SUPERRIGIDITY WITH MAPPING CLASS GROUP TARGET VIA JORDAN DECOMPOSITION

GYUJIN OH

Abstract. In this short note, we reprove the super-rigidity of homomorphisms from $\mathrm{SL}_n(\mathbb{Z})$ to a mapping class group, a weaker version of [FM98], using Steinberg's algebraic proof of super-rigidity in [Ste85].

Contents

The goal of this note is to prove the "super-rigidity of $\mathrm{SL}_n(\mathbb{Z})$ with mapping class group target" in the sense of [FM98] (see Theorem 2.1 for a precise statement that we prove) using a very soft argument utilizing the interplay between the Jordan decomposition in $\mathrm{SL}_n(\mathbb{Z})$ and the Nielsen–Thurston decomposition in the mapping class group. Even though the said super-rigidity result can now be proved in various ways, we believe the method of proof presented in this note is still interesting.

1. Lemmas on Nielsen–Thurston decomposition

We first prove some lemmas on Nielsen–Thurston decomposition, emphasizing the viewpoint that this is the mapping class group analogue of Jordan decomposition. We freely use the definitions of [FM12].

Proposition 1.1 ([FM12, Corollary 13.3]). *Let $f \in \mathrm{Mod}(S_{g,n})$ be a mapping class, and let $\{c_1, \cdots, c_m\}$ be its canonical reduction system. Let $R_1, \cdots, R_m$ be tubular neighborhoods of $c_1, \cdots, c_m$, respectively, and $R_{m+1}, \cdots, R_{m+p}$ be the closures of the connected components of $S_{g,n} - \cup_{i=1}^m R_i$. Then, there is a representative ϕ of f that permutes R_i. Therefore, there is a positive integer $k > 0$ such that $\phi^k(R_i) = R_i$ for all i.*

Let $\eta_i : \mathrm{Mod}(R_i) \to \mathrm{Mod}(S_{g,n})$ be the natural map. Then, for $1 \le i \le m$, there is a power of Dehn twist $f_i \in \mathrm{Mod}(R_i)$, and for $m < i \le m+p$, there is a pseudo-Anosov element $f_i \in \mathrm{Mod}(R_i)$, such that

$$f^k = \prod_{i=1}^{m+p} \eta_i(f_i).$$

Theorem 1.2 (Ivanov, [Iva92, Corollary 1.8]). *For $f \in \mathrm{Mod}(S_{g,n})[m]$ with $m \ge 3$, the integer k in Proposition 1.1 can be taken to be 1.*

Note that $\mathrm{Mod}(S_{g,n})$ acts linearly on $H^1(S_{g,n}, \mathbb{Z})$, and $f \in \mathrm{Mod}(S_{g,n})[m]$ means that f acts trivially on $H^1(S_{g,n}, \mathbb{Z}/m\mathbb{Z})$.

Definition 1.3. For $f \in \mathrm{Mod}(S_{g,n})[m]$, $m \geq 3$, we define f_s, f_u as

$$f_u = \prod_{i=1}^{m} \eta_i(f_i),$$

$$f_s = \prod_{i=m+1}^{m+p} \eta_i(f_i),$$

using the notation of Proposition 1.1.

By definition, f_u is a multitwist (i.e. a product of powers of mutually commuting Dehn twists), and $f_u f_s = f_s f_u$. The notation mimics that of the Jordan decomposition.

Lemma 1.4. *Let $m \geq 3$. If $x, y \in \mathrm{Mod}(S_{g,n})[m]$ commute with each other (i.e. $xy = yx$), then x_u, y_u, x_s, y_s all commute with each other. Furthermore, $(xy)_u = x_u y_u = y_u x_u$, and $(xy)_s = x_s y_s = y_s x_s$.*

Proof. Since $xyx^{-1} = y$, $x\,\mathrm{CRS}(y) = \mathrm{CRS}(xyx^{-1}) = \mathrm{CRS}(y)$. Therefore, there is large $N \gg 0$ such that each curve $c \in \mathrm{CRS}(y)$ is fixed by x^N. This implies that c is in a maximal reduction system of x^N, so c does not intersect (up to isotopy) any curve in $\mathrm{CRS}(x)$.

Let

$$\mathrm{CRS}(x) = \{c_1, \cdots, c_m, c_{m+1}, \cdots, c_{m+a}\}, \quad \mathrm{CRS}(y) = \{c_1, \cdots, c_m, c'_{m+1}, \cdots, c'_{m+b}\},$$

where $\mathrm{CRS}(x) \cap \mathrm{CRS}(y) = \{c_1, \cdots, c_m\}$ is the set of common isotopy classes. For notational simplicity, we denote $c'_i = c_i$ for $1 \leq i \leq m$. Then,

$$x_u = \prod_{i=1}^{m+a} T_{c_i}^{n_i}, \quad y_u = \prod_{i=1}^{m+b} T_{c'_i}^{n'_i},$$

where T_c is the Dehn twist along c. The argument in the previous paragraph shows that any pair of curves in $S := \{c_1, \cdots, c_{m+a}, c'_{m+1}, \cdots, c'_{m+b}\}$ does not intersect (up to isotopy). Therefore, the Dehn twists $T_{c_1}, \cdots, T_{c_{m+a}}, T_{c'_{m+1}}, \cdots, T_{c'_{m+b}}$ all commute with each other. This implies that $x_u y_u = y_u x_u$.

Let us now fix more notations. Let R_i (R'_i, respectively) be a tubular neighborhood of c_i (c'_i, respectively), and let $R_{m+a+1}, \cdots, R_{m+p}$ ($R'_{m+b+1}, \cdots, R'_{m+q}$, respectively) be the closures of the connected components of $S_{g,n} - \cup_{i=1}^{m+a} R_i$ ($S_{g,n} - \cup_{i=1}^{m+b} R'_i$, respectively). For $1 \leq i \leq m+p$ ($1 \leq i \leq m+q$, respectively), let $\eta_i : \mathrm{Mod}(R_i) \to \mathrm{Mod}(S_{g,n})$ ($\eta'_i : \mathrm{Mod}(R'_i) \to \mathrm{Mod}(S_{g,n})$, respectively) be the natural map. Let

$$x_s = \prod_{i=m+a+1}^{m+p} \eta_i(x_i) \qquad \left(y_s = \prod_{i=m+b+1}^{m+q} \eta'_i(y_i), \text{ respectively} \right).$$

Here, the element $x_i \in \mathrm{Mod}(R_i)$ for $m+a+1 \leq i \leq m+p$ ($y_i \in \mathrm{Mod}(R'_i)$ for $m+b+1 \leq i \leq m+q$, respectively), is either the identity or pseudo-Anosov.

As x fixes both $\mathrm{CRS}(x)$ and $\mathrm{CRS}(y)$, S is a reduction system for x, and also for y by symmetry. Therefore, for $m+a+1 \leq i \leq m+p$, x_i fixes the set

$$S_i := \{c'_j \in \mathrm{CRS}(y) \setminus \mathrm{CRS}(x) \mid c'_j \subset R_i\}.$$

Therefore, by the Nielsen–Thurston classification, if $S_i \neq \emptyset$, x_i cannot be pseudo-Anosov, so x_i has to be the identity. This immediately implies that x_s and y_u have different supports, so they commute, and similarly for x_u and y_s.

We now prove that $x_s y_s = y_s x_s$. Let

$$\operatorname{Supp}(x_s) := \bigcup_{m+a+1 \leq i \leq m+p,\ x_i \neq \mathrm{id}} R_i, \qquad \operatorname{Supp}(y_s) := \bigcup_{m+b+1 \leq i \leq m+q,\ y_i \neq \mathrm{id}} R_i'.$$

We claim that

$$\operatorname{Supp}(x_s) \cap \operatorname{Supp}(y_s) = \bigcup_{R_i = R_j',\ m+a+1 \leq i \leq m+p,\ m+b+1 \leq j \leq m+q} R_i.$$

Suppose that there are $m+a+1 \leq i \leq m+p$ and $m+b+1 \leq j \leq m+q$ such that $R_i \cap R_j' \neq \emptyset$, $x_i \neq \mathrm{id}$ and $y_j \neq \mathrm{id}$. As $x_i \neq \mathrm{id}$, none of the boundary curves of R_j' is contained in R_i. This implies that $R_i \subset R_j'$. By symmetry, $R_i \supset R_j'$, so this impies that $R_i = R_j'$. In this case, all the factors in the canonical decompositions in x or y other than x_i and y_j have the supports disjoint from $R_i = R_j'$. Thus, $xy = yx$ implies that $x_i y_j = y_j x_i$. Outside $\operatorname{Supp}(x_s) \cap \operatorname{Supp}(y_s)$, either x_i or y_j is the identity, so $x_i y_j = y_j x_i$. All in all, this implies that $x_s y_s = y_s x_s$.

Now note that, over $R_i = R_j'$, $x_i y_j$ is pseudo-Anosov; this is because $x_i y_j = y_j x_i$ is in the centralizer of x_i. Therefore, $x_u y_u$ is a multitwist, and $x_s y_s$ is the product of pseudo-Anosov elements over subsurfaces. Thus, $xy = (x_u y_u)(x_s y_s)$ is the decomposition of $xy = (xy)_u (xy)_s$. ☐

The following is analogous to [Ste85, (1), pg. 340].

Lemma 1.5. *Let $m \geq 3$. If $x, y, z \in \operatorname{Mod}(S_{g,n})[m]$ are such that $[x, y] = z$ (here $[x, y] = xyx^{-1}y^{-1}$ is the commutator) and that z commutes with x, y. Then, the following holds.*

(1) $[x_u, y] = z_u$.
(2) $[x, y_s] = z_s$.
(3) $[x_u, y_s] = 1$.
(4) $z_s^N = 1$ *for some* N.

Proof.

(1) Note that $xyx^{-1} = zy = yz$, so $y^{-1}xy = zx$. Since $\operatorname{CRS}(y^{-1}xy) = y^{-1}\operatorname{CRS}(x)$, $(y^{-1}xy)_u = y^{-1}x_u y$, which implies that $y^{-1}x_u y = (xz)_u = x_u z_u$ by Lemma 1.4.
(2) Note that $xyx^{-1} = zy$. Since $\operatorname{CRS}(xyx^{-1}) = x\operatorname{CRS}(y)$, $(xyx^{-1})_s = xy_s x^{-1}$, which implies that $xy_s x^{-1} = (zy)_s = z_s y_s$ by Lemma 1.4.
(3) Since z_u commutes with both x_u and y, one can apply (2) to (1) and obtain (3).
(4) This will follow if we show that $x^N y_s x^{-N} = y_s$ for some N. Firstly, as x permutes $\operatorname{CRS}(y)$, x^N fixes $\operatorname{CRS}(y)$ elementwise for some N. Thus, without loss of generality, we may assume that x fixes $\operatorname{CRS}(y)$ elementwise. As y, z_s commute with each other by Lemma 1.4, we may also assume without loss of generality that z_s fixes $\operatorname{CRS}(y)$ elementwise.

    Now let $\operatorname{CRS}(y) = \{c_1, \cdots, c_m\}$, $R_1, \cdots, R_m$ be tubular neighborhoods of $c_1, \cdots, c_m$, respectively, and let $R_{m+1}, \cdots, R_{m+p}$ be the closures of the connected components of $S_{g,n} - \cup_{i=1}^m R_i$. Let $y_s = \prod_{i=m+1}^{m+p} \eta_i(y_i)$, where $y_i \in \operatorname{Mod}(R_i)$ and $\eta_i : \operatorname{Mod}(R_i) \to \operatorname{Mod}(S_{g,n})$. We may also assume that x fixes each of $R_{m+1}, \cdots, R_{m+p}$. For $m+1 \leq i \leq m+p$, let $x_i \in \operatorname{Mod}(R_i)$ be the restriction of x to R_i, and $z_i \in \operatorname{Mod}(R_i)$ be the restriction of z to R_i. Then, y_i and z_i commute with each other, and $x_i y_i x_i^{-1} = y_i z_i$. We claim that

$x_i^{l_i} y_i x_i^{-l_i} = y_i$ for some $l_i > 0$. If the claim is true, then taking N to be the lcm of all l_i's for $m+1 \le i \le m+p$ and $z_i \neq \mathrm{id}$ will give the desired result.

If z_i is pseudo-Anosov, by the result of McCarthy [McC82, Theorem 1], a power of y_i is a power of z_i. Thus, taking a large enough power of x_i, one has $x_i^{n_i} y_i x_i^{-n_i} = y_i^{m_i}$ for some $n_i > 0, m_i \neq 0$. By Lemma 1 of *op. cit.*, $m_i = \pm 1$, so we may take $l_i = 2n_i$ and the Claim is indeed true.

□

2. Proof of superrigidity of $\mathrm{SL}_n(\mathbb{Z})$ with mapping class group target

Theorem 2.1. *Let $m, n \ge 3$ and $\varphi : \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Mod}(S)[m]$ be a homomorphism. Then, φ has finite image.*

Proof. By [Ste85], $\mathrm{SL}_n(\mathbb{Z})$ is generated by $x_{ij} = I + E_{ij}$, $i \neq j$, subject to the following relations.

- $[x_{ij}, x_{kl}] = x_{il}$ if $i \neq l$, $j = k$,
- $[x_{ij}, x_{kl}] = 1$, if $i \neq l$, $j \neq k$,
- $(x_{12} x_{21}^{-1} x_{12})^4 = 1$.

Let $\varphi(x_{ij}) =: a_{ij} \in \mathrm{Mod}(S)[m]$. Let $b_{ij} = (a_{ij})_u$. Then, the b_{ij}'s satisfy the relations

- $[b_{ij}, b_{kl}] = b_{il}$ if $i \neq l$, $j = k$,
- $[b_{ij}, b_{kl}] = 1$ if $i \neq l$, $j \neq k$.

We claim the following.

Claim. For all $i \neq j$, $b_{ij} = 1$.

To prove the Claim, without loss of generality, after reindexing, it suffices to prove $b_{32} = 1$. Note that we have the relation

$$b_{31} b_{12} b_{31}^{-1} = b_{12} b_{32} = b_{32} b_{12}.$$

Let $b_{12} = \prod_{i=1}^s T_{c_i}^{p_i}$, where T_{c_i} is the Dehn twist along c_i. Then, $b_{31} b_{12} b_{31}^{-1} = \prod_{i=1}^s T_{b_{31}(c_i)}^{p_i}$. Thus, $b_{32} = b_{31} b_{12} b_{31}^{-1} b_{12}^{-1} = \prod_{i=1}^s T_{b_{31}(c_i)}^{p_i} \prod_{i=1}^s T_{c_i}^{-p_i}$ commutes with both b_{12} and b_{31}. Note that this expression expresses b_{32} as a multitwist, as $b_{31} b_{12} b_{31}^{-1}$ and b_{12}^{-1} commute with each other. Thus, for any $1 \le i, j \le s$, $i(b_{31}(c_i), c_j) = 0$.

Let $b_{31} = \prod_{i=1}^t T_{d_i}^{q_i}$. If $b_{32} \neq 1$, then there is $1 \le x_1 \le s$, $1 \le y_1 \le t$ such that $i(c_{x_1}, d_{y_1}) \neq 0$. As b_{31} and $b_{32} = \prod_{i=1}^s T_{b_{31}(c_i)}^{p_i} \prod_{i=1}^s T_{c_i}^{-p_i}$ commute, this implies that there is $1 \le x_2 \le s$ such that $b_{31}(c_{x_2}) = c_{x_1}$ and $p_{x_2} = p_{x_1}$. Note that by the definition of x_2, $x_2 \neq x_1$. This implies that c_{x_2} is not fixed by b_{31}, or there is some $1 \le y_2 \le t$ such that $i(c_{x_2}, d_{y_2}) \neq 0$. We can thus inductively define $1 \le x_\alpha \le s$, $\alpha = 1, 2, 3, \cdots$, such that $b_{31}(c_{x_\alpha}) = c_{x_{\alpha-1}}$. Therefore, there is some $N \gg 0$ such that $b_{31}^N(c_{x_1}) = c_{x_1}$, which is a contradiction as $\mathrm{CRS}(b_{31}^N) = \mathrm{CRS}(b_{31})$. Therefore, $b_{32} = 1$, which proves the Claim.

From the Claim, we now know that $a_{ij} = (a_{ij})_s$ for all $i \neq j$. By Lemma 1.5(4), for each $i \neq j$, a_{ij} is finite order. This implies that there is a large $N \gg 0$ such that $\varphi(x_{ij})^N = 1$ for all $i \neq j$. This implies that $\ker(\varphi)$ contains x_{ij}^N for all N. As the congruence subgroup $\Gamma(N) \subset \mathrm{SL}_n(\mathbb{Z})$ is the normal subgroup generated by x_{ij}^N's, $\ker(\varphi) \supset \Gamma(N)$, which implies that φ is of finite image. □

References

[FM98] Benson Farb and Howard Masur. Superrigidity and mapping class groups. *Topology*, 37(6):1169–1176, 1998.

[FM12] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[Iva92] Nikolai V. Ivanov. *Subgroups of Teichmüller modular groups*, volume 115 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992. Translated from the Russian by E. J. F. Primrose and revised by the author.

[McC82] John McCarthy. Normalizers and centralizers of pseudo-Anosov mapping classes, 1982.

[Ste85] Robert Steinberg. Some consequences of the elementary relations in SL_n. In *Finite groups—coming of age (Montreal, Que., 1982)*, volume 45 of *Contemp. Math.*, pages 335–350. Amer. Math. Soc., Providence, RI, 1985.

Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027

E-mail address: gyujinoh@math.columbia.edu