HW #1

ALGEBRAIC NUMBER THEORY, GU4043; INSTRUCTOR: GYUJIN OH

Reading Homework. Try Exercises 1.1 and 1.4 in the textbook. Read their solutions in the back.

Question 1. For an R-algebra S, we say that S is **integral** over R if every element $s \in S$ is integral over R.

Suppose that S is an R-algebra and T is an S-algebra. Show that if S is integral as an R-algebra and if T is integral as an S-algebra, then T is integral as an R-algebra.

Question 2. Let K be a number field, and L be a subfield. For $a \in \mathcal{O}_K$, let $p_a(X) \in L[X]$ be the minimal polynomial of a over L. Show that $p_a(X) \in \mathcal{O}_L[X]$.

Question 3. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, which is a number field with $[K : \mathbb{Q}] = 4$. Show that

$$\mathcal{O}_K = \left\{ a + \frac{b\sqrt{2}}{2} + c\sqrt{3} + \frac{d\sqrt{6}}{2} \ : \ a, b, c, d \in \mathbb{Z}, b \equiv d \pmod 2 \right\}.$$

Hint. One way to simplify the calculation is to use Question 2, using the subfields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ of K.