Proof.* Since every map in $\mathcal{S}_*$ can be factored as a weak equivalence composed with a Kan fibration, we may assume that $Y \to Z$ is a Kan fibration. Then by [Ker, Theorem 3.2.6.1.] we win. $\square$

**Definition 8.7.** We define the $\infty$-category of spectra Sp as the following limit of simplicial sets

$$\mathrm{Eq}(\mathcal{S}_*) \xrightarrow{d_0} \mathrm{Eq}(\mathcal{S}_*) \xleftarrow{\Omega \circ d_1} \mathrm{Eq}(\mathcal{S}_*) \xrightarrow{d_0} \mathrm{Eq}(\mathcal{S}_*) \xleftarrow{\Omega \circ d_1} \dots$$

where $\mathcal{S}_* \simeq \mathrm{Eq}(\mathcal{S}_*) \subset \mathrm{Fun}(\Delta^1, \mathcal{S}_*)$ is the full subcategory of equivalences; and $d_0$ gives the target and $d_1$ gives the source. In other words,

$$\mathrm{Sp} := \mathrm{Eq}(\mathcal{S}_*) \times_{\mathcal{S}_*} \mathrm{Eq}(\mathcal{S}_*) \times_{\mathcal{S}_*} \mathrm{Eq}(\mathcal{S}_*) \times_{\mathcal{S}_*} \dots.$$

An object in Sp is given by a sequence of pointed spaces $X(i)$ such that $X(i) \xrightarrow{\simeq} \Omega X(i+1)$ with $i \in \mathbb{N}$. We will call such an object spectrum.

The next Proposition shows that Sp is an $\infty$-category and gives a more conceptual expression.

**Proposition 8.8.** *The $\infty$-category of spectra Sp can be realized as the sequential limit in $\mathrm{Cat}_\infty$*

$$\mathrm{Sp} \simeq \lim(\dots \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*).$$

*Proof.* This is essentially the same argument as Example 3.9 and Example 3.9. Again let $\overline{I}$ be the nerve of the totally ordered set $(\mathbb{N}, \leq)$ and $I = \Delta^1 \sqcup_{\Delta^0} \Delta^1 \sqcup_{\Delta^0} \Delta^1 \sqcup_{\Delta^0} \dots$ given by $0 \to 1 \to 2 \to 3 \to \dots$. Hence to compute the sequential limit $\mathrm{Cat}_\infty$ given by $\overline{I}$, it suffices to prove it by restricting to $I$ which has no nondegenerate $n$-simplices for $n \geq 2$. Let $\overline{I} \to \mathrm{Cat}_\infty$ be the desired limit diagram. Restricting to $I$, the limit is the equalizer of

$$\prod_{\mathbb{N}} \mathcal{S}_* \underset{\Omega}{\overset{\mathrm{Id}}{\rightrightarrows}} \prod_{\mathbb{N}} \mathcal{S}_* \ .$$

But this is given by the pullback diagrams of Definition 8.7 using the homotopy coherent nerve description of $\mathrm{Cat}_\infty$. □

**Remark 8.9.** The proposition above can be summarized in the following general conclusion (see also [Ker, Variant 7.6.6.12.]). For every sequential diagram

$$\dots \mathcal{C}_3 \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

of $\infty$-categories. One can compute its limit with a levelwise equivalent diagram of isofibrant tower given by iterated homotopy fibre products

$$\dots \mathcal{C}_2 \times^h_{\mathcal{C}_1} \mathcal{C}_1 \times^h_{\mathcal{C}_0} \mathcal{C}_0 \to \mathcal{C}_1 \times^h_{\mathcal{C}_0} \mathcal{C}_0 \to \mathcal{C}_0.$$

The limit of this tower in $\mathrm{Set}_\Delta$ is given by the infinite product

$$\mathcal{C}_0 \times^h_{\mathcal{C}_0} \mathcal{C}_1 \times^h_{\mathcal{C}_1} \mathcal{C}_2 \times^h_{\mathcal{C}_2} \mathcal{C}_3 \dots$$

whose objects are sequences of pairs $\{(c_i, \alpha_i)\}_{i \in \mathbb{N}}$ where $c_i \in \mathcal{C}_i$ and $\alpha_i : F_i(c_{i+1}) \simeq c_i$ is an isomorphism in $\mathcal{C}_i$ given by the description of the homotopy pullback (argue similarly as Example 2.14 or see [Ker, Remark 7.6.4.6.]).

The mapping spaces of spectra can be described by the following lemma.

**Lemma 8.10.** *Let $F : I \to \mathrm{Cat}_\infty$ be a small diagram. Let $\mathcal{C}$ be the limit $\lim_I F(i)$. For any two objects $x, y$ of $\mathcal{C}$ and $i \in I$, let $x_i, y_i$ be the corresponding objects in $\mathcal{C}_i = F(i)$. Then the mapping space of $x, y$ is given by*

$$\mathrm{Map}_{\mathcal{C}}(x, y) \simeq \lim_I \mathrm{Map}_{\mathcal{C}_i}(x_i, y_i)$$

*where the latter is the induced limit from $I \to \hat{S}$.*

*Proof.* For any infinite category $\mathcal{D}$, by [Ker, Corollary 4.4.5.3.], the map $\mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathcal{D} \times \mathcal{D}$ is an isofibration. Hence by [Ker, Proposition 4.5.2.26.], the mapping space $\mathrm{Map}_{\mathcal{D}}(a, b)$ is in fact given by the pullback in $\mathrm{Cat}_\infty$

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{D}}(a, b) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{D}) \\ \downarrow & \lrcorner & \downarrow \\ \Delta^0 & \xrightarrow{(a,b)} & \mathcal{D} \times \mathcal{D} \end{array} .$$

Since limits commute and $\mathrm{Fun}(\Delta^1, -)$ commutes with limits, we conclude that the mapping space in $\mathcal{C}$ is given by the limits of mapping spaces in $\mathcal{C}_i$. □

**Example 8.11** (Mapping Spaces of Spectra)**.** Given two spectrum $X, Y$. According to Lemma 8.10, the mapping space $\mathrm{Map}_{\mathrm{Sp}}(X, Y)$ is given by

$$\mathrm{Map}_{\mathrm{Sp}}(X, Y) \simeq \lim_{i \in \mathbb{N}} \mathrm{Map}_{\mathcal{S}_*}(X(i), Y(i))$$

where the transition map is induced by $\Omega$. Informally speaking, it is given by a sequence of maps $f_i : X(i) \to Y(i)$ and homotopies between $X(i) \to Y(i) \to \Omega Y(i+1)$ and $X(i) \to \Omega X(i+1) \to \Omega Y(i+1)$.

**Remark 8.12.** The definition of the $\infty$-category of spectra in [Lur17, Definition 1.4.2.8.] is described by the following. Let $\mathcal{S}_*^{\mathrm{fin}}$ be the smallest full subcategory of $\mathcal{S}_*$ contains $*$ and is stable under finite colimits. This is the same as saying that the objects in $\mathcal{S}_*^{\mathrm{fin}}$ have finite underlying simplicial set as in Definition 3.10. In particular, it contains all the pointed $n$-sphere $S^n$ by Example 3.7. Let $\mathcal{C}$ be an $\infty$-category with finite limits. A functor

$$F : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$$

is said to be excisive (resp. reduced) if $F$ sends pushouts to pullbacks (resp. sends initial objects to terminal objects). We define the $\infty$-category of spectrum objects in $\mathcal{C}$ to be

$$\mathrm{Sp}(\mathcal{C}) \coloneqq \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \subset \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$$

the full subcategory of reduced excisive functors. Roughly speaking, these functors are determined by its value on any infinite sequence of spheres of strictly increasing dimension. If one choose the sequence to be the canonical one $S^0, S^1, S^2, S^3, \dots$, then the induced map

$$\mathrm{Sp}(\mathcal{C}) \coloneqq \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \to \lim(\dots \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*)$$

is an equivalence of $\infty$-categories by [Lur17, Remark 1.4.2.25.]. At least on the level of objects, one can see that the sequence of pointed objects $\{F(S^n)\}_{n \in \mathbb{N}}$ is given by $F(S^{n+1}) \simeq F(\Sigma S^n) \xrightarrow{\simeq} \Omega F(S^n)$ which is the same as Definition 8.7.

We now give some examples.

**Example 8.13** (Eilenberg-MacLane Spectrum)**.** For any abelian group $A$, we define the Eilenberg-MacLane spectrum $HA$ to be

$$(HA)(i) \coloneqq K(A, i) = B^i A$$

the $i$th Eilenberg-MacLane space/classifying space. Note that $\Omega K(A, i+1) \simeq K(A, i)$ is an homotopic equivalence by Corollary 8.5.

**Example 8.14** (Suspension Spectrum)**.** Let $X \in \mathcal{S}_*$ be a pointed space, then the associated suspension spectrum $\Sigma^\infty X$ is given by

$$(\Sigma^\infty X)(i) = \operatorname*{colim}_k \Omega^k \Sigma^{k+i} X$$

where the transition maps are given by the unit $\mathrm{Id} \to \Omega\Sigma$ acting on $\Sigma^{k+i} X$ and composing with $\Omega^k$. The first attempt to associate $X$ to a spectrum, is to take all the suspensions of $X$, namely $X(i) = \Sigma^i X$. In this way, one get a map $X(i) \to \Omega X(i+1)$ given by the adjoint of $\Sigma X(i) \xrightarrow{\simeq} X(i+1)$. But this map is not an equivalence. And the construction of $\Sigma^\infty X$ is to force it to be an equivalence. Note that the functor $\Omega$ is given by a pullback, it commutes with filtered colimits, hence $\Omega(\Sigma^\infty X)(i+1) = \Omega\operatorname{colim}_k \Omega^k\Sigma^{k+i+1}X \simeq \operatorname{colim}_k \Omega^{k+1}\Sigma^{k+i+1}X$ as desired. The suspension spectrum defines a functor

$$\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}.$$

An important special case is when $X = S^0$, then we get the sphere spectrum $\mathbb{S} := \Sigma^\infty S^0$. We will also write $\Sigma^\infty_+$ as the composition of

$$\mathcal{S} \xrightarrow{\text{can}} \mathcal{S}_* \xrightarrow{\Sigma^\infty} \mathrm{Sp}.$$

The importance of the suspension spectra is revealed by the following definition and lemma.

**Definition 8.15.** We define the infinite looping functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}_*$ by projection

$$\mathrm{Sp} \simeq \lim(\dots \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*) \to \mathcal{S}_*$$

on to the degree 0 factor.

**Lemma 8.16.** *The suspension spectrum functor $\Sigma^\infty$ is left adjoint to the infinite looping functor $\Omega^\infty$, i.e. for any $X$ in $\mathcal{S}_*$ and $Y$ in $\mathrm{Sp}$, there is an equivalence*

$$\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, Y) \simeq \mathrm{Map}_{\mathcal{S}_*}(X, \Omega^\infty Y).$$

*Sketch.* We only construct the map here. Note that $Y(0) \simeq \Omega^{k+i}Y(i+k)$ for every $i$. Then by the adjuntion $\Sigma \dashv \Omega$ we have

$$\mathrm{Map}_{\mathcal{S}_*}(X, Y(0)) \simeq \mathrm{Map}_{\mathcal{S}_*}(X, \Omega^{k+i}Y(i+k)) \simeq \mathrm{Map}_{\mathcal{S}_*}(\Sigma^{k+i}X, Y(i+k)) \xrightarrow{\Omega^k} \mathrm{Map}_{\mathcal{S}_*}(\Omega^k\Sigma^{k+i}X, Y(i))$$

where the last map is given by $\Omega^k$. Regard the left hand side as a constant diagram and take limit with respect to $k$ on both sides we get

$$\mathrm{Map}_{\mathcal{S}_*}(X, Y(0)) \to \lim_k \mathrm{Map}_{\mathcal{S}_*}(\Omega^k\Sigma^{k+i}X, Y(i)) \simeq \mathrm{Map}_{\mathcal{S}_*}(\operatorname*{colim}_k \Omega^k\Sigma^{k+i}X, Y(i)) = \mathrm{Map}_{\mathcal{S}_*}((\Sigma^\infty X)(i), Y(i)).$$

Then take limit with respect to $i$ and use Example 8.11 we get

$$\mathrm{Map}_{\mathcal{S}_*}(X, \Omega^\infty Y) \to \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, Y).$$

The inverse is just by given projecting $\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, Y)$ onto the degree 0 factor. □

**Remark 8.17.** It is not obvious that the map constructed above are inverse to each other. Some machinery needs to be developed, namely excisive approximation, c.f. [Lur17, 6.1.1.] (or similarly spectratification). First of all, one needs to enlarge $\mathcal{S}_*$ by the category of all functors $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}_*)$. Obverse that evaluated at $*$ gives an equivalence between the right exact functors in $\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}_*)$ and $\mathcal{S}_*$ since $\mathcal{S}_*^{\mathrm{fin}}$ is freely generated by $*$ under finite colimits. By slightly abusing notations, we also denote the functor

$$T : \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}_*) \to \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}_*)$$

given by $F \mapsto \operatorname{colim}\Omega \circ F \circ \Sigma$ as $T$. Informally, it sends a prespectra $(X, \Sigma X, \Sigma^2 X, \dots)$ to $(\Omega\Sigma X, \Omega\Sigma^2 X, \Omega\Sigma^3 X, \dots)$. Then $\Sigma^\infty$ can be interpreted as $F \mapsto \operatorname{colim}_k T^k(F) = \operatorname{colim}_k \Omega^k \circ F \circ \Sigma^k$. It is not hard to see that the essential image of $\Sigma^\infty$ lands in Sp. Now the claim is the natural transformation $\theta : \mathrm{Id} \to \Sigma^\infty$ realizes $\Sigma^\infty$ as a Bousfields localization in the sense of Remark 7.2. By [Lur06, Proposition 5.2.7.4.], it is equivalent of checking for every $F$, the canonical maps

$$\Sigma^\infty\theta_F, \theta_{\Sigma^\infty F} : \Sigma^\infty F \to \Sigma^\infty\Sigma^\infty F$$

are equivalences, which are not hard to check.

In the light of Lemma 8.16, we will often think Sp as the infinite delooping of $\mathcal{S}_*$. We now give an example.

**Example 8.18.** Let $K$ be a object in $\mathcal{S}_*^{\text{fin}}$[1] and $X$ be in $\mathcal{S}_*$, then we have

$$\begin{aligned}\operatorname{Map}_{\mathrm{Sp}}(\Sigma^\infty K, \Sigma^\infty X) &\simeq \operatorname{Map}_{\mathcal{S}_*}(K, \Omega^\infty\Sigma^\infty X)\\ &\simeq \operatorname{Map}_{\mathcal{S}_*}(K, \operatorname*{colim}_k \Omega^k\Sigma^k X)\\ &\simeq \operatorname*{colim}_k \operatorname{Map}_{\mathcal{S}_*}(K, \Omega^k\Sigma^k X)\\ &\simeq \operatorname*{colim}_k \operatorname{Map}_{\mathcal{S}_*}(\Sigma^k K, \Sigma^k X)\end{aligned}$$

The third equivalence uses that $K$ is compact and others are by adjunction introduced previously. In particular, if $K = S^0$, then we get the stable homotopy groups of $X$

$$\pi_n(\operatorname{Map}_{\mathcal{S}_*}(\mathbb{S}, \Sigma^\infty X)) = \pi_n^S(X).$$

**Remark 8.19.** Let $\mathcal{C}$ be a pointed infinite category with finite colimits. Then by the universal property of the pushout, we have

$$\operatorname{Map}_{\mathcal{C}}(\Sigma X, Y) \simeq \Omega\operatorname{Map}_{\mathcal{C}}(X, Y).$$

Hence, $\pi_n(\operatorname{Map}_{\mathcal{C}}(\Sigma X, Y)) \simeq \pi_{n+1}(\operatorname{Map}_{\mathcal{C}}(X, Y))$. In particular, the homotopy class of maps between two spetra $\pi_0(\operatorname{Map}_{\mathrm{Sp}}(X, Y))$ is always an abelian group. This follows from the following bijection

$$\begin{aligned}\pi_0(\operatorname{Map}_{\mathrm{Sp}}(X, Y)) &\simeq \lim_i \pi_0(\operatorname{Map}_{\mathcal{S}_*}(X(i), Y(i)))\\ &\simeq \lim_i \pi_0(\operatorname{Map}_{\mathcal{S}_*}(X(i), \Omega^2 Y(i+2)))\\ &\simeq \lim_i \pi_0(\operatorname{Map}_{\mathcal{S}_*}(\Sigma^2 X(i), Y(i+2)))\\ &\simeq \lim_i \pi_2(\operatorname{Map}_{\mathcal{S}_*}(X(i), Y(i+2)))\end{aligned}$$

and the fact that $\pi_2(-)$ is always an abelian group by [Ker, Theorem 3.2.2.10.].

**Lemma 8.20.** *The category of spectra* Sp *has all (small) limits, filtered colimits and a zero object.*

*Proof.* The zero object is given by the $(X(i))_{i\in\mathbb{N}}$ where $X(i) = *$. Note that limits and filtered colimits with the loop functor $\Omega$. Hence taking limits and filtered colimits levelwise give spectra. Now Example 8.11 shows that taking limits and filtered colimits levelwise satisfies the universal properties respectively. ☐

We need the following lemma to show that (small) colimits exist in Sp.

**Lemma 8.21.** *The functor* $\Omega : \mathrm{Sp} \to \mathrm{Sp}$ *given by the same formation as in Construction 8.1 is an equivalence.*

[1]It is warned in [Lur17, Warning 1.4.2.7.] that compact objects in $\mathcal{S}_*$ are not just $\mathcal{S}_*^{\text{fin}}$.

*Proof.* Since $\Omega$ is given by pullbacks, by Lemma 8.20, it is computed levelwise. Hence $\Omega$ sends $(X(0), X(1), X(2), \dots)$ to $(\Omega X(0), X(0), X(1), \dots)$. Since $\Omega X(0)$ is uniquely determined by $X(0)$, the inverse can be given sending $(X(0), X(1), X(2), \dots)$ to $(X(1), X(2), X(3), \dots)$. In the setting of Remark 8.12, the inverse is given by $F \mapsto F \circ \Sigma$ for $F$ in $\mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$. □

**Proposition 8.22** ([Lur17, Proposition 1.4.2.11.])**.** *The category of spectra* Sp *has all (small) colimits. Moreover, a diagram is a pushout if and only if it is a pullback.*

The proposition suggests the following class of $\infty$-categories.

**Definition 8.23.** A pointed $\infty$-category is called a stable $\infty$-category if it admits all (small) limits and colimits and pullback squares are exact pushout squares.

**Example 8.24.** One can check by using explicit formulas of the pushouts and pullbacks that the derived category $\mathcal{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is stable.

**Remark 8.25.** Proposition 8.22 comes from a more general statement. If an $\infty$-category has all finite limits, an zero object and the loop functor $\Omega$ is an equivalence, then it has pushouts and pushout squares are exact pullback squares. For example, since $\Omega$ is an equivalence and the suspension is adjoint to it, the suspension functor $\Sigma$ has to be its inverse. But this is very different from taking the suspension levelwise. Moreover, note that given a spectrum $Y$, the degree $i$ underlying pointed space $Y(i)$ is given by $\Omega^\infty \Sigma^i Y$.

We give two immediate examples by the suspension functor $\Sigma$ in Sp.

**Example 8.26.** Let $A$ be an abelian group and $X$ be a pointed space. Then we have

$$[\Sigma^\infty X, \Sigma^n HA]_{\mathrm{Sp}} \simeq [X, K(A, n)]_{\mathcal{S}_*} \simeq H^n_{\mathrm{sing}}(|X|, A).$$

where $|X|$ is the geometric realization of $X$ and the last isomorphism is given by the characterization of the Eilenberg-MacLane Space.

**Example 8.27.** For any spectrum $Y$, we have

$$[\Sigma^n \mathbb{S}, Y]_{\mathrm{Sp}} \simeq [S^n, \Omega^\infty Y]_{\mathcal{S}_*} = \pi_n(\Omega^\infty Y).$$

Note that in the first isomorphism we uses the commutativity of $\Sigma$ and $\Sigma^\infty$ in the first factor. This is done by using adjunction and deducing from the fact that $\Omega$ and $\Omega^\infty$ commutes.

**Remark 8.28.** In the examples above, one can get Mayer–Vietoris sequence type of sequences for both cohomologies and homotopy groups. This is because the stability of Sp allows one to compute a fibre sequence and a cofibre sequence at the same time. Hence, $[-,-]_{\mathrm{Sp}}$ commutes with fibre sequences and cofibre sequences in both factors. This makes the category Sp extremely useful.

The following lemma allows us to think about spectra not only as the infinite delooping of $\mathcal{S}_*$ but also as infinite suspension of $\mathcal{S}_*$.

**Lemma 8.29.** *For any spectrum $Y$, we have $Y \simeq \mathrm{colim}_i\, \Sigma^{-i}\Sigma^\infty Y(i) \coloneqq \mathrm{colim}_i\, \Omega^i \Sigma^\infty Y(i)$.*

*Proof.* By (co)Yoneda lemma, it suffices to check that for any spectrum $X$, we have

$$\mathrm{Map}_{\mathrm{Sp}}(Y, X) \simeq \mathrm{Map}_{\mathrm{Sp}}(\operatorname*{colim}_i \Sigma^{-i}\Sigma^\infty Y(i), X).$$

By Remark 8.25 $X(i) = \Omega^\infty \Sigma^i X$. Then by the adjunction $\Sigma^\infty \dashv \Omega^\infty$ and the fact that the suspension functor $\Sigma$ is an equivalence, we have

$$\begin{aligned}\mathrm{Map}_{\mathrm{Sp}}(Y, X) &\simeq \lim_i \mathrm{Map}_{\mathcal{S}_*}(Y(i), X(i)) \\ &\simeq \lim_i \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty Y(i), \Sigma^i X) \\ &\simeq \lim_i \mathrm{Map}_{\mathrm{Sp}}(\Sigma^{-i}\Sigma^\infty Y(i), X) \\ &\simeq \mathrm{Map}_{\mathrm{Sp}}(\operatorname{colim}_i \Sigma^{-i}\Sigma^\infty Y(i), X)\end{aligned}$$

Note that the map $\Sigma^{-i}\Sigma^\infty Y(i) \to \Sigma^{-i-1}\Sigma^\infty Y(i+1)$ is given by the adjunction of the following composition

$$Y(i) \to \Omega^\infty \Sigma^i \Omega^i \Sigma^\infty Y(i) \to \Omega^\infty \Sigma^i \Omega^{i+1} \Sigma^\infty Y(i+1).$$

□

We have seen the relation of spectra and cohomology and stable homotopy groups. Now we come to the relation of homology.

**Construction 8.30.** We construct the singular chain (or homology) functor as follows. Let $C_*$ be the functor

$$C_* : \mathrm{Sp} \to \mathcal{D}(\mathbb{Z})$$

given by $Y \mapsto \operatorname{colim}_i C_*(|Y(i)|, \mathbb{Z})[-i]$ where $C_*(|Y(i)|, \mathbb{Z})$ is the (reduced) singular chain complex of $|Y(i)|$ and maps are given by Lemma 8.29. This functor is characterize by preserving colimits and taking $\Sigma^\infty X$ to $C_*(X, \mathbb{Z})$. In particular, it sends $\mathbb{S}$ to $\mathbb{Z}[0]$.

The above construction suggests the analogy of spectra being the complexes over the sphere spectrum and $C_*$ being the base change functor. To be more precise, we have the following.

**Definition 8.31.** Let $\mathrm{Sp}^{\mathrm{fin}} \subset \mathrm{Sp}$ be the full subcategory of spectra whose objects are of the form $\Sigma^{-n}\Sigma^\infty K$ where $K \in (\mathcal{S}_*^{\mathrm{fin}})_0$. It is called the Spanier-Whitehead category.

**Lemma 8.32.** $\mathrm{Sp}^{\mathrm{fin}}$ *is closed under finite colimits.*

*Proof.* In fact, $\mathrm{Sp}^{\mathrm{fin}} \subset \mathrm{Sp}$ is generated by $\mathbb{S}$ under finite colimits and shifts. Note that $\mathcal{S}_*^{\mathrm{fin}}$ is generated by $S^0$ by finite colimits. Since $\Sigma^\infty$ is the left adjoint of $\Omega^\infty$, it preserves finite colimits. Clearly $\Sigma^{-n}$ preserves finite colimits since it is an equivalence. Hence every object of $\mathrm{Sp}^{\mathrm{fin}}$ is a finite colimit of $\Sigma^{-n}\Sigma^\infty S^0 \simeq \Sigma^{-n}\mathbb{S}$. □

**Proposition 8.33.** *The category of spectra* Sp *is compactly generated by the Spanier-Whitehead category* $\mathrm{Sp}^{\mathrm{fin}}$*, i.e.* $\mathrm{Sp} \simeq \mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}})$.

*Proof.* Note that objects in $\mathrm{Sp}^{\mathrm{fin}}$ are compact, namely for any $\Sigma^{-n}\Sigma^\infty K$ in $\mathrm{Sp}^{\mathrm{fin}}$ and filtered colimits $\operatorname{colim}_k Y_k$ in Sp, we have

$$\begin{aligned}\mathrm{Map}_{\mathrm{Sp}}(\Sigma^{-n}\Sigma^\infty K, \operatorname*{colim}_k Y_k) &\simeq \mathrm{Map}_{\mathcal{S}_*}(K, \Omega^\infty\Sigma^n \operatorname*{colim}_k Y_k)\\ &\simeq \mathrm{Map}_{\mathcal{S}_*}(K, \operatorname*{colim}_k \Omega^\infty\Sigma^n Y_k)\\ &\simeq \operatorname*{colim}_k \mathrm{Map}_{\mathcal{S}_*}(K, \Omega^\infty\Sigma^n Y_k)\\ &\simeq \operatorname*{colim}_k \mathrm{Map}_{\mathrm{Sp}}(\Sigma^{-n}\Sigma^\infty K, Y_k).\end{aligned}$$

Note that by Lemma 8.29, for any spectrum $Y$, we have $Y \simeq \operatorname{colim}_i \Sigma^{-i}\Sigma^\infty Y(i) := \operatorname{colim}_i \Omega^i\Sigma^\infty Y(i)$. Then we can filter $Y(i)$ by finite spaces. □

**Remark 8.34.** In the derived category $\mathcal{D}(R)$, we have the full subcategory of perfect complexes $\mathrm{Perf}(R)$ which are the compact generators of $\mathcal{D}(R)$. Analogously, in the category of spectra Sp, we have the full subcategory of finite spectra $\mathrm{Sp}^{\mathrm{fin}}$. In other words, one can think about Sp as "$\mathcal{D}(\mathbb{S})$". But of course the mapping space $\mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}, \mathbb{S})$ is not discrete.

**Definition 8.35.** We define the $n$th homotopy group of a spectrum $X$ as $\pi_n(X) := [\Sigma^n\mathbb{S}, X]_{\mathrm{Sp}}$. Moreover, we say $X$ is $n$-connective if $\pi_m(X) = 0$ for $m < n$. We say $X$ is $n$-coconnective if $\pi_m(X) = 0$ for $m > n$. And we denote $\mathrm{Sp}_{\geq n}$ and $\mathrm{Sp}_{\leq n}$ as the full subcategory of $n$-connective objects and $n$-coconnective objects respectively.

Note that by Example 8.27, $X$ is $n$-connective if $X(i)$ is $(i+n)$-connective for all $i \in \mathbb{N}$. Similarly, $X$ is $n$-coconnective if $X(i)$ is $(i+n)$-truncated (or coconnective) for all $i \in \mathbb{N}$.

**Lemma 8.36.** *Let $X \to Y \to Z$ be a fibre sequence of spectra, then we have the following long exact sequence of homotopy groups*

$$\cdots \to \pi_n(X) \to \pi_n(Y) \to \pi_n(Z) \to \pi_{n-1}(X) \to \pi_{n-1}(Y) \to \pi_{n-1}(Z) \to \dots$$

*Proof.* Let $W = \mathrm{cofib}(Y \to Z)$. Then we have the following pushout diagrams

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & Z & \longrightarrow & W \end{array}.$$

By the university of pushouts, the outer rectangle is also a pushout, hence $W \simeq \Sigma X$. By Proposition 8.22, the above squares are all pullbacks. Argue similarly, we have $\mathrm{fib}(X \to Y) \simeq \Omega Z$. Iterate this process, we get a long fibre sequence, i.e. each adjacent three terms form a fibre sequence

$$\cdots \to \Omega X \to \Omega Y \to \Omega Z \to X \to Y \to Z \to \Sigma X \to \Sigma Y \to \Sigma Z \to \dots$$

Apply $\mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}, -)$ one gets a long fibre sequences of Kan complexes. Then we win by applying $\pi_0$ and invoking Lemma 8.6 □

**Proposition 8.37.** *The inclusion $\mathrm{Sp}_{\leq n} \subset \mathrm{Sp}$ has a left adjoint $\tau_{\leq n}$.*

*Proof.* Let $X$ be a spectrum. Set $X_n = X$ and we build $X_k$ inductively as follows. Pick a generating set of $\pi_k(X_{k-1})$ which gives a corresponding map

$$\bigoplus \Sigma^k \mathbb{S} \to X_{k-1}.$$

And we let $X_k \coloneqq \mathrm{cofib}(\bigoplus \Sigma^k \mathbb{S} \to X_{k-1})$ be the cofibre. And we claim that $\tau_{\le n} X = \mathrm{colim}_k X_k$ is $n$-coconnective and $\pi_m(\tau_{\le n} X) = \pi_m(X)$ for $m \le n$. First note that $\pi_m(\mathbb{S}) \simeq 0$ for $m < 0$ since $\mathbb{S}$ is a suspension spectrum. Hence by Lemma 8.36, $\pi_m(X_k) \simeq \pi_m(X_{k-1})$ for $m \le n$ and $\pi_{n+k}(X_k) \simeq 0$. Therefore, the desired construction suffices the need.

To see that $\tau_{\le n}$ is actually a left adjoint, one needs to prove that

$$\mathrm{Map}_{\mathrm{Sp}}(\tau_{\le n} X, Y) \simeq \mathrm{Map}_{\mathrm{Sp}}(X, Y)$$

for every $n$-coconnective $Y$. By Example 8.11, it suffices to show that $\mathrm{Map}_{\mathcal{S}_*}((\tau_{\le n} X)(i), Y(i)) \simeq \mathrm{Map}_{\mathcal{S}_*}(X(i), Y(i))$. Note that by the observation in Definition 8.35, we have $\pi_m((\tau_{\le n} X)(0)) \simeq \pi_m(X(0))$ for $m \le n$ and $\pi_m((\tau_{\le n} X)(0)) \simeq 0$ for $m > n$. Hence $(\tau_{\le n} X)(0)$ are homotopic to $\tau_{\le n} X(0)$ on the connected component of the base point by Whitehead's theorem 8.38. Since both $\pi_0((\tau_{\le n} X)(0))$ and $\pi_0(\tau_{\le n} X(0))$ are groups, all other components are homotopic. Argue inductively, one sees that

$$(\tau_{\le n} X)(i) \simeq \tau_{\le i+n} X(i).$$

In other words, the spectrum $\tau_{\le n} X$ is given by the sequence of pointed spaces $\{\tau_{\le i+n} X(i)\}$. Note that by the observation in 8.35, $Y(i)$ is $(i+n)$-coconnective. Hence it remains to show that $\tau_{\le m}$ is left adjoint for the inclusion $(\mathcal{S}_*)_{\le m} \subset \mathcal{S}_*$. This is well-known, c.f.[Ker, Proposition 3.5.7.29.]. Roughly speaking the quotient map $\mathrm{cosk}_{m+1}(S) \twoheadrightarrow \tau_{\le m}(S)$ is a trivial Kan fibration by [Ker, Corollary 3.5.6.15.]. Hence it suffices to prove the corresponding mapping spaces of coskeleton are isomorphic in $\mathrm{Kan}_*$. Use Yoneda embedding and $\mathrm{Map}_{\mathcal{S}_*}(-,-) \simeq \underline{\mathrm{Hom}}_{\mathrm{Set}_\Delta}(-,-)$ is the internal Hom, one then reduces to the corresponding statement for $\mathrm{Hom}_{\mathrm{Set}_\Delta}(-,-)$. Then it is done by the adjointness of skeleton and coskeleton.

□

Recall the Whitehead's theorem.

**Theorem 8.38** ([Ker, Theorem 3.2.7.1.])**.** *Let $f : X \to Y$ be a morphism of Kan complexes. Then $f$ is a homotopy equivalence if and only if $\pi_0(f)$ is a bijection and*

$$\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$$

*is an isomorphism for all $n \ge 1$ and all $x \in X_0$.*

**Notation 8.39.** For any spectrum $X$, we let

$$\tau_{\ge n} X \coloneqq \mathrm{fib}(X \to \tau_{\le n-1} X)$$

denotes the fibre of $X \to \tau_{\le n-1} X$. And we call $\tau_{\le n}$ and $\tau_{\ge n}$ the truncation functors.

The following lemma essentially gives the $t$-structure on Sp.

**Lemma 8.40.** *We have the following regards the truncation functors.*

*(1) If $X$ is a $n$-connective and $Y$ is a $(n-1)$-coconnective spectrum, then*

$$\mathrm{Map}_{\mathrm{Sp}}(X,Y) \simeq *.$$

*(2) The full subcategory of objects which are $n$-coconnective and $n$-connective at the same time have discrete mapping spaces. In fact, $\mathrm{Sp}_{\geq n} \cap \mathrm{Sp}_{\leq n} \simeq N(\mathrm{Ab})$.*

*Proof.* For (1), note that $\mathrm{Map}_{\mathrm{Sp}}(X,Y) \simeq \mathrm{Map}_{\mathrm{Sp}}(\tau_{\leq n-1}X, Y)$ by Proposition 8.37. By Whitehead's theorem, $\tau_{\leq n-1}X \simeq *$ since all the homotopy groups of $(\tau_{\leq n-1}X)(i)$ are zero. Hence the conclusion.

For (2), we first observe that mapping spaces of $\mathrm{Sp}_{\geq n} \cap \mathrm{Sp}_{\leq n}$ are discrete. Indeed, since $\Sigma^i X$ is $(n+1)$-connective for $i > 0$, then by (1) we have

$$\pi_i(\mathrm{Map}_{\mathrm{Sp}}(X,Y)) \simeq \pi_0(\mathrm{Map}_{\mathrm{Sp}}(\Sigma^i X, Y)) \simeq 0.$$

This shows that the base point component of $\mathrm{Map}_{\mathrm{Sp}}(X,Y)$ is discrete. But the group structure of $\pi_0$ gives that the other components are also discrete. Therefore, $\mathrm{Sp}_{\geq n} \cap \mathrm{Sp}_{\leq n}$ is equivalent to a 1-category by [Ker, Proposition 4.8.3.1.].

We claim that the functor $\pi_n$ gives an equivalence of $\infty$-categories

$$\pi_n : \mathrm{Sp}_{\geq n} \cap \mathrm{Sp}_{\leq n} \to N(\mathrm{Ab}).$$

If $X$ is both $n$-connective and $n$-coconnective, then $\pi_m(X(0)) \simeq 0$ for $m \neq n$. Hence $X(0)$ is the Eilenberg-MacLane space $K(\pi_n(X), n)$ by Whitehead's theorem. It follows that $X$ is a Eilenberg MacLane spectrum. Conversely, an Eilenberg MacLane spectrum with $X(0) = K(A,n)$ is both $n$-connective and $n$-coconnective. It is easy to see that these two functors give an equivalence of categories.

□

**Remark 8.41.** The proof of Lemma 8.40 actually shows that there is an accessible $t$-structure on Sp defined by $n$-connective and $n$-coconnective truncations. Moreover, the heart $\mathrm{Sp}^{\heartsuit}$ is given by the inverse limit of the tower of $\infty$-categories

$$\ldots \xrightarrow{\Omega} \mathrm{EM}_1(\mathcal{S}_*) \xrightarrow{\Omega} \mathrm{EM}_0(\mathcal{S}_*)$$

where $\mathrm{EM}_i(\mathcal{S}_*)$ denotes the full subcategory of $\mathcal{S}_*$ spanned by the Eilenberg-MacLane spaces of degree $i$. The tower is constant after the second term.

One way to to get from the spectra concentrated at certain degree to all spectra is the following construction.

**Construction 8.42** (Postnikov Tower and Whitehead Tower)**.** For every spectrum $X$, there is a tower given by (coconnective) truncations of $X$,

$$\cdots \to \tau_{\leq n+1}X \to \tau_{\leq n}X \to \tau_{\leq n-1} \ldots$$

which is called the Postnikov tower of $X$. We say the Postnikov tower of $X$ in Sp converge if $X \simeq \lim_n \tau_{\leq n} X$ for all spectrum $X$. Note that $\mathrm{fib}(\tau_{\leq n+1}X \to \tau_{\leq n}X) \simeq \Sigma^{n+1} H\pi_{n+1}(X)$ is the shift of the Eilenberg-MacLane spectrum of $\pi_{n+1}(X)$.

The above concept can also be dualized, namely for every spectrum $X$, there is a tower given by (connective) truncations of $X$,

$$\cdots \to \tau_{\geq n-1}X \to \tau_{\geq n}X \to \tau_{\geq n+1} \ldots$$

which is called the Whitehead tower of $X$. We say the Whitehead tower of $X$ in Sp converge if $X \simeq \operatorname{colim}_n \tau_{\geq n} X$ for all spectrum $X$.

**Proposition 8.43.** *The category of* Sp *is both left complete and right complete, i.e. all Postnikov towers and Whitehead towers converge.*

*Proof.* Recall from [Lur17, Proposition 1.2.1.19.] that a stable $\infty$-cateogry $\mathcal{C}$ with t-structure is left complete if

(1) $\mathcal{C}$ admits countable products and $\mathcal{C}_{\geq 0}$ is stable under countable products;

(2) the full subcategory $\mathcal{C}_{\geq \infty} := \bigcap \mathcal{C}_{\geq n}$ consists only of zero objects of $\mathcal{C}$.

And dually we have for right completeness. If $X$ is in either $\bigcap \mathcal{C}_{\geq n}$ or $\bigcap \mathcal{C}_{\leq -n}$, then all the homotopy groups of $X$ vanish and therefore all the homotopy groups of $X(i)$ vanish for all $i$. Hence $X$ is a zero object by Whitehead's theorem. Now it suffices to show that $\mathrm{Sp}_{\geq 0}$ and $\mathrm{Sp}_{\leq 0}$ are stable under products and coproducts respectively. It then reduces to show that $\pi_n$ preserves products and coproducts. Shifting if necessary, we may assume $n = 0$. Now $\pi_0$ is the composition of

$$\mathrm{Sp} \xrightarrow{\Omega^\infty} \mathcal{S}_* \xrightarrow{\pi_0} N(\mathrm{Set}).$$

which clearly preserves products and coproducts. $\square$

## 9. Symmetric Monoidal $\infty$-Categories

We will introduce symmetric monoidal $\infty$-categories in this section which will be crucial to the contruction of topological cyclic homology. Most of the notions work the same as the ordinary categories. We first recall the notion of ordinary symmetric monoidal categories.

**Definition 9.1.** A symmetric monoidal 1-category consists of the following datum:

(1) a category $\mathcal{C}$;
(2) a tensor product functor $-\otimes - : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$;
(3) a tensor unit $1_{\mathcal{C}}\in\mathcal{C}$;
(4) and natural isomorphisms:
  (a) associativity $(a\otimes b)\otimes c \xrightarrow{\simeq} a\otimes(b\otimes c)$;
  (b) unitality $1_{\mathcal{C}}\otimes a \xrightarrow{\simeq} a \xleftarrow{\simeq} a\otimes 1_{\mathcal{C}}$;
  (c) symmetry $a\otimes b\xrightarrow{\simeq} b\otimes a$.

such that the following datum of coherence are satisfied:

(1) the commutative diagram of pentagon identity,

$$\begin{array}{ccc} & ((a\otimes b)\otimes c)\otimes d & \\ \swarrow\simeq & & \simeq\searrow \\ (a\otimes(b\otimes c))\otimes d & & (a\otimes b)\otimes(c\otimes d) \\ \downarrow\simeq & & \downarrow\simeq \\ a\otimes((b\otimes c)\otimes d) & \xrightarrow{\simeq} & a\otimes(b\otimes(c\otimes d)) \end{array}$$

(2) the commutative diagram of triangle identity,

$$\begin{array}{ccc} (a\otimes 1_{\mathcal{C}})\otimes b & \xrightarrow{\simeq} & a\otimes(1_{\mathcal{C}}\otimes b) \\ & \searrow\simeq \quad \simeq\swarrow & \\ & a\otimes b & \end{array}$$

(3) the commutative diagram of hexagon identity,

$$\begin{array}{ccccc} (a\otimes b)\otimes c & \xrightarrow{\simeq} & a\otimes(b\otimes c) & \xrightarrow{\simeq} & (b\otimes c)\otimes a \\ \downarrow\simeq & & & & \downarrow\simeq \\ (b\otimes a)\otimes c & \xrightarrow{\simeq} & b\otimes(a\otimes c) & \xrightarrow{\simeq} & b\otimes(c\otimes a) \end{array}$$

(4) and finally $\mathrm{Id}_{x\otimes y} : x\otimes y\xrightarrow{\simeq} y\otimes x\xrightarrow{\simeq} x\otimes y$ is the identity.

And we will write it as $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ or simply $\mathcal{C}$ when there is no confusion on the symmetric monoidal structure.

As soon as we have the definition of symmetric monoidal categories, we can require the functors that preserve the corresponding structures.

**Definition 9.2.** A lax symmetric monoidal functor between symmetric monoidal categories $(\mathcal{C}, \otimes_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}})$ is given by a functor $F : \mathcal{C} \to \mathcal{D}$ such that there are natural morphisms

(1) $F(c) \otimes_{\mathcal{D}} F(c') \to F(c \otimes_{\mathcal{C}} c')$;

(2) $1_{\mathcal{D}} \to F(1_{\mathcal{C}})$

satisfies compatibility conditions with the associativities, unitalities and symmetries. And a lax symmetric monoidal functor is called (strong) symmetric monoidal if the natural morphisms

(1) $F(c) \otimes_{\mathcal{D}} F(c') \xrightarrow{\simeq} F(c \otimes_{\mathcal{C}} c')$;

(2) $1_{\mathcal{D}} \xrightarrow{\simeq} F(1_{\mathcal{C}})$

are isomorphisms.

We need one last notion.

**Definition 9.3.** A symmetric monoidal category $(\mathcal{C}, \otimes)$ is called closed if for every object $c \in \mathcal{C}$ the functor $- \otimes c : \mathcal{C} \to \mathcal{C}$ admits a right adjoint, which will be denoted as $\underline{\mathrm{Hom}}_{\mathcal{C}}(c, -)$ the internal Hom. In particular, $- \otimes c$ preserves colimits.

**Proposition 9.4.** *Assume that $R$ is a commutative ring. Then the category of $R$-modules $\mathrm{Mod}_R$ admits a essentially unique symmetric monoidal structure which is closed and with the tensor unit $R$.*

*Proof.* The existence is clear. Assume there is another symmetric monoidal structure $- \otimes -$ which is closed and with the tensor unit $R$. Then for any $M \in \mathrm{Mod}_R$, we have

$$(\bigoplus_I R) \otimes M \simeq \bigoplus_I (R \otimes M)$$

since $- \otimes -$ is closed and the tensor unit is $R$. However, if $N$ is an arbitrary $R$-module, then we can resolve $N$ by

$$\bigoplus_J R \xrightarrow{f} \bigoplus_I R \to N \to 0$$

and write $N \simeq \mathrm{coker}(f)$. Use the right exactness of $- \otimes -$ again, we have

$$N \otimes M \simeq \mathrm{coker}(f) \otimes M \simeq \mathrm{coker}(\bigoplus_J M \to \bigoplus_I M).$$

Then Eckmann-hilton argument shows that this is the same as $- \otimes_R -$.

□

**Remark 9.5.** The above proposition is a more general phenomenon. Let $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ be a symmetric monoidal category. Assume further that $1_{\mathcal{C}}$ is a generator. Then the Yoneda functor

$$\mathrm{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, ) : \mathcal{C} \to \mathrm{Mod}_{\mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}})}$$

is fully faithful. Hence we implicitly used that $\mathrm{End}_R(R) \simeq R$.

**Remark 9.6.** The functor of induction (or pullback) for any map of rings $R \to S$,

$$- \otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S$$

is symmetric monoidal, i.e. inherits canonical refinement to a symmetric functor

$$(M \otimes_R N) \otimes_R S \simeq (M \otimes_R S) \otimes_S (N \otimes_R S).$$

We will then generalize the above notions to $\infty$-categories. We will define the notion of symmetric monoidal $\infty$-categories such that $N(\mathcal{C})$ is a symmetric monoidal $\infty$-category if and only $\mathcal{C}$ is a symmetric monoidal 1-category. Moreover, we will define lax and strong symmetric monoidal functors. The key takeaway is two important examples, Sp and $\mathcal{D}(R)$.

**Theorem 9.7** ([Lur17, Corollary 4.8.2.19.])**.** *(1) The $\infty$-categories $\mathcal{D}(R)$ admits a essentially unique closed symmetric monoidal structure with tensor unit $R[0]$, which will be denoted as*

$$- \otimes_R^L - : \mathcal{D}(R) \times \mathcal{D}(R) \to \mathcal{D}(R).$$

*The functor of induction*

$$- \otimes_R^L S : \mathcal{D}(R) \to \mathcal{D}(S)$$

*inherits canonical strong symmetric monoidal structure.*

*(2) The $\infty$-categories* Sp *admits a essentially unique closed symmetric monoidal structure with tensor unit $\mathbb{S}$, which will be denoted as*

$$- \otimes_{\mathbb{S}} - : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}.$$

*The singular chain functor and the suspension spectrum functor*

$$C_* : \mathrm{Sp} \to \mathcal{D}(\mathbb{Z}),$$

$$\Sigma^\infty_+ : \mathcal{S} \to \mathrm{Sp}$$

*inherits canonical strong symmetric monoidal structure.*

*Sketch of Theorem 9.7.* Again, we just give an intuition instead of a rigorous proof here. Let $X \in \mathrm{Sp}_0$, then we have $\mathbb{S} \otimes X \simeq X$. For any $Y \in \mathcal{S}_0$, we can write $Y$ as the colimit of $Y$-shaped diagram with constant value $*$, i.e.

$$Y \simeq \mathrm{colim}(Y \to \Delta^0 \xrightarrow{*} \mathcal{S}).$$

This follows from [Lur06, Corollary 3.3.4.6.] by straightening-unstraightening equivalence. Hence we have

$$\Sigma^\infty_+ Y \otimes X \simeq \Sigma^\infty_+ (\operatorname*{colim}_Y *) \otimes X \simeq \operatorname*{colim}_Y ((\Sigma^\infty_+ *) \otimes X) \simeq \operatorname*{colim}_Y (\mathbb{S} \otimes X) \simeq \operatorname*{colim}_Y X.$$

Namely, it is given by the colimit of $Y \to \Delta^0 \xrightarrow{X} \mathrm{Sp}$. Observe also that $(\Sigma^{-n} Y) \otimes X \simeq \Sigma^{-n}(Y \otimes X)$. Therefore, in general, write $Y$ as $\mathrm{colim}_i \, \Sigma^{-n} \Sigma^\infty Y(i)$ and we can compute $X \otimes Y$.

□

**Remark 9.8.** The symmetric monoidal adjunction theorem

$$\widehat{\Theta}_* : \mathrm{Pr}^{\mathrm{Alg}} \to \mathrm{Pr}^{\mathrm{Mod}}_{\mathfrak{M}/}$$

in [Lur17, Theorem 4.8.5.11. and Theorem 4.8.5.16.] roughly states that closed symmetric monoidal structure on $\mathrm{Mod}_R$ with tensor unit $R$ is equivalent to the data of an $\mathbb{E}_\infty$-structure on $R$. In particular, if the induced commutative structure on $\mathrm{End}_{\mathrm{Mod}_R}(1)$ is the given commutative algebra structure on $R$, then the closed symmetric monoidal structure with tensor unit $R \simeq \mathrm{End}_{\mathrm{Mod}_R}(1)$ is essentially unique.

**Notation 9.9.** Let $\mathrm{Fin}_*$ be the category of finite pointed sets. And we will denote objects in $\mathrm{Fin}_*$ as $\langle n\rangle = \{0, 1, \ldots, n\}$ with 0 as the base point. And we will denote $\{1, \ldots, n\}$ as $\langle n\rangle^\circ$. Write $N(\mathrm{Fin}_*)$ as the associated $\infty$-category. For every $1 \le i \le n$, we denote

$$\rho^i : \langle n\rangle \to \langle 1\rangle$$

as the functor sends $k$ to 1 if $k = i$ otherwise 0.

This is the first definition that goes back to Segal.

**Definition 9.10.** A symmetric monoidal $\infty$-category is a functor

$$\underline{\mathcal{C}} : N(\mathrm{Fin}_*) \to \mathrm{Cat}_\infty$$

such that the induced maps

$$\underline{\mathcal{C}}(\langle n\rangle) \xrightarrow{(\rho^i_*)} \prod_{i=1}^{n} \underline{\mathcal{C}}(\langle 1\rangle)$$

are equivalences for all $n \in \mathbb{N}$. We denote $\mathcal{C} = \underline{\mathcal{C}}(\langle 1\rangle)$ as the underlying $\infty$-category and the tensor functor is given by

$$- \otimes - : \mathcal{C} \times \mathcal{C} \xleftarrow{\simeq} \underline{\mathcal{C}}(\langle 2\rangle) \xrightarrow{m_*} \underline{\mathcal{C}}(\langle 1\rangle) = \mathcal{C}$$

where $m : \langle 2\rangle \to \langle 1\rangle$ sends 0 to 0 and others to 1. The tensor unit is given by

$$1_{\mathcal{C}} : * = \underline{\mathcal{C}}(\langle 0\rangle) \xrightarrow{i_*} \underline{\mathcal{C}}(\langle 1\rangle) = \mathcal{C}$$

where $i : \langle 0\rangle \to \langle 1\rangle$ is the inclusion.

This is a coherent way of describing all the associativity, unitality, symmetry and other constraints.

Once we have the definition of symmetric monoidal $\infty$-categories we can define functors between them.

**Definition 9.11.** A (strong) symmetric monoidal functor between symmetric monoidal $\infty$-categories $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ is given by a natural transformation between them.

One drawback of Definition 9.10 is that to define lax symmetric monoidal functors, one really need to use the notion of "lax functors", which requires to think about $\mathcal{C}_\infty$ as a $(\infty, 2)$-category. Another drawback is that some choices have to be make in order to define associativity, unitality, symmetry etc. All those drawbacks can be overcome by the following refined definition by Lurie.

**Definition 9.12.** A symmetric monoidal $\infty$-category is a functor from the category of operators to finite pointed sets

$$\mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$$

satisfying the following:

(1) $\mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$ is a cocartesian fibration. In particular the pullback diagram in $\mathrm{Cat}_\infty$

$$\begin{array}{ccc} \mathcal{C}^{\otimes}_{\langle n\rangle} & \longrightarrow & \mathcal{C}^{\otimes} \\ \downarrow & \lrcorner & \downarrow \\ \Delta^0 & \xrightarrow{\langle n\rangle} & N(\mathrm{Fin}_*) \end{array} .$$

is given by the pullback diagram in $\mathrm{Set}_\Delta$. Moreover, since it is a cocartesian fibration, for every morphism $f : \langle n\rangle \to \langle m\rangle$ there is an induced functor $f_! : \mathcal{C}^{\otimes}_{\langle n\rangle} \to \mathcal{C}^{\otimes}_{\langle m\rangle}$.

(2) The induced maps

$$\mathcal{C}^{\otimes}_{\langle n\rangle} \xrightarrow{(\rho^i_!)} \prod_{i=1}^{n} \mathcal{C}^{\otimes}_{\langle 1\rangle}$$

are an equivalence for all $n \in \mathbb{N}$.

Again we will denote $\mathcal{C}$ as $\mathcal{C}^{\otimes}_{\langle 1\rangle}$ and the tensor product functor as

$$-\otimes - : \mathcal{C} \times \mathcal{C} \xleftarrow{\simeq} \mathcal{C}^{\otimes}_{\langle 2\rangle} \xrightarrow{m_!} \mathcal{C}^{\otimes}_{\langle 1\rangle} = \mathcal{C}$$

where $m : \langle 2\rangle \to \langle 1\rangle$ sends 0 to 0 and others to 1.

**Remark 9.13.** The two notions agree by the straightening-unstraightening equivalence, c.f. Remark 5.6.

We give some examples besides Theorem 9.7.

**Example 9.14.** If $\mathcal{C}$ is a symmetric monoidal 1-category, then $N(\mathcal{C})$ is a symmetric monoidal $\infty$-category. More generally, if $\mathcal{C}$ is a symmetric monoidal simplicial enriched category, then the homotopy coherent nerve $N_\Delta(\mathcal{C})$ will be a symmetric monoidal $\infty$-category. We will see the 1-category case in detail latter in this section.

**Example 9.15.** The $\infty$-category of spaces $\mathcal{S}$ has a cartesian symmetric monoidal structure given by the products. This can be generalized to arbitrary $\infty$-category with finite cartesian products.

**Example 9.16.** The $\infty$-category of pointed spaces $\mathcal{S}_*$ has a symmetric monoidal structure given by the smash products. Recall that the smash product of $X$ and $Y$ is given by $X \times Y / X \wedge Y$.

We will decode the meaning of the fibration approach in the context of ordinary categories. We first recall the notion of Grothendieck opfibrations which corresponds to cocartesian fibrations in the context of $\infty$-categories. The following is taken from [Gro10, 4.1.].

**Definition 9.17.** Let $p : \mathcal{C} \to \mathcal{D}$ be a functor and $f : c_1 \to c_2$ be a morphism in $\mathcal{C}$ with image $p(f) = \alpha : d_1 \to d_2$. The morphism $f$ is called $p$-cocartesian if for every $h : c_1 \to c_3$ in $\mathcal{C}$ with image

$p(h) = \gamma : d_1 \to d_3$ and every $\beta : d_2 \to d_3$ such that $\gamma = \beta \circ \alpha$ there is a unique $g : c_2 \to c_3$ in $\mathcal{C}$ such that $\beta = p(g)$ and $h = g \circ f$, namely every lift problem

$$\begin{tikzcd} c_1 \arrow[r, "f"] \arrow[rr, bend left, "\forall"] \arrow[d, dash] & c_2 \arrow[r, dashed, "\exists!"] \arrow[d, dash] & c_3 \arrow[d, dash] \\ d_1 \arrow[r, "\alpha"] \arrow[rr, bend right] & d_2 \arrow[r, "\forall"] & d_3 \end{tikzcd}$$

admits a unique solution. In other words, any lifting problem

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "f"] & \Lambda^2_0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\ & \Delta^2 \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

admits a unique solution. One can also interpretate this as the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(c_2, c_3) \arrow[r, "f^*"] \arrow[d, "p"] & \mathrm{Hom}_{\mathcal{C}}(c_1, c_3) \arrow[d, "p"] \\ \mathrm{Hom}_{\mathcal{D}}(p(c_2), p(c_3)) \arrow[r, "p(f)^*"] & \mathrm{Hom}_{\mathcal{D}}(p(c_1), p(c_3)) \end{tikzcd}$$

being a pullback diagram. We say $p : \mathcal{C} \to \mathcal{D}$ is a Grothendieck opfibration if for all $c_1 \in \mathcal{C}$ and for all morphisms $\alpha$ in $\mathcal{D}$ with codomain $p(c_1)$ there is a $p$-cocartesian lift.

The most important property of being a Grothendieck opfibration is the following construction.

**Construction 9.18.** Let $p : \mathcal{C} \to \mathcal{D}$ be a Grothendieck opfibration. Then for any morphism $\alpha : d_1 \to d_2$, we define

$$\alpha_! : \mathcal{C}_{/d_1} \to \mathcal{C}_{/d_2}$$

by sending $c_1$ to the target of the $p$-cocartesian lift $f : c_1 \to c_2$ of $\alpha$. Since $f : c_1 \to c_2$ is $p$-cocartesian, $\alpha_!$ does define a functor. Moreover, there is a unique natural isomorphism $\beta_! \circ \alpha_! \simeq (\beta \circ \alpha)_!$.

We now come to the example of the Grothendieck opfibration for symmetric monoidal categories.

**Example 9.19.** Let $(\mathcal{C}, \otimes)$ be an ordinary symmetric monoidal category. We define $\mathcal{C}^{\otimes}$ as follows:

(1) objects are given by $(c_1, \dots, c_n)$ with $c_i \in \mathcal{C}$ for $n \in \mathbb{N}$;
(2) morphisms from $(c_1, \dots, c_n)$ to $(d_1, \dots, d_m)$ are given by
  (a) a map $\alpha : \langle n \rangle \to \langle m \rangle$ in $\mathrm{Fin}_*$;

(b) and for each $0 \neq k \in \langle m \rangle$, a map in $\mathcal{C}$

$$\bigotimes_{i \in \alpha^{-1}(k)} c_i \to d_k.$$

Note that the composition in $\mathcal{C}^{\otimes}$ is defined by the obvious way. Moreover, we have a forgetful functor $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ sending $(c_1, \dots, c_n)$ to $\langle n \rangle$ and a morphism $(c_1, \dots, c_n) \to (d_1, \dots, d_m)$ to the underlying map $\alpha$. It is not hard to see that $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ is a Grothendieck opfibration. Indeed, for every $(c_1, \dots, c_n) \in \mathcal{C}^{\otimes}$ with $\alpha : \langle n \rangle \to \langle m \rangle$ in $\mathrm{Fin}_*$, we define

$$d_j \coloneqq \bigotimes_{i \in \alpha^{-1}(k)} c_i$$

for $1 \leq k \leq m$. The obvious map from $(c_1, \dots, c_n)$ to $(d_1, \dots, d_m)$ gives a $p$-cocartesian lift. Moreover, $\mathcal{C}^{\otimes}_{\langle 1 \rangle} \simeq \mathcal{C}$ and

$$\mathcal{C}^{\otimes}_{\langle n \rangle} \xrightarrow{(\rho^i_!)} \prod_{i=1}^{n} \mathcal{C}$$

is an equivalence. Note that all the constraints can be encoded this way. For example, let $\sigma : \langle 2 \rangle \to \langle 2 \rangle$ be the automorphism exchanging 1 and 2. Then $m \circ \sigma = m$, which gives the symmetry by the corresponding functor $m_! \circ \sigma_! \simeq m_!$.

We can now define the notion of lax symmetric monoidal functors of $\infty$-categories.

**Definition 9.20.** A morphism $\alpha : \langle n \rangle \to \langle m \rangle$ in $\mathrm{Fin}_*$ is called inert if the induced map

$$\alpha : \alpha^{-1}(\langle m \rangle^{\circ}) \to \langle m \rangle^{\circ}$$

is a bijection. In other words, for every $k \in \langle m \rangle^{\circ}$, $\alpha^{-1}(\{k\})$ has exactly 1 element.

**Definition 9.21.** A lax symmetric monoidal functor between symmetric monoidal $\infty$-categories $\mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$ and $\mathcal{D}^{\otimes} \to N(\mathrm{Fin}_*)$ is a commutative diagram in $\mathrm{Cat}_{\infty}$

$$\begin{array}{ccc} \mathcal{C}^{\otimes} & \xrightarrow{F^{\otimes}} & \mathcal{D}^{\otimes} \\ & \searrow \quad \swarrow & \\ & N(\mathrm{Fin}_*) & \end{array}$$

such that $F^{\otimes}$ sends cocartesian lifts of inert morphism to corcartesian lifts. We will denote all the lax symmetric monoidal functors between $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ as $\mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ or $\mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathcal{C}, \mathcal{D})$ by slightly abusing notations.

A (strong) monoidal functor if it preserves arbitrary cocartesian lifts. We will denote all the lax symmetric monoidal functors between $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ as $\mathrm{Fun}^{\otimes}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ or $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$.

**Remark 9.22.** the diagram in $\mathrm{Cat}_{\infty}$ commutes is the same as it commutes in $\mathrm{Set}_{\Delta}$ by fibration techniques.

Again we explain the idea of strong (resp. lax) symmetric monoidal functors in the context of 1-category.

**Example 9.23.** Let $\mathcal{C}$, $\mathcal{D}$ be symmetric monoidal categories. Then giving a strong (resp. lax) symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is equivalent of giving a functor $F^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ which sends cocartesian lifts (resp. of inert morphisms) to corcartesian lifts. For simplicity, we explain how to get $F$ from $F^{\otimes}$. First note that $F^{\otimes}_{\langle 1 \rangle}$ gives the underlying functor $F : \mathcal{C} \to \mathcal{D}$. Note that all projections $\rho^i : \langle n \rangle \to \langle 1 \rangle$ are inert but the multiplication $m : \langle 2 \rangle \to \langle 1 \rangle$ and the inclusion $i : \langle 0 \rangle \to \langle 1 \rangle$ not. Hence, by the equivalence

$$\mathcal{C}^{\otimes}_{\langle n \rangle} \xrightarrow{(\rho^i_!)} \prod_{i=1}^{n} \mathcal{C}$$

one can always identify $F^{\otimes}_{\langle n \rangle}(c_1, \ldots, c_n)$ with $(F(c_1), \ldots, F(c_n))$. Assume first that $F^{\otimes}$ preserves all cocartesian lift, then the cocartesian lifts of $m$ and $i$ given by $(c_1, c_2) \to c$ and $1_{\mathcal{C}}$, respectively, are preserved. Note that $(c_1, c_2) \to c$ and $\varnothing \to 1_{\mathcal{C}}$ being cocartesian means that $c_1 \otimes c_2 \simeq c$ and $1_{\mathcal{C}}$ is the tensor unit. But their image $(F(c_1), F(c_2)) \to F(c)$ and $F(1_{\mathcal{C}})$ are again cocartesian, hence $F(c_1) \otimes F(c_2) \simeq F(c)$ and $F(1_{\mathcal{C}}) \simeq 1_{\mathcal{D}}$. Now if $F^{\otimes}$ only preserves cocartesian lifts of inert morphisms, then one get a unique map

$$F(c_1) \otimes F(c_2) \to F(c) \simeq F(c_1 \otimes c_2)$$

by using the universal property of the cocartesian lift $(F(c_1), F(c_2)) \to F(c_1) \otimes F(c_2)$ to the image $F^{\otimes}_{\langle 2 \rangle}(c_1, c_2) = (F(c_1), F(c_2)) \to F(c) \simeq F(c_1 \otimes c_2)$.

## 10. $\mathbb{E}_n$-Algebras

We will introduce the notion $\mathbb{E}_n$-algebras in a symmetric monoidal $\infty$-category in this section. We will see that $\mathbb{E}_1$-algebras recover associative algebras and $\mathbb{E}_\infty$-algebras recover commutative algebras. Throughout this section, $\mathcal{C}$ will be a symmetric monoidal $\infty$-category. We start by defining associative algebra objects.

**Definition 10.1.** We define the associative tensor active (1-)category $\mathrm{Assoc}^{\otimes}_{\mathrm{act}}$ as follows:

(1) Objects $\mathrm{Assoc}^{\otimes}_{\mathrm{act}}$ are finite sets;
(2) Morphisms consist of the following datum a map of finite set $f : S \to T$ together with a total ordering on each $f^{-1}(\{t\})$. In particular, the set of all morphisms from $S$ to a singleton is given by all possible total ordering on $S$, which has cardinality $n!$ where $n = |S|$.
(3) Composition of $f : S \to T$ and $g : T \to U$ uses the lexicographic ordering on each

$$(g \circ f)^{-1}(\{u\}) = \bigsqcup_{t \in g^{-1}(\{u\})} f^{-1}(\{t\}),$$

i.e. we first compare two elements by the respective $t$, and if they agree, by the ordering on $f^{-1}(\{t\})$.

We also endow $\mathrm{Assoc}^{\otimes}_{\mathrm{act}}$ with the symmetric monoidal structure given by disjoint union.

**Example 10.2.** Write $\langle 1 \rangle \in \mathrm{Assoc}^{\otimes}_{\mathrm{act}}$ for the 1-element set[2]. In $\mathrm{Assoc}^{\otimes}_{\mathrm{act}}$, $\langle 1 \rangle$ is an associated algebra object. The multiplication map $m$ is given by the canonical map

$$\langle 1 \rangle \sqcup \langle 1 \rangle \to \langle 1 \rangle$$

where the ordering is given by $* < *'$ where $*$ is the point on the left hand side and $*'$ is the point on the right hand side. The unit map is given by

$$\varnothing \to \langle 1 \rangle.$$

Note that the associativity is given by the commutative diagram

$$\begin{array}{ccc} \langle 1 \rangle \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle & \xrightarrow{m \sqcup \mathrm{Id}} & \langle 1 \rangle \sqcup \langle 1 \rangle \\ \big\downarrow{\scriptstyle \mathrm{Id} \sqcup m} & & \big\downarrow{\scriptstyle m} \\ \langle 1 \rangle \sqcup \langle 1 \rangle & \xrightarrow{\quad m \quad} & \langle 1 \rangle \end{array}$$

where we define the left singleton to be the smallest and the right singleton to be the largest. Hence one can think of $\mathrm{Assoc}^{\otimes}_{\mathrm{act}}$ as the "free" symmetric monoidal category on one algebra object. Indeed, all the maps in $\mathrm{Assoc}^{\otimes}_{\mathrm{act}}$ are given by certain association of the multiplication $m$.

With this in mind, we make the following definition.

[2] We slightly abused the notations here since $\langle 1 \rangle \in \mathrm{Fin}_*$ denotes the 2-element set. But one can view as the 1-element set added a base point.

**Definition 10.3.** An associative algebra (object) in a symmetric monoidal $\infty$-category $\mathcal{C}$ is given by a symmetric monoidal functor

$$F : N(\mathrm{Assoc}^{\otimes}_{\mathrm{act}}) \to \mathcal{C}.$$

The underlying object is given by $F(\langle 1 \rangle)$. The algebra structure is then induced by the one in $\mathrm{Assoc}^{\otimes}_{\mathrm{act}}$ in a coherent way. We denote the $\infty$-category of associative algebra of $\mathcal{C}$ as

$$\mathrm{Alg}(\mathcal{C}) := \mathrm{Fun}^{\otimes}(N(\mathrm{Assoc}^{\otimes}_{\mathrm{act}}), \mathcal{C}).$$

**Example 10.4.** The category of symmetric monoidal functors from $\mathrm{Assoc}^{\otimes}_{\mathrm{act}}$ to the category of abelian groups Ab is equivalent to the category of associated rings.

**todo:** exercise

The definition of commutative algebra object is much more simplier.

**Definition 10.5.** Let $\mathrm{Comm}^{\otimes}_{\mathrm{act}}$ denote the category of finite sets with symmetric monoidal structure given by the disjoint union.

**Definition 10.6.** A commutative algebra (object) in a symmetric monoidal $\infty$-category $\mathcal{C}$ is given by a symmetric monoidal functor

$$F : N(\mathrm{Comm}^{\otimes}_{\mathrm{act}}) \to \mathcal{C}.$$

The underlying object is given by $F(\langle 1 \rangle)$. The algebra structure is then induced by the one in $\mathrm{Comm}^{\otimes}_{\mathrm{act}}$ in a coherent way. We denote the $\infty$-category of commutative algebras of $\mathcal{C}$ as

$$\mathrm{CAlg}(\mathcal{C}) := \mathrm{Fun}^{\otimes}(N(\mathrm{Comm}^{\otimes}_{\mathrm{act}}), \mathcal{C}).$$

**Example 10.7.** The category of symmetric monoidal functors from $\mathrm{Comm}^{\otimes}_{\mathrm{act}}$ to the category of abelian groups Ab is equivalent to the category of commutative rings.

**todo:** exercise

**Remark 10.8** (Warning)**.** For a symmetric monoidal 1-category, the commutative algebra objects form a full subcategory of the associated algebra objects. In other words, it is just a property for an associated algebra to be commutative and the multiplications are the same. But this is false in the world of $\infty$-categories. Indeed, the forgetful functor (which is symmetric monoidal) $\mathrm{Assoc}^{\otimes}_{\mathrm{act}} \to \mathrm{Comm}^{\otimes}_{\mathrm{act}}$ induces a functor

$$\mathrm{CAlg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$$

for a symmetric monoidal $\infty$-category $\mathcal{C}$. But this functor is far from being fully faithful. More precisely, lifting an associated algebra to a commutative algebra is not just a property of an extra structure.

The goal of introducing $\mathbb{E}_n$-algebras is to provide an interpolation between $\mathrm{CAlg}(\mathcal{C})$ and $\mathrm{Alg}(\mathcal{C})$.

**Definition 10.9.** Let $0 \leq n < \infty$. We define a symmetric monoidal $\infty$-category $\mathbb{E}^{\otimes}_{n,\mathrm{act}}$ as the homotopy coherent nerve of the following symmetric monoidal topological-enriched category:

(1) objects are given by finite disjoint union of disks $\coprod_{i=1}^{k} D_i^n$ for $k \in \mathbb{N}$ where $D_i^n := (-1,1)^n$ for all $i$;

(2) morphisms from $\bigsqcup_{i=1}^k D_i^n$ to $\bigsqcup_{j=1}^l D_j^n$ are given by rectilinear embeddings, i.e. embeddings given by the formula

$$f(x_1, \dots, x_n) = (\alpha_1 x_1 + \beta_1, \dots, \alpha_n x_n + \beta_n)$$

with $\alpha_1, \dots, \alpha_n \in \mathbb{R}_{>0}$ on each unit disk of the domain; compositions are defined by the obvious ones;

(3) finally to endow the Hom sets with a topological structure, note that

$$\mathrm{Hom}(\bigsqcup_{i=1}^k D_i^n, D^n) \subset (\mathbb{R}^{2n})^k$$

is an open subset, hence we can just take the subspace topology; in general, we can decompose

$$\mathrm{Hom}(\bigsqcup_{i=1}^k D_i^n, \bigsqcup_{j=1}^l D_j^n) = \bigsqcup_{f:\langle k\rangle \to \langle l\rangle} \prod_{j=1}^n \mathrm{Hom}(\bigsqcup_{f(i)=j} D_i^n, D_j^n)$$

and take the canonical topology;

(4) the symmetric monoidal structure is given by the disjoint union.

Now we can define $\mathbb{E}_n$-algebras.

**Definition 10.10.** An $\mathbb{E}_n$-algebra in a symmetric monoidal $\infty$-category $\mathcal{C}$ is given by a symmetric monoidal functor

$$F : \mathbb{E}_{n,\mathrm{act}}^{\otimes} \to \mathcal{C}.$$

The underlying object is given by $F(D^n)$. We denote the $\infty$-category of $\mathbb{E}_n$-algebras of $\mathcal{C}$ as

$$\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) := \mathrm{Fun}^{\otimes}(\mathbb{E}_{n,\mathrm{act}}^{\otimes}, \mathcal{C}).$$

To understand $\mathbb{E}_n$-algebras, we first try to understand the mapping spaces of $\mathbb{E}_{n,\mathrm{act}}^{\otimes}$.

**Remark 10.11** (Configuration Spaces)**.** By the observation in Definition 10.9, to under stand the mapping spaces in $\mathbb{E}_{n,\mathrm{act}}^{\otimes}$, it suffices to understand

$$\mathrm{Map}_{\mathbb{E}_{n,\mathrm{act}}^{\otimes}}(\bigsqcup_{i=1}^k D_i^n, D^n).$$

Let $\mathrm{Conf}_k(D^n)$ be the subspace of $(D^n)^k$ with $k$ different points, which is called the configuration space of $k$ points in $D^n$. In this way, we get a map

$$\mathrm{Map}_{\mathbb{E}_{n,\mathrm{act}}^{\otimes}}(\bigsqcup_{i=1}^k D_i^n, D^n) \to \mathrm{Conf}_k(D^n)$$

by evaluation at the center points. In fact, this is a homotopy equivalence by the following lemma. Granting this fact, this means that

$$\mathrm{Map}_{\mathbb{E}_{n,\mathrm{act}}^{\otimes}}(D^n, D^n) \simeq \mathrm{Conf}_1(D^n) \simeq *$$

is contractible. Hence there will be no nontrivial structural endomorphisms on the underlying $\mathbb{E}_n$-algebras. Moreover, we have

$$\mathrm{Map}_{\mathbb{E}^{\otimes}_{n,\mathrm{act}}}(D^n \sqcup D^n, D^n) \simeq \mathrm{Conf}_2(D^n) \simeq S^{n-1}$$

where the second equivalence is by using a linear homotopy to move one point to the center and the other point to a point on the $(n-1)$-sphere with distance $1/2$ to the center. Now let

$$\underline{A} : \mathbb{E}^{\otimes}_{n,\mathrm{act}} \to \mathcal{C}$$

be an $\mathbb{E}_n$-algebra in a symmetric monoidal $\infty$-category $\mathcal{C}$ with underlying object $A := \underline{A}(D^n)$. There is a map

$$\mathrm{Map}_{\mathbb{E}^{\otimes}_{n,\mathrm{act}}}(D^n \sqcup D^n, D^n \simeq S^{n-1}) \to \mathrm{Map}_{\mathcal{C}}(A \otimes A, A).$$

The 0-cell of $S^{n-1}$ will give the multiplication of $A$.

**Lemma 10.12.** *Evaluation at the origin*

$$\mathrm{ev}_0 : \mathrm{Map}_{\mathbb{E}^{\otimes}_{n,\mathrm{act}}}(\bigsqcup_{i=1}^{k} D^n_i, D^n) \to \mathrm{Conf}_k(D^n)$$

*determines a homotopy equivalence.*

*Proof.*

**todo:** exercise: Hint write down an inverse: HA Lemma 5.1.1.3.

□

In order to understand $\mathbb{E}_n$-algebras better, we first look at the case where $n = 1$.

**Proposition 10.13.** *There is an equivalence of symmetric monoidal $\infty$-categories*

$$N(\mathrm{Assoc}^{\otimes}_{\mathrm{act}}) \to \mathbb{E}^{\otimes}_{1,\mathrm{act}}$$

*which sends $\langle k \rangle$ to $\bigsqcup_{i=1}^{k} D^n_i$.*

*Proof.* First observe that $\mathbb{E}^{\otimes}_{1,\mathrm{act}}$ is essentially a 1-category and is equivalent to $\mathrm{Assoc}^{\otimes}_{\mathrm{act}}$ since a rectilinear map from $D^1 \sqcup \ldots D^1 \to D^1$ is given by a linear order of $k$ points on $(-1, 1)$ which is exactly the definition of $\mathrm{Assoc}^{\otimes}_{\mathrm{act}}$.

**todo:** HA Example 5.1.0.7.

□

## 11. $p$-Adic Completion

## 12. The Tate Construction

## 13. The Tate Diagonal

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