DERIVED HECKE ALGEBRA FOR MODULAR FORMS OF WEIGHT ONE VIA CLASSICALITY

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Abstract. We construct the action of the derived diamond operators at p on the space of weight one (classical) modular forms in two different ways, one using p-adic modular forms and another using Λ-adic Sen operator. This is the p-adic analogue of the operation of taking complex conjugation to modular forms, and it originates from the (obstruction to) *classicality* of overconvergent modular forms of weight one. Our construction uses a recent work of Pan [Pan22] which gives a geometric interpretation of the classicality. Using this construction, we formulate a p-adic version of the conjecture of Harris–Venkatesh [HV19].

Contents

1. Introduction 1
2. Derived diamond action via classicality of weight one modular forms 2
3. Derived Sen action via Λ-adic Sen operator 6
4. The p-adic analogue of the Harris–Venkatesh conjecture 8
References 8

1. Introduction

The archimedean derived Hecke operator in the context of weight one modular forms is the complex conjugation. This is folklore, but was first explicitly spelled out in [Hor23, Oh22]. It exploits the fact that the algebraic cohomology groups $H^0(X,\omega)$ and $H^1(X,\omega)$ can be also computed in a transcendental way using the Dolbeault complex:

$$0 \to H^0(X,\omega) \to \mathscr{A}^{0,0}(X,\omega) \to \mathscr{A}^{0,1}(X,\omega) \to H^1(X,\omega) \to 0.$$

The complex conjugation is then done by seeing a modular form as a real analytic section in $\mathscr{A}^{0,0}(X,\omega)$, taking the complex conjugation, which yields a real analytic section in $\mathscr{A}^{0,1}(X,\omega)$, and projecting to $H^1(X,\omega)$.

We aim to explain a similar picture in the p-adic context. Namely, the algebraic cohomology groups $H^0(X,\omega)$ and $H^1(X,\omega)$ can be computed in a p-adic analytic way using the works of Pan [Pan22]:

$$0 \to H^0(X,\omega) \to H^0_0 \to H^1_w \to H^1(X,\omega) \to 0.$$

Here, H^0_0 and H^1_w are in an appropriate sense the weight one specializations of *Hida/Coleman theory* and *higher Hida/Coleman theory* (in the sense of Boxer–Pilloni [BP21, Pil24]), respectively. The four-term exact sequence above is equivariant under many actions such as prime-to-p Hecke operators, U_p-operators, and the diamond operators, and the derived action of the diamond operators is the p-adic analogue of the archimedean picture we had alluded above in the first paragraph.

This can be seen as the weight-one-modular-form analogue of the work of [KR23] on derived diamond operators on the Betti cohomology of locally symmetric spaces with $\operatorname{rank}_{\mathbb{R}} G \neq \operatorname{rank}_{\mathbb{R}} K$.

Acknowledgements. To be added later.

Notation. Let p be a rational prime, $C = \mathbb{C}_p$ and $K^p \subset \mathrm{GL}_2(\mathbb{A}_f^p)$ be a tame level. Let $\mathcal{X}_{K^pK_p} = X(K^pK_p)_C^{\mathrm{ad}}$ be the adic space associated with the modular curve of level K^pK_p over C, and let $\mathcal{X}_{K^p} \sim \varprojlim_{K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)} \mathcal{X}_{K^pK_p}$ be the perfectoid modular curve of tame level K^p. Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $B \subset G$ be the upper triangular Borel subgroup, $N \subset B$ be the unipotent radical of B, $\mathfrak{g} = \operatorname{Lie} G \otimes_{\mathbb{Q}_p} C$, $\mathfrak{b} = \operatorname{Lie} B \otimes_{\mathbb{Q}_p} C$, $\mathfrak{n} = \operatorname{Lie} N \otimes_{\mathbb{Q}_p} C$. There is the Hodge–Tate period map

$$\pi_{\mathrm{HT}} : \mathcal{X}_{K^p} \to \mathbb{P}^1_C,$$

that is equivariant under the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ (which acts on $\mathbb{P}^1_C$ on the right) and the Hecke operators away from p (which acts trivially on $\mathbb{P}^1_C$). Let $\mathcal{O}_{K^p} = \pi_{\mathrm{HT},*}\mathcal{O}_{\mathcal{X}_{K^p}}$, and ω_{K^p} be line bundle over $\mathcal{X}_{K^p}$ that comes from the Hodge line bundles ω over $X(K^pK_p)_C$.

2. Derived diamond action via classicality of weight one modular forms

We briefly review the work of Pan [Pan22] on the geometric interpretation of classicality of weight one overconvergent modular forms. The theory starts with the observation that there is a nice notion of the functor of "taking locally analytic vectors" of an admissible Banach representation of a p-adic Lie group. In particular, with respect to the $\mathrm{GL}_2(\mathbb{Q}_p)$-action on $\mathcal{O}_{K^p}$, there is the subsheaf $\mathcal{O}_{K^p}^{\mathrm{la}} \subset \mathcal{O}_{K^p}$ of "locally analytic vectors". One key property of the sheaf of locally analytic vectors is that the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ can be differentiated, so that there is a C-linear action of $\mathfrak{g}$ on $\mathcal{O}_{K^p}^{\mathrm{la}}$ that commutes with the G-action.

Regarding $\mathbb{P}^1_C$ as the flag variety of $\mathrm{GL}_{2,C}$, for $x \in \mathbb{P}^1_C(C)$, let $\mathfrak{b}_x \subset \mathfrak{g}$ be the corresponding Borel subalgebra, and $\mathfrak{n}_x \subset \mathfrak{b}_x$ be its nilpotent radical. We define

$$\mathfrak{g}^0 := \mathcal{O}_{\mathbb{P}^1_C} \otimes_C \mathfrak{g},$$

$$\mathfrak{b}^0 := \{f \in \mathfrak{g}^0 \mid f_x \in \mathfrak{b}_x\},$$

$$\mathfrak{n}^0 := \{f \in \mathfrak{g}^0 \mid f_x \in \mathfrak{n}_x\}.$$

The action of $\mathfrak{g}$ on $\mathcal{O}_{K^p}^{\mathrm{la}}$ extends to an action of $\mathfrak{g}^0$ via $(f \otimes z) \cdot s := f(z \cdot s)$.

Theorem 2.1 (Geometric Sen theory)**.** *The action of* $\mathfrak{n}^0$ *annihilates* $\mathcal{O}_{K^p}^{\mathrm{la}}$.

Proof. See [Pan22, Theorem 4.2.7]. $\square$

Let $\mathfrak{h}$ be the Cartan subalgebra of diagonal matrices of $\mathfrak{g}$. Then, by Theorem 2.1, $\mathfrak{b}^0/\mathfrak{n}^0 \cong \mathcal{O}_{\mathbb{P}^1_C} \otimes_C \mathfrak{h}$ is a trivial vector bundle over $\mathbb{P}^1_C$ acting on $\mathcal{O}_{K^p}^{\mathrm{la}}$. A specific trivialization can be chosen by fixing $H^0(\mathbb{P}^1_C, \mathfrak{b}^0/\mathfrak{n}^0) \cong \mathfrak{h}$, and we can identify $\mathfrak{h}$ with the constant sections of $\mathfrak{b}^0/\mathfrak{n}^0$. Let the action of $\mathfrak{h}$ on $\mathcal{O}_{K^p}^{\mathrm{la}}$ defined as the action of the constant sections of $\mathfrak{b}^0/\mathfrak{n}^0$ be denoted as $\theta_{\mathfrak{h}}$. We will call this the *horizontal action* of $\mathfrak{h}$ to distinguish it from the action of $\mathfrak{h} \subset \mathfrak{g}$.

Theorem 2.2 (Arithmetic Sen theory)**.** *The operator* $\theta_{\mathfrak{h}}\left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)$ *is the Sen operator in the sense of* [Pan22, Definition 5.1.5].

Proof. See [Pan22, Theorem 5.1.11]. $\square$

Let us emphasize that $\mathcal{O}^{\mathrm{la}}_{K^p}$ has the actions of G, $\mathfrak{g}$, $\theta_{\mathfrak{h}}$, and the Hecke operators away from p, all commuting with each other. For a character $\chi : \mathfrak{h} \to C$, let $\mathcal{O}^{\mathrm{la},\chi}_{K^p}$ be the subsheaf of $\mathcal{O}^{\mathrm{la}}_{K^p}$ of sections of weight χ (i.e., local sections f of $\mathcal{O}^{\mathrm{la}}_{K^p}$ where $\theta_{\mathfrak{h}}(h)f = \chi(h)f$ for $h \in \mathfrak{h}$). Then, $\mathcal{O}^{\mathrm{la},\chi}_{K^p}$ has mutually commuting actions of G, $\mathfrak{g}$, and the prime-to-p Hecke operators.

Theorem 2.3 (Pan). *For $n_1, n_2 \in C$, let $(n_1, n_2) : \mathfrak{h} \to C$ be the character defined by*

$$(n_1, n_2)\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) := an_1 + dn_2,$$

1. *The $\mathfrak{n}$-coinvariant $(\mathcal{O}^{\mathrm{la},(n_1,n_2)}_{K^p})_{\mathfrak{n}}$ (where the action of $\mathfrak{n}$ on $\mathcal{O}^{\mathrm{la},(n_1,n_2)}_{K^p}$ is via the $\mathfrak{g}$-action) is supported at $\infty \in \mathbb{P}^1_C$.*
2. *If $n_1 - n_2 \in \overline{\mathbb{Q}} - \{0, -1, -2, \cdots\}$, then $(\mathcal{O}^{\mathrm{la},(n_1,n_2)}_{K^p})_{\mathfrak{n}} \cong (\mathcal{O}^{\mathrm{la},(n_1,n_2)}_{K^p})_{\infty}$.*
3. *Suppose that n_1, n_2 are integers. The stalk $(\mathcal{O}^{\mathrm{la},(n_1,n_2)}_{K^p})_{\infty}$ can be explicitly identified with the space of B-overconvergent modular forms of weight $n_1 - n_2$ and tame level K^p, namely*

$$(\mathcal{O}^{\mathrm{la},(n_1,n_2)}_{K^p})_{\infty} \cong \varinjlim_{n\to\infty} M^{\dagger}_{n_1-n_2}(K^p\Gamma(p^n)) =: M^{B\text{-}\dagger}_{n_1-n_2}(K^p),$$

where $M^{\dagger}_{n_1-n_2}(K^p\Gamma(p^n))$ is the space of overconvergent sections of $\omega^{n_1-n_2}$ on a strict neighborhood of the canonical locus of $X_{K^p\Gamma(p^n)}$.

Proof. For (1), see [Pan22, Proposition 5.2.10 (1)]. For (2), see [Pan22, Proposition 5.2.10 (3)]. For (3), see [Pan22, Proposition 5.2.6]. □

Note that $(\mathcal{O}^{\mathrm{la},\chi}_{K^p})_{\infty}$ has mutually commuting actions of B (the stabilizer of $\infty \in \mathbb{P}^1_C$ in G), $\mathfrak{g}$, and the prime-to-p Hecke operators.

In the above Theorem, we used a non-standard terminology of *B-overconvergent modular forms.* The name originates from the fact that the space of B-overconvergent modular forms has an action of $B(\mathbb{Q}_p)$. The traditional definition of overconvergent modular forms involve taking the limit over deeper (balanced) Γ_1-levels. Namely, if we define

$$\Gamma_1(p^n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \mid a-1, d-1, c \in p^n\mathbb{Z}_p \right\},$$

then the *space of overconvergent modular forms* of weight $k \in \mathbb{Z}$ and tame level K^p is defined as

$$M^{\dagger}_k(K^p) := \varinjlim_{n\to\infty} M^{\dagger}_k(K^p\Gamma_1(p^n)).$$

Note that $M^{\dagger}_k(K^p)$ has the action of $H(\mathbb{Z}_p) = \left\{ \begin{pmatrix} \mathbb{Z}_p^{\times} & 0 \\ 0 & \mathbb{Z}_p^{\times} \end{pmatrix} \right\}$, where $H \subset G$ is the diagonal Cartan. Moreover, there is an action of the *U_p-operator*, defined by

$$U_p := \sum_{i=0}^{p-1} \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix}.$$

One can recover $M^{\dagger}_k(K^p)$ from $M^{B\text{-}\dagger}_k(K^p)$ and vice versa, as follows. Firstly, $M^{\dagger}_k(K^p) = M^{B\text{-}\dagger}_k(K^p)^{N(\mathbb{Z}_p)}$, with the action of $H(\mathbb{Z}_p)$ and U_p induced from the action of G . Conversely, $M^{B\text{-}\dagger}_k(K^p)$ is recovered from $M^{\dagger}_k(K^p)$ by inverting the action of $\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}$.

Let $k \in \mathbb{N}$, and $\omega^{k,\mathrm{sm}}_{K^p} \subset \pi_{\mathrm{HT},*}\omega^{\otimes k}_{K^p}$ be the subsheaf of G-smooth sections. This sheaf encodes the "classical" modular forms:

Lemma 2.1 (Pan). *For $i = 0, 1$, there is a natural isomorphism*

$$H^i(\mathbb{P}^1_C, \omega^{k,\mathrm{sm}}_{K^p}) \cong \varinjlim_{K_p \subset G} H^i(X(K^p K_p)_C, \omega^{\otimes k}).$$

Proof. This is a consequence of the rigid-analytic GAGA and [Pan22, Lemma 5.3.5]. □

By the observations in [Pan22, 5.3.10], multiplication by $t^{-k}e_1^k$ gives rise to an injection

$$\mathcal{O}^{\mathrm{la},(k,0),\mathfrak{n}}_{K^p} \hookrightarrow \omega^{k,\mathrm{sm}}_{K^p},$$

that is an isomorphism away from $\infty \in \mathbb{P}^1$. Namely, we have a short exact sequence of sheaves

$$0 \to \mathcal{O}^{\mathrm{la},(k,0),\mathfrak{n}}_{K^p} \xrightarrow{\times t^{-k}e_1^k} \omega^{k,\mathrm{sm}}_{K^p} \to (i_\infty)_* M^{B\text{-}\dagger}_k(K^p) \to 0,$$

where i_∞ is the inclusion map of ∞ into $\mathbb{P}^1_C$. Taking the sheaf cohomology, we obtain an exact sequence

$$0 \to H^0(\mathbb{P}^1_C, \mathcal{O}^{\mathrm{la},(k,0),\mathfrak{n}}_{K^p}) \to H^0(\mathbb{P}^1_C, \omega^{k,\mathrm{sm}}_{K^p}) \to M^{B\text{-}\dagger}_k(K^p) \to H^1(\mathbb{P}^1_C, \mathcal{O}^{\mathrm{la},(k,0),\mathfrak{n}}_{K^p}) \to H^1(\mathbb{P}^1_C, \omega^{k,\mathrm{sm}}_{K^p}) \to 0.$$

We have used that $(i_\infty)_* M^{B\text{-}\dagger}_k(K^p)$ is supported at ∞ and thus has no higher cohomology. Furthermore, $H^0(\mathbb{P}^1_C, \mathcal{O}^{\mathrm{la},(k,0),\mathfrak{n}}_{K^p}) = 0$, as $\mathcal{O}^{\mathrm{la},(k,0),\mathfrak{n}}_{K^p}$ has zero stalk at ∞. Therefore, we have a four-term exact sequence

$$0 \to H^0(\mathbb{P}^1_C, \omega^{k,\mathrm{sm}}_{K^p}) \to M^{B\text{-}\dagger}_k(K^p) \to H^1(\mathbb{P}^1_C, \mathcal{O}^{\mathrm{la},(k,0),\mathfrak{n}}_{K^p}) \to H^1(\mathbb{P}^1_C, \omega^{k,\mathrm{sm}}_{K^p}) \to 0.$$

This four-term exact sequence is equivariant under mutually commuting actions of B, $\mathfrak{h} = \mathfrak{b}/\mathfrak{n}$, and the prime-to-$p$ Hecke operators. This short exact sequence may be regarded as the local cohomology exact sequence of the sheaf $\omega^{k,\mathrm{sm}}_{K^p}$ with respect to $\infty \hookrightarrow \mathbb{P}^1_C \hookleftarrow \mathbb{A}^1_C$.

By Lemma 2.1, both the first and the fourth terms of the four-term exact sequence are not zero only if $k = 1$. For $i = 0, 1$, we define the *space of classical H^i-modular forms of weight one* as

$$H^i(\omega) := H^i(\mathbb{P}^1_C, \omega^{1,\mathrm{sm}}_{K^p}).$$

Furthermore, we define the *space of B-overconvergent H^i-modular forms of weight one* as

$$H^0(\omega^{B\text{-}\dagger}) := M^{B\text{-}\dagger}_1(K^p),$$

$$H^1(\omega^{B\text{-}\dagger}) := H^1(\mathbb{P}^1_C, \mathcal{O}^{\mathrm{la},(1,0),\mathfrak{n}}_{K^p}).$$

We may drop the decoration "H^i-" in the above definitions when $i = 0$.

The four-term exact sequence when $k = 1$ then becomes

$$0 \to H^0(\omega) \to H^0(\omega^{B\text{-}\dagger}) \to H^1(\omega^{B\text{-}\dagger}) \to H^1(\omega) \to 0.$$

Definition 2.1. We define the *boundary operator*

$$\delta : H^0(\omega^{B\text{-}\dagger}) \to H^1(\omega^{B\text{-}\dagger})$$

be the middle map of the four-term exact sequence when $k = 1$. The boundary operator is equivariant under mutually commuting actions of B, $\mathfrak{h}$, and the prime-to-p Hecke operators.

The boundary operator may be defined on the overconvergent modular forms as well.

Definition 2.2. We define the space of *overconvergent H^i-modular forms of weight one* as

$$H^i(\omega^\dagger) := H^i(\omega^{B\text{-}\dagger})^{N(\mathbb{Z}_p)}.$$

Indeed, $H^0(\omega^\dagger) = M_1^\dagger(K^p)$. Taking $N(\mathbb{Z}_p)$-invariants, we obtain

$$\delta : H^0(\omega^\dagger) \to H^1(\omega^\dagger),$$

and we also refer to it as the *boundary operator*. This boundary operator is equivariant under mutually commuting actions of $H(\mathbb{Z}_p)$, U_p, $\mathfrak{h}$, and the prime-to-p Hecke operators. As these cohomology classes can be computed using the Čech-cohomology with an affine cover U_1, U_2 of $\mathbb{P}^1_C$ such that $0 \in U_1$, $\infty \in U_2$ and U_1, U_2 are stable under the $N(\mathbb{Z}_p)$-action, there is a four-term exact sequence

$$0 \to H^0(\omega)^{N(\mathbb{Z}_p)} \to H^0(\omega^\dagger) \xrightarrow{\delta} H^1(\omega^\dagger) \to H^1(\omega)^{N(\mathbb{Z}_p)} \to 0.$$

Remark 2.1. The space of overconvergent H^1-modular forms is closely related to the *higher Coleman theory* of Boxer–Pilloni (see [BP21, BP22, Pil24]). In this context, the boundary operator is closely related to a differential in the *Cousin complex*.

We can now define the *derived diamond operator* as follows.

Definition 2.3. Let $H_0 = \left\{ \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix} \right\} \subset H$. Let $K^\bullet = [H^0(\omega^\dagger) \xrightarrow{\delta} H^1(\omega^\dagger)]$ be the complex of C-Banach representations of H_0, with H^i in cohomological degree i. For $n \geq 1$, we define the *derived diamond action at $\Gamma_1(p^n)$-level* to be the natural action

$$\begin{aligned} &\mathrm{Ext}^1_{C[1+p^n\mathbb{Z}_p]}(C, C) \times \mathrm{Hom}_{C[1+p^n\mathbb{Z}_p]}(C, H^0(K^\bullet)) \\ \cong\, &\mathrm{Ext}^1_{C[1+p^n\mathbb{Z}_p]}(C, C) \times \mathrm{Hom}_{C[1+p^n\mathbb{Z}_p]}(C, K^\bullet) \\ \to\, &\mathrm{Ext}^1_{C[1+p^n\mathbb{Z}_p]}(C, K^\bullet) \\ \to\, &\mathrm{Hom}_{C[1+p^n\mathbb{Z}_p]}(C, H^1(K^\bullet)), \end{aligned}$$

where the last map is the natural connecting map, $C[1+p^n\mathbb{Z}_p] := \mathcal{O}_C[[1+p^n\mathbb{Z}_p]][1/p]$, and $1+p^n\mathbb{Z}_p$ acts by $1+p^n\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p^\times \cong H_0$.

The relation of the derived diamond action with the classical modular forms comes from the following.

Lemma 2.2. *For $i = 0, 1$ and $n \geq 1$, we have natural isomorphisms*

$$\mathrm{Hom}_{C[1+p^n\mathbb{Z}_p]}(C, H^i(K^\bullet)) \cong H^i(X(K^p\Gamma_1(p^n))_C, \omega),$$

equivariant under mutually commuting actions of U_p and the prime-to-p Hecke operators.

Proof. This is immediate from the four-term exact sequence. □

Therefore, the derived diamond action at $\Gamma_1(p^n)$-level is the U_p-equivariant Hecke-equivariant action

$$\mathrm{Ext}^1_{C[1+p^n\mathbb{Z}_p]}(C, C) \times H^0(X(K^p\Gamma_1(p^n))_C, \omega) \to H^1(X(K^p\Gamma_1(p^n))_C, \omega),$$

where the Hecke action and the U_p-action on $\mathrm{Ext}^1_{C[1+p^n\mathbb{Z}_p]}(C, C)$ are trivial.

3. Derived Sen action via Λ-adic Sen operator

Using the geometric interpretation of the (obstruction to) classicality developed in §2, we can give another action similar to the derived dimaond action, where a Tor-group acts in the reverse direction to the derived diamond operator, using the Λ-adic Galois representation attached to a Hida family. This duality between the Tor-action and the Ext-action is natural in the context of derived Hecke operators; see [CG18, GV18, Ven19]. Note that, even though the construction of this section itself does not require the results on classicality as in §2, that it is related to classical H^i-modular forms *a posteriori* uses the results on classicality. Another advantage is that the construction in this section yields an *integral* operator.

Before the construction, we need to recall the notion of Λ-adic Galois representation attached to a Hida theory and related constructions, as clarified in [Cai18a, Cai18b].

Let $N \geq 1$ be such that $(N,p)=1$. Consider

$$H^1_{\text{ét}} := \varprojlim_r H^1_{\text{ét}}(X_1(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p),$$

where $X_1(Np^r)$ is the modular curve corresponding to the level group

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) \mid a-1, c \in Np^r\widehat{\mathbb{Z}} \right\}.$$

The diamond operators give rise to the $\Lambda := \mathbb{Z}_p[[1+p\mathbb{Z}_p]]$-module structure on $H^1_{\text{ét}}$. Furthermore, if we let e be Hida's ordinary projector (arising as the limit of the powers of U_p), a theorem of Hida tells that $eH^1_{\text{ét}}$ is finite and free as a Λ-module. Moreover, this gives rise to a Λ-adic Galois representation

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}_\Lambda(eH^1_{\text{ét}})$$

which interpolates the Galois representations attached to $eH^1_{\text{ét}}(X_1(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)$. This Galois representation is *ordinary* at p. Namely, upon restricting to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $eH^1_{\text{ét}}$ has a filtration

$$0 \to (eH^1_{\text{ét}})^I \to eH^1_{\text{ét}} \to (eH^1_{\text{ét}})_I \to 0,$$

where $I \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the inertia subgroup, and $(-)^I$ and $(-)_I$ are the I-invariants and the I-coinvariants functors, respectively. It turns out that $(eH^1_{\text{ét}})^I$ and $(eH^1_{\text{ét}})_I$ are also finite and free as Λ-modules, and $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts as explicit characters on $(eH^1_{\text{ét}})^I$ and $(eH^1_{\text{ét}})_I$ (see [Cai18b, Corollary 1.2.7]).

The Λ-adic Eichler–Shimura isomorphism [Oht95, Cai18b] gives a canonical isomorphism

$$(eH^1_{\text{ét}})^I \otimes_\Lambda \Lambda_{W(\overline{\mathbb{F}}_p)[\mu_N]} \cong e\mathfrak{H} \otimes_\Lambda \Lambda_{W(\overline{\mathbb{F}}_p)[\mu_N]},$$
$$(eH^1_{\text{ét}})_I \otimes_\Lambda \Lambda_{W(\overline{\mathbb{F}}_p)[\mu_N]} \cong eS(N,\Lambda) \otimes_\Lambda \Lambda_{W(\overline{\mathbb{F}}_p)[\mu_N]},$$

where $eS(N,\Lambda)$ is the space of ordinary Λ-adic cusp forms of tame level N, and $e\mathfrak{H} := \varprojlim_r e\mathfrak{H}_r$ is the ordinary Hida Hecke algebra.

Given a Λ-adic cusp form $\mathbf{f} \in eS(N,\Lambda)$, cutting ρ out by the corresponding Hecke eigensystem $\lambda_{\mathbf{f}} : e\mathfrak{H} \to \Lambda$ gives rise to a 2-dimensional Λ-adic Galois representation $\rho_{\mathbf{f}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\Lambda)$. This interpolates the Galois representations attached to specializations of $\mathbf{f}$. Namely, given an integer k and a primitive p^r-th root of unity ζ, consider the arithmetic point $x_{k,\zeta} : \Lambda \to \mathbb{Z}_p[\zeta]$ given by $[\gamma] \mapsto \zeta\varepsilon_{\text{cyc}}^k(\gamma)$, where ε_{cyc} is the p-adic cyclotomic character and $1+p\mathbb{Z}_p$ is identified with $\mathrm{Gal}(\mathbb{Q}^{\text{cyc}}/\mathbb{Q})$ (here $\mathbb{Q}^{\text{cyc}}$ is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$). Then, for $k \geq 2$, $x_{k,\zeta} \circ \lambda_{\mathbf{f}}$ gives rise to a Hecke eigensystem of a normalized cuspidal Hecke eigen-new-form $\mathbf{f}_{x,\zeta} \in$

$S_k(\Gamma_1(Np^r), \mathbb{Z}_p[\zeta])$, and its associated Galois representation $\rho_{\mathbf{f}_{k,\zeta}}$ coincides with $x_{k,\zeta} \circ \rho_{\mathbf{f}}$. We also see $\mathbb{Z}_p[\zeta]$ ($\mathbb{Q}_p[\zeta]$, respectively) as a Λ-module via $x_{k,\zeta}$, which we denote by $\mathcal{O}_{k,\zeta}$ ($\mathcal{O}_{k,\zeta}[1/p]$, respectively).

It is famously known that the specialization of $\mathbf{f}$ for $k = 1$ is not necessarily a classical modular form of weight one. However, the Fontaine–Mazur conjecture tells us when $\mathbf{f}_{1,\zeta}$ is a classical modular form of weight one, based on the Galois representation.

Theorem 3.1. *Let ζ be a p-power root of unity.*

(1) *The weight one specialization $x_{1,\zeta} \circ \rho_{\mathbf{f}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_p[\zeta])$ is almost Hodge–Tate of weight $0, 0$ at p (in the sense of* [Fon04]*).*

(2) *The ordinary filtration*

$$0 \to N_{\mathbf{f}} \to \rho_{\mathbf{f}} \to M_{\mathbf{f}} \to 0,$$

where $N_{\mathbf{f}} := (eH^1_{\text{ét}})^I[\lambda_{\mathbf{f}}]$ and $M_{\mathbf{f}} := (eH^1_{\text{ét}})_I[\lambda_{\mathbf{f}}]$, gives rise to an extension

$$c_{\mathbf{f}} \in \mathrm{Ext}^1_{\Lambda[\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)]}(M_{\mathbf{f}}, N_{\mathbf{f}}).$$

Then, $\mathbf{f}_{1,\zeta}$ is a classical modular form of weight one if and only if $c_{\mathbf{f}} \otimes_\Lambda \mathcal{O}_{1,\zeta} = 0$, i.e., the extension $0 \to N_{\mathbf{f}} \otimes_\Lambda \mathcal{O}_{1,\zeta} \to \rho_{\mathbf{f}} \otimes_\Lambda \mathcal{O}_{1,\zeta} \to M_{\mathbf{f}} \otimes_\Lambda \mathcal{O}_{1,\zeta} \to 0$ splits. This is also equivalent to $x_{1,\zeta} \circ \rho_{\mathbf{f}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_p[\zeta])$ being Hodge–Tate of weight $0, 0$ at p.

Proof. This is the Fontaine–Mazur conjecture in the ordinary case, see [CG18, Pan22]. □

Theorem 3.1 leads us to consider the following.

Definition 3.1 (Λ-adic Sen operator)**.** Let $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$. Then, $(\rho_{\mathbf{f}} \otimes_\Lambda \Lambda_{W(C^\flat)})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\zeta_{p^\infty}))}$ inherits an action of Γ. Let $(\rho_{\mathbf{f}} \otimes_\Lambda \Lambda_{W(C^\flat)})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\zeta_{p^\infty})),\Gamma\text{-an}}$ be the largest submodule where Γ acts analytically (in this case, unipotently), which now inherits the action of $\mathrm{Lie}(\Gamma) \cong \mathbb{Z}_p$. The action of $1 \in \mathbb{Z}_p \cong \mathrm{Lie}(\Gamma)$ induces a Λ-linear map $\theta : M_{\mathbf{f}} \to N_{\mathbf{f}}$, which we call the *$\Lambda$-adic Sen operator.*

Lemma 3.1. *Let ζ be a p-power root of unity. Then, $\mathbf{f}_{1,\zeta}$ is a classical modular form of weight one if and only if $\theta \otimes_\Lambda \mathcal{O}_{1,\zeta} = 0$.*

Proof. This follows immediately from the definition of the Sen operator that an almost Hodge–Tate representation of weight $0, 0$ is Hodge–Tate if and only if the Sen operator vanishes. See [Fon04]. □

We are now ready to construct the derived Sen action.

Definition 3.2 (Derived Sen action)**.** Let $\widetilde{K}^\bullet_{\mathbf{f}} := [M_{\mathbf{f}} \xrightarrow{\theta} N_{\mathbf{f}}]$ be the complex of Λ-modules in homological degrees $[0, 1]$. Let ζ be a primitive p^r-th root of unity. We define the *derived Sen action* to be the action

$$\begin{aligned} &\mathrm{Tor}^1_\Lambda(\mathcal{O}_{1,\zeta}, \mathcal{O}_{1,\zeta}) \times \operatorname{coker}\theta \otimes \mathcal{O}_{1,\zeta} \\ \cong\, &\mathrm{Tor}^1_\Lambda(\mathcal{O}_{1,\zeta}, \mathcal{O}_{1,\zeta}) \times \mathrm{Tor}^0_\Lambda(\widetilde{K}^\bullet_{\mathbf{f}}, \mathcal{O}_{1,\zeta}) \\ \to\, &\mathrm{Tor}^1_\Lambda(\widetilde{K}^\bullet_{\mathbf{f}}, \mathcal{O}_{1,\zeta}) \\ \to\, &\ker\theta \otimes \mathcal{O}_{1,\zeta}. \end{aligned}$$

The relation of the derived Sen action with the classical modular forms comes from the following.

Lemma 3.2. *For a primitive p^r-th root of unity ζ, we have natural isomorphisms*

$$\ker\theta \otimes \mathcal{O}_{1,\zeta} \cong H^0(X_1(Np^r),\omega)_{\mathbb{Z}_p[\zeta]}[\lambda_{f_{1,\zeta}}],$$

$$\operatorname{coker}\theta \otimes \mathcal{O}_{1,\zeta} \cong H^1(X_1(Np^r)\omega)_{\mathbb{Z}_p[\zeta]}[\lambda_{f_{1,\zeta}}],$$

where $H^*(X_1(Np^r),\omega)_{\mathbb{Z}_p[\zeta]}$ is a lattice of $H^*(X_1(Np^r)_{\mathbb{Q}_p[\zeta]},\omega)$ given by the specialization of $H^0(\omega)$ or $H^1(\mathcal{O})$ of [Cai18a].

Proof. This follows from the control theorem [Cai18a, Theorem 1.2.1], [Cai18b, Theorem 1.2.4], and the interpretation of $N_{\mathbf{f}}$ as the higher Hida theory [BP22], interpolating $H^1(X_1(Np^r)_{\mathbb{Z}_p[\zeta]},\omega^k)$. □

4. The p-adic analogue of the Harris–Venkatesh conjecture

We formulate the p-adic analogue of the Harris–Venkatesh conjecture [HV19]. We fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ throughout this section.

Conjecture 4.1 (p-adic Harris–Venkatesh conjecture)**.** *Let $(N,p)=1$. Let $f \in S_1(\Gamma_1(Np),C)$ be a normalized cuspidal Hecke eigen-new-form. Let $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$ be the Galois representation associated to f, and let H be the number field cut out by $\mathrm{Ad}^0\rho_f$. We let*

$$U_f := \mathrm{Hom}_{\overline{\mathbb{Q}}[\mathrm{Gal}(H/\mathbb{Q})]}(\mathrm{Ad}^0\rho_f, \mathcal{O}_H^\times \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}),$$

which is a 1-dimensional $\overline{\mathbb{Q}}$-vector space. Let $U_f^\vee := \mathrm{Hom}_{\overline{\mathbb{Q}}}(U_f,\overline{\mathbb{Q}})$. Consider the homomorphism $U_f^\vee \to \mathrm{Ext}^1_{C[1+p\mathbb{Z}_p]}(C,C)$ given by $\varphi \mapsto \frac{\varphi(u_f)}{\log_p(u_f)}\log_p$ for $u_f \in U_f$ (the definition is independent of the choice of u_f), where $\log_p \in \mathrm{Ext}^1_{C[1+p\mathbb{Z}_p]}(C,C)$ is the element corresponding to the p-adic logarithm $\log_p \in \mathrm{Hom}(1+p\mathbb{Z}_p, C)$. Then, the derived diamond action

$$\mathrm{Ext}^1_{C[1+p\mathbb{Z}_p]}(C,C) \times H^0(X_1(Np)_C,\omega) \to H^1(X_1(Np)_C,\omega)$$

restricts to

$$U_f^\vee \times H^0(X_1(Np)_{\overline{\mathbb{Q}}},\omega) \to H^1(X_1(Np)_{\overline{\mathbb{Q}}},\omega)$$

along the homomorphism $U_f^\vee \to \mathrm{Ext}^1_{C[1+p\mathbb{Z}_p]}(C,C)$.

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