CONGRUENCES OF EISENSTEIN SERIES AND THE BLOCH-KATO CONJECTURE

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1. Introduction

For an elliptic curve $E/\mathbb{Q}$, the Birch-Swinnerton-Dyer conjecture asserts that

$$\mathrm{rk}\, E(\mathbb{Q}) = \mathrm{ord}_{s=1} L(E, s),$$

where the $L(E, s)$ is known to have good analytic properties thanks to the modularity theorem. We know a few things towards BSD, but mostly about the analytic rank ≤ 1 case: $r_{\mathrm{an}} \leq 1$ implies $r_E = r_{\mathrm{an}}$, which is a result of Gross-Zagier and Kolyvagin.

Recall how this is proved.

(1) Choose an imaginary quadratic field K such that $\mathrm{ord}_{s=1} L(E/K, s) = 1$. Note that $L(E/K, s) = L(E, s)L(E^K, s)$, where K is the twist of E by the quadratic character of K. This also uses modularity.

(2) The Gross-Zagier formula tells something about $L'(E/K,1)$, namely

$$L'(E/K,1) \sim \langle y_K, y_K\rangle_{\mathrm{NT}},$$

for $y_K \in E(K)$, the so-called Heegner point.

The two above points imply that $y_K \in E(K)$ is a non-torsion point. This uses the non-degeneracy of the Néron-Tate height pairing.

(3) Kolyvagin's method of Euler systems shows that, if y_K is non-torsion, then $\operatorname{cork} \mathrm{Sel}_{p^\infty}(E/K) = 1$. Recall that we have a fundamental exact sequence

$$0 \longrightarrow E(F) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{\operatorname{rk} E(F)} \longrightarrow \mathrm{Sel}_{p^\infty}(E/F) \longrightarrow Ш(E/F)[p^\infty] \longrightarrow 0,$$

for any number field F. Thus the fundamental exact sequence implies that $Ш(E/K)[p^\infty]$ is finite as well as $\operatorname{rk} E(K) = 1$. Keeping track of Galois action of $\mathrm{Gal}(K/\mathbb{Q})$ we can manage to make this to come from E, not from E^K.

How far can one go from the Gross-Zagier-Kolyvagin argument?

(1) Can E be replaced with an abelian variety A?
(2) $\operatorname{ord}_{s=1} L(E,s) \geq 2$?
(3) Results for $L(f,s)$ with $f \in S_{2k}(\Gamma_0(N))$ for $k \geq 2$?

Some answers:

Check this.

(1) For a weight 2 newform $f \in S_2(\Gamma_0(N))^{\mathrm{new}}$, Shimura constructed a $\mathbb{Q}$-abelian variety A_f of dimension $[\mathbb{Q}(f) : \mathbb{Q}]$ and $\mathrm{End}_{\mathbb{Q}}(A_f) \otimes \mathbb{Q} \supset \mathbb{Q}(f)$ (note $\mathbb{Q}(f)$ is totally real because f is a newform of trivial Nebentype). It has the L-function $L(A_f,s) = \prod_{\sigma:\mathbb{Q}(f)\hookrightarrow\mathbb{R}} L(f^\sigma,s)$. Thus $A_f(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$ has an action of $\mathbb{Q}(f)$, and it turns out that if $r_{\mathrm{an}} = \operatorname{ord}_{s=1} L(f,s) \leq 1$, then $\dim_{\mathbb{Q}(f)} A_f(\mathbb{Q}) \otimes \mathbb{Q} = r_{\mathrm{an}}$ (so that $\operatorname{ord}_{s=1} L(A_f,s) = r_{\mathrm{an}}[\mathbb{Q}(f) : \mathbb{Q}]$). Essentially the same proof as Gross-Zagier-Kolyvagin.
(2) Very little is known.
(3) Take the associated Galois representation $\rho_f : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}(V)$ where V is a 2-dimensional L-vector space, $L \supset \mathbb{Q}(f)$ a finite extension of $\mathbb{Q}_p$, such that $L(\rho_f,s) = L(f,s)$ (Note: we use the geometric convention that $L(V,s) = \prod_\ell \det(1 - \ell^{-s}\,\mathrm{Frob}_\ell|_{V^{I_\ell}})^{-1}$ (at p we use either the L-factor of a compatible ℓ'-adic Galois representation, $\ell' \neq p$, or the L-factor of the Weil-Deligne representation of the local Galois reprsentation at p, coming from the p-adic monodromy theorem), with Frob_ℓ being the geometric Frobenius). Suppose $\operatorname{ord}_{s=k} L(E,s) \leq 1$. We can try to choose an imaginary quadratic field K such that $\operatorname{ord}_{s=k} L(f/K,s) = 1$, where $L(f/K,s) = L(f,s)L(f\otimes\eta_K,s)$ (here we really mean the newform associated to $f \otimes \eta_K$).
    - This is not always possible, when f is a level 1 modular form, as all twists have the same sign.

    If this is possible, then there is an analogue of Heegner point, the Heegner cycle z_K, a codimension k cohomologously trivial Chow cycle of the Kuga-Sato variety, say denoted as KS (canonical nonsingular compactification over $X_0(N)$ of the $(2k-2)$-fold self-product of the universal elliptic curve over $Y_0(N)$). As the cycle class map is sent to 0 (cohomologously trivial!), its image in the absolute étale cohomology lies in the next filtration, whose subquotient is $H^1(K, H^{2k-1}(\mathrm{KS}_{\overline{\mathbb{Q}}}, \mathbb{Q}_p(k)))$ ("Abel-Jacobi map"), and as V is found as a subquotient of $H^{2k-1}(\mathrm{KS}_{\overline{\mathbb{Q}}}, \mathbb{Q}_p(k))$, we obtain a cycle $c_K \in H^1_f(K, V(k))$. Show-Wu Zhang proved that $L'(f/K,k) = \langle z_K, z_K\rangle_{\mathrm{BB}}$, so the analytic rank 1 implies that z_K is nonzero in $\mathrm{CH}^k(\mathrm{KS})_0 \otimes \mathbb{Q}$. Also, Nekovar proved that, if c_K is nonzero, then $\dim_L H^1_f(K, V(k)) = 1$. The problem is that we do not know $z_K \neq 0$ implies $c_K \neq 0$ (tied to the standard conjectures).

Question to think about: is there a holomorphic eigenform of weight > 2 whose L-function has a zero at the central value of order ≥ 2? This would give an effective version of Gauss' class number formula (argument by Dorian Goldfeld), but we do not know partly because we do not know how to detect $z_K = 0$, not just $c_K = 0$.

The rest of the course will explain many of the details of a proof of a result towards (2) + (3).

Theorem 1.1 (Skinner-Urban). *Let $f \in S_{2k}(\Gamma_0(N))^{\mathrm{new}}$ be a newform, and $p \nmid N$.*

(1) *If $L(f,k) = 0$, then $\dim_L H^1_f(\mathbb{Q}, V(k)) > 0$.*
(2) *If $L(f,k) = 0$ and $\varepsilon(f)$, the root number of f (the functional equation is $L(f,s) = \varepsilon(f)L(f, 2k - s)$), is $+1$ (so that $\mathrm{ord}_{s=k} L(f,s)$ is even), then $\dim_L H^1_f(\mathbb{Q}, V(k)) \geq 2$.*

Remark 1.1. Some cases of Theorem 1.1 can be deduced from other results.

- If f is ordinary at p, p is odd, $p - 1 \mid 2k - 2$ (+some hypothesis on residual Galois representation), then (1) is a consequence of the Iwasawa Main Conjecture for f, and (2) also is, after combining with Nekovar's work on the "parity conjecture" (i.e. the analytic rank is congruent mod 2 to the corank of the Selmer group, which here is $\dim_L H^1_f(\mathbb{Q}, V(k))$).

Theorem 1.1 is a special case of a more general theorem of the form

$$L(V,0) = 0 \Longrightarrow \dim H^1_f(K,V) > 0,$$

for V a Galois representation associated to some cuspidal automorphic representation of a unitary group over $\mathbb{Q}$.

Example 1.1. Let's take the simplest example: the Riemann zeta function. The trivial simple zeros at $-2m$, $2m \geq 2$, correspond to $\dim_{\mathbb{Q}_p} H^1_f(\mathbb{Q}, \mathbb{Q}_p(2m+1)) = H^1_{\mathrm{rel}}(\mathbb{Q}, \mathbb{Q}_p(2m+1))$, the classes trivial at all $\ell \neq p$.

Automorphic L-functions arise as constant terms of Eisenstein series. Let

$$G_{2k}(z;s) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} j(\gamma, z)^{-2k} |j(\gamma, z)|^{-s},$$

where $\Gamma_\infty = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\}$. If $\mathrm{Re}(s+2k) \gg 0$, this series converges absolutely and defines a holomorphic function in s. The general theory of Eisenstein series (by Selberg, Langlands, $\cdots$) says that $G_{2k}(z;s)$ has a meromorphic continuation in s. It also has a Fourier expansion

$$G_{2k}(z;s) = \sum_{n \in \mathbb{Z}} c_n(y;s) e(nx),$$

with

$$c_0(y;s) = 1 + (-1)^k y^{1-(2k+s)} \frac{\Gamma\left(1 - \frac{s+2k}{2}\right)\Gamma\left(\frac{s+2k}{2}\right)^2 \zeta(2-(s+2k))}{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+2k\right)\Gamma\left(\frac{1-(s+2k)}{2}\right)\zeta(1-(s+2k))}.$$

By the theory of constant terms, $G_{2k}(z;s)$ is holomorphic at $s = s_0$ iff $c_0(y;s)$ is, and $G_{2k}(z;s)$ is holomorphic in z iff $c_0(y;s)$ is.

At $s = 0$, if $2k \geq 4$, then there is $\zeta(2-(s+2k))$ factor in the numerator which gives zero, so that $c_0(y,0) = 1$, so $c_0(y,s)$ is holomorphic in z at $s = 0$. If $2k = 2$, then there is no zero nor pole, so $c_0(y,s) = 1 + (*)y^{-1}$, which is not holomorphic. This explains the connection between simple trivial zeros at $-2k \leq -2$ and existence of **holomorphic** Eisenstein series $G_{2k+2}(z;0)$ of weight $2k+2$ and of level 1. This Eisenstein series is related to the usual classical Eisenstein series by

$$\frac{\zeta(1-2k)}{2} G_{2k}(z;0) =: E_{2k}(z) = \frac{\zeta(1-2k)}{2} + \sum_{n=1}^{\infty} \sigma_{2n-1}(n) q^n,$$

where $q = e^{2\pi i z}$.

The associated Galois representation to Eisenstein series $G_{2k}(z;0)$ is of form $\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon_{\mathrm{cyc}}^{1-2k} \end{pmatrix}$ ($\because$ we chose to work with geometric Frobenius). Take the critical p-stabilization $G_{2k}^*(z) = G_{2k}(z) - G_{2k}(pz)$, then as this is **holomorphic**, this can be put into a p-adic family of eigenforms that are generally cuspidal (Coleman family). This gives a non-split extension by degenerating the Galois representations of the family, which is a nonzero element in $H_f^1(\mathbb{Q}, \mathbb{Q}_p(2k-1))$, i.e. a nonsplit extension $\begin{pmatrix} 1 & * \\ 0 & \varepsilon_{\mathrm{cyc}}^{1-2k} \end{pmatrix}$ that are split at all primes $\ell \neq p$.

What about $2k = 2$? Indeed y^{-1} is not a holomorphic function on the upper half plane, but it is **nearly holomorphic** (in the sense of Shimura). Thus the critical p-stabilization of $E_{2k}(z)$, $E_{2k}^*(z) = E_{2k}(z) - E_{2k}(pz) = \sum_{n=1}^{\infty} \sigma_{2k-1}^*(n) q^n$ where $\sigma_{2k-1}^*(n) = p^{N(2k-1)} \sum_{d|n, p\nmid d} d^{2k-1}$ when $p^N \| n$, is not homolomorphic but just nearly holomorphic. On the other hand, the other p-stabilization, the **ordinary stabilization**, $E_{2k}^{\mathrm{ord}}(z) = E_{2k}(z) - p^{2k-1}E_{2k}(pz)$, is holomorphic even at $2k = 2$.

The critical p-stabilization $E_{2k}^*(z)$, having vanishing constant term on the level of q-expansions, defines a p-adic **cusp** form, i.e. p-adic limit of classical cusp forms, even for $2k = 2$ (although it is not an overconvergent modular form if $2k = 2$).

Coleman family gives an interpretation of overconvergent p-adic modular forms as "nice" analytic family of eigenforms. The variable we would be varying is the weight, seeing as varying over ($\overline{\mathbb{Q}}_p$-points of) a ball

$$B_r(\overline{\mathbb{Q}}_p) = \{\alpha \in \overline{\mathbb{Q}}_p \mid |\alpha - 1|_p \leq p^{-r}\}.$$

Note that B_r can be also thought as a rigid analytic space. Let $A_r = \mathcal{O}(B_r) \cong \overline{\mathbb{Q}}_p\langle p^{-r}T\rangle$ ($T(\alpha) = \alpha - 1$).

Theorem 1.2 (Coleman family). *For some $r \gg 0$, there exists a finite normal A_r-algebra R and a normalized (i.e. $a_1 = 1$) formal q-expansion $F = \sum_{n=1}^{\infty} a_n q^n \in qR[[q]]$ such that the following are satisfied.*

1. *For some $\phi_0 \in \mathrm{Hom}_{\mathrm{cont}}(R, \overline{\mathbb{Q}}_p)$ such that $\phi_0(T) = 0$, and its stabilization at ϕ_0 gives the q-expansion of E_{2k}^*.*
2. *For any $m \gg 0$, $(p-1) \mid m$ and any $\phi \in \mathrm{Hom}_{\mathrm{cont}}(R, \overline{\mathbb{Q}}_p)$ such that $\phi(1+T) = (1+p)^m$, the stabilization of F at ϕ is a classical eigenform of weight $2k+m$, level p, trivial Nebentype with slope $2k-1$ (i.e. $|\phi(a_p)|_p = p^{1-2k}$).*

Note that if $m \gg 0$, then F_ϕ is a cuspform, because an Eisenstein series of weight $2k'$ and level 1 can only have slopes $\infty, 0, 2k'-1$ (think about what can be achieved for divisor sums). Thus if

$1 - 2k \neq 1 - 2k'$, the stabilization of F at weight $2k' = 2k + m$ cannot be an Eisenstein series (i.e. it is cuspidal).

Each such F_ϕ has an ssociated 2-dimensional p-adic Galois representation $\rho_\phi : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(L_\phi)$, unramified away from p, such that $\operatorname{tr}\rho_\phi(\mathrm{Frob}_\ell) = \phi(a_\ell)$ for all $\ell \neq p$, where $L_\phi = \phi(R) \subset \overline{\mathbb{Q}}_p$ is a finite extension of $\mathbb{Q}_p$. How do we package these into a family? There is no canonical choice of lattices, but we can use **pseudorepresentations**. That all $\ker\phi$'s form a Zariski dense subset in R implies that the limit of traces of Frobenii can be taken in R, and as all ρ_ϕ factors through $\mathrm{Gal}(\mathbb{Q}_{\{p,\infty\}}/\mathbb{Q})$, we can use Chebotarev density to define a pseudorepresentation $T : G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_{\{p,\infty\}}/\mathbb{Q}) \longrightarrow R$ such that $\phi \circ T = \operatorname{tr}\rho_\phi$. The theory of pseudorepresentations (that pseudorepresentations over a field are traces of genuine representations) gives us a continuous Galois representation $\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_F(V)$ for a 2-dimensional F-vector space, where $F = \mathrm{Frac}(R)$, such that $\operatorname{tr}\rho = T$.

Remark 1.2. The pseudorepresentation T does not always yield a Galois representation into $\mathrm{GL}_2(R)$ unless you have extra assumption on T (e.g. residually irreducible).

With respect to some basis v_1, v_2 of V, we can write ρ as

$$\rho(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix},$$

where $a_\sigma, b_\sigma, c_\sigma, d_\sigma \in F$, for $\sigma \in R[G_{\mathbb{Q}}]$. Note that as specialization at ϕ_0 gives a reducible pseudorerpesentation, $T \bmod \mathfrak{p} = 1 + \varepsilon_{\mathrm{cyc}}^{1-2k}$, where we denote $\mathfrak{p} := \ker\phi_0$, we can choose $\sigma_0 \in I_p$ such that $\varepsilon_{\mathrm{cyc}}^{1-2k}(\sigma_0) \neq 1$.

We can then diagonalize the specific element $\rho(\sigma_0)$ (after possibly enlarging R and m), so that WLOG we can assume that

$$\rho(\sigma_0) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

such that $\phi_0(\alpha) = 1$ and $\phi_0(\beta) = \varepsilon_{\mathrm{cyc}}(\sigma_0)^{1-2k}$. Define the idempotents $\varepsilon_1 = \frac{1}{\alpha-\beta}(\sigma_0 - \beta), \varepsilon_2 = \frac{1}{\beta-\alpha}(\sigma_0 - \alpha) \in R[G_{\mathbb{Q}}]$ (also after possibly enlarging R and m; take the neighborhood where $\alpha - \beta$ is invertible, and this is possible without ruining desired conditions like normality because R is 1-dimensional). These are idempotents in a sense that

$$\rho(\varepsilon_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \rho(\varepsilon_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

- $\operatorname{tr}\rho(\varepsilon_1\sigma) = a_\sigma, \operatorname{tr}\rho(\varepsilon_2\sigma) \in R$,
- and thereby $b_\sigma c_\tau = a_{\sigma\tau} - a_\sigma a_\tau \in R$,
- $a_\sigma (\bmod \mathfrak{p}) = 1(\sigma)$,
- $d_\sigma (\bmod \mathfrak{p}) = \varepsilon_{\mathrm{cyc}}^{1-2k}(\sigma)$,

for all $\sigma, \tau \in R[G_{\mathbb{Q}}]$ (by manipulating with these relations).

Note that b, c take some nonzero values, because otherwise it violates the cuspidality of F_ϕ (i.e. ρ_{F_ϕ} being irreducible). Take B to be the $R_{\mathfrak{p}}$-module generated by b_σ's in F. This is a nonzero fractional ideal of $R_{\mathfrak{p}}$ ($\mathfrak{p}$ is height 1 so the localization is in fact a PID). This is a fractional ideal (i.e. not everything) because $G_{\mathbb{Q}}$ is compact. Let $M_1 = Bv_1, M_2 = R_{\mathfrak{p}}v_2, M = M_1 \oplus M_2 \subset V$. By irreducibility $M = R[G_{\mathbb{Q}}]v_2$, and as $R_{\mathfrak{p}}$ is PID, M is a free $R_{\mathfrak{p}}$-module of rank 2. For $m_1 \in M_1$, it is of form bv_1 for some $b \in B$, and

$$\rho(\sigma)m_1 = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \begin{pmatrix} b \\ 0 \end{pmatrix} = \begin{pmatrix} a_\sigma b \\ c_\sigma b \end{pmatrix} = a_\sigma m_1 + c_\sigma b v_2.$$

As $c_\sigma b \in \mathfrak{p}R_{\mathfrak{p}}$, and $a_\sigma \equiv 1(\sigma) \bmod \mathfrak{p}$, we have $\rho(\sigma)m_1 \in 1(\sigma)m_1 + \mathfrak{p}M$. In other words, $\overline{M}_1 = M_1/\mathfrak{p}M_1$ is a $G_{\mathbb{Q}}$-fixed line in $\overline{M} = M/\mathfrak{p}M$. It sits inside an exact sequence

$$0 \longrightarrow \overline{M}_1 \longrightarrow \overline{M} \longrightarrow \overline{M}/\overline{M}_1 = M_2/\mathfrak{p}M_2 = \overline{M}_2 \longrightarrow 0.$$

As a Galois module, $\overline{M}_2 \cong E(1-2k)$, where E is the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$, as

$$\rho(\sigma)v_2 = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in d_\sigma v_2 + M_1,$$

or mod $\mathfrak{p}$, $\rho(\sigma)v_2 \in \varepsilon_{\mathrm{cyc}}^{1-2k}(\sigma)v_2 + \mathfrak{p}$. Thus we get an exact sequence of Galois modules

$$0 \longrightarrow E \longrightarrow \overline{M} \longrightarrow E(1-2k) \longrightarrow 0.$$

As v_2 generates M (and $\overline{M}$), there is no way that this could be split. Taking a twist, we get a nonzero extension

$$0 \longrightarrow E(2k-1) \longrightarrow \overline{M}(2k-1) \longrightarrow E \longrightarrow 0.$$

This is unramified at places away from p because we already had trivial situations away from p. This thus gives a nonzero element in $H^1_f(\mathbb{Q}, E(2k-1))$.

Remark 1.3. There is no condition at p because $H^1(\mathbb{Q}_p, \mathbb{Q}_p(n)) = H^1_f(\mathbb{Q}_p, \mathbb{Q}_p(n))$ for $n > 1$ (or $2k-1 > 1$). This breaks down as $\dim_{\mathbb{Q}_p} H^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) = 2$ whereas $\dim_{\mathbb{Q}_p} H^1_f(\mathbb{Q}_p, \mathbb{Q}_p(1)) = 1$. In general cases one analyzes how crystalline periods vary in a family.

2. Bloch-Kato conjecture

Let K be a number field and $L/\mathbb{Q}_p$ be a finite extension. Let V be a finite dimensional L-vector space on which G_K acts continuously and L-linearly, namely a continuous homomorphism $\rho : G_K \longrightarrow \mathrm{Aut}_L(V)$. We assume the following conditions on ρ.

- (geom) ρ is geometric, which means it is unramified away from finitely many places and $\rho_{G_{K_v}}$ is de Rham for all $v \mid p$.
- (reg) For all $v \mid p$, the Hodge-Tate weights of $\rho|_{G_{K_v}}$ are regular, i.e. occur with multiplicity one.

If V is (absolutely) irreducible, the Fontaine-Mazur conjecture is that V occurs in the étale cohomology of a proper smooth variety X/K. Also it will follow from Langlands conjectures that V is associated with a cuspidal automorphic representation of some reductive group; by functoriality also expects that one can find one from $\mathrm{GL}_{\dim V, K}$.

Choose an embedding $L \hookrightarrow \mathbb{C}$ and let q_v be the number of residue field at v. We define $L(V,s) = \prod_v P_v(q_v^{-s})^{-1}$, where $P_v(X) \in L[X]$ is defined as

$$P_v(X) = \begin{cases} \det(1 - X\,\mathrm{Frob}_v \mid_{V^{I_v}}) & v \nmid p \\ \det(1 - X\phi_v|_{D_{\mathrm{cris}}(V)}) & v|p \end{cases}$$

where ϕ_v is the crystalline Frobenius.

Remark 2.1. As $D_{\mathrm{cris}}(V)$ technically is a vector space over K_v^0, one needs to a little more extra work. We will only face the cases of $K_v = \mathbb{Q}_p$, so there is no ambiguity here.

By Fontaine-Mazur + Langlands, we expect that the Euler product has a half-plane of absolute convergence and has a meromorphic continuation to the whole $\mathbb{C}$. Furthermore, it is analytic if V is not a twist of trivial representation.

Example 2.1.
- $V = L(m)$, then $L(V,s) = \zeta_K(s+m)$.
- For E/K an elliptic curve and $V = V_pE$, $L(V^\vee, s) = L(E,s)$ (geometric Frobenius!). As the Weil pairing gives an isomorphism $V^\vee \cong V(1)$, $L(V,s) = L(E, s+1)$.
- Let $\chi : K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ be an algebraic Hecke character, which means that $\chi_\infty|_{(K\otimes_{\mathbb{Q}}\mathbb{R})_0^\times} = \xi|_{(K\otimes_{\mathbb{Q}}\mathbb{R})_0^\times}$, for an algebraic character ξ of $\mathrm{Res}_{K/\mathbb{Q}}\,\mathbb{G}_m =: G$. In terms of a character on $G(\overline{\mathbb{Q}}) = (K \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^\times = \prod_{\sigma : K \hookrightarrow \overline{\mathbb{Q}}} \overline{\mathbb{Q}}^\times$, it is of form $(a_\sigma) \mapsto \prod a_\sigma^{n_\sigma}$ for integers n_σ. As the class field theory gives an isomorphism $G_K^{\mathrm{ab}} \cong K^\times \backslash \mathbb{A}_K^\times / \overline{(K \otimes_{\mathbb{Q}} \mathbb{R})_0^\times}$m we can build a 1-dimensional p-adic Galois representation over K, σ_χ, by

  $$\sigma_\chi((x_v)) = \chi((x_v))\xi^{-1}(x_\infty)\xi(x_p),$$

  upon choosing embeddings into $\overline{\mathbb{Q}}_p$, $\cdots$. These corrections are made to make sure that the map is stable on $K^\times$ and $\overline{(K \otimes_{\mathbb{Q}} \mathbb{R})_0^\times}$. Then the value, a priori in $\overline{\mathbb{Q}}_p$ (after of course choosing all the embeddings), is defined over a finite extension of $\mathbb{Q}_p$, and it satisfies $L(\sigma_\chi, s) = L(\chi, s)$. Also here it is important to chooes the convention that the geometric Frobenius corresponds to a uniformizer.

Definition 2.1 (Local Bloch-Kato f-condition)**.** *We define*

$$H_f^1(K_v, V) = \begin{cases} \ker(H^1(K_v, V) \longrightarrow H^1(I_v, V)) & v \nmid p \\ \ker(H^1(K_v) \longrightarrow H^1(K_v, V \otimes B_{\mathrm{cris}})) & v|p \end{cases}$$

What does the condition at p exactly mean? Note that we have, for any extension $0 \longrightarrow V \longrightarrow X \longrightarrow L \longrightarrow 0$ (here L is as before a finite extension of $\mathbb{Q}_p$ with trivial Galois action)

$$\begin{tikzcd} 0 \arrow[r] & (V \otimes B_{\mathrm{cris}})^{G_{K_v}} \arrow[r] & (X \otimes B_{\mathrm{cris}})^{G_{K_v}} \arrow[r] & (L \otimes B_{\mathrm{cris}})^{G_{K_v}} = (L \otimes K_v^0) \arrow[r] & H^1(K_v, V \otimes B_{\mathrm{cris}}) \\ 0 \arrow[r] & V \arrow[r] \arrow[u] & X \arrow[r] \arrow[u] & L \arrow[r] \arrow[u] & H^1(K_v, V) \arrow[u] \end{tikzcd}$$

Thus that the extension class $[X]$ is in $H_f^1(K_v, V)$ means that $D_{\mathrm{cris}}(X)$ loses dimension only from $D_{\mathrm{cris}}(V)$.

For v not dividing p,

$$H^1_f(K, V) = H^1_{\mathrm{nr}}(K_v, V) = H^1(G_{K_v}/I_v, V^{I_v}),$$

and as G_{K_v}/I_v is pro-cyclic generated by Frob_v, by evaluating at Frob_v we get $H^1_f(K_v, V) \cong V^{I_v}/(\mathrm{Frob}_v - 1)V^{I_v}$, which makes it viable to be computed.

Example 2.2. (1) For $m \neq 0$, $H^1_{\mathrm{nr}}(K_v, L(m)) \cong L/(q_v^m - 1)L = 0$.

(2) For V the p-adic Tate module of an elliptic curve, $H^1_{\mathrm{nr}}(K_v, V) = 0$, at least for v not dividing $N_E p$. This is because of the Riemann Hypothesis part of the Weil conjecture that 1 cannot be a Frobenius eigenvalue of V. This is expected as this is expected to be the same local condition as $E(K_v) \otimes \mathbb{Q}_p \longrightarrow H^1(K_v, V_pE)$ (map from Kummer theory), and as v does not divide p, say if v is ℓ-adic, then $E(K_v)$ is pro-ℓ, which means $E(K_v) \otimes \mathbb{Q}_p = 0$.

Example 2.3. For $v|p$, we have the following.

(1) We have

$$H^1_f(K_v, L(m)) = \begin{cases} 0 & \text{if } m < 0 \\ \text{a little complicated (explicit)} & \text{if } m = 0, 1 \\ H^1(K_v, L(m)) & \text{if } m > 1 \end{cases}$$

What about $m = 0, 1$? Note that $H^1(K_v, L) = \mathrm{Hom}_{\mathrm{cont}}(G_K^{\mathrm{ab}}, L)$, and $H^1(K_v, L(1)) \cong K_v^\times \otimes L$. One needs to explicitly calculate which extensions are crystalline.

(2) For $V = V_pE$, $H^1_f(K_v, V) \cong E(K_v) \otimes \mathbb{Z}_p \mathbb{C}_p$ through the Kummer theory map, which is proved by Bloch-Kato themselves.

Definition 2.2 (Bloch-Kato Selmer group)**.** *We define the* ***Bloch-Kato Selmer group*** $H^1_f(K, V)$ *as*

$$H^1_f(K, V) = \{c \in H^1(K, V) \mid \mathrm{loc}_v\, c \in H^1_f(K_v, V) \textit{for all } v\}.$$

Example 2.4. (1) $H^1_f(K, \mathbb{Q}_p(m)) = \{\text{classes trivial at } v \mid p\}$, for $m > 1$.

(2) $H^1_f(K, V_pE)$ is the characteristic 0 p-adic Slemer group of E/K.

Remark 2.2. For integral or mod p Galois representations, we can propagate our definition of local Bloch-Kato Selmer condition. Namely, let $T \subset V$ be a G_K-stable lattice inside a rational p-adic Galois representation, and $A = V/T$ (e.g. $T = T_pE, V = V_pE, A = E[p^\infty]$). Then from $0 \longrightarrow T \longrightarrow V \longrightarrow A \longrightarrow 0$, we have a sequence of maps $H^1(K_v, T) \longrightarrow H^1(K_v, V) \longrightarrow H^1(K_v, A)$. We then just define $H^1_f(K_v, A) = \mathrm{im}(H^1_f(K_v, V) \subset H^1(K_v, V) \longrightarrow H^1(K_v, A))$ and $H^1_f(K_v, T)$ be the preimage of $H^1_f(K_v, V)$. We then define the global Bloch-Kato Selmer group in the analogous way.

Example 2.5. $H^1_f(K, T_pE) = \varprojlim \mathrm{Sel}_{p^n}(E/K)$, $H^1_f(K, E[p^\infty]) = \varinjlim \mathrm{Sel}_{p^n}(E/K)$.

In this vein, we can define the **Bloch-Kato Tate-Shafarevich group** $\text{Ш}_{\mathrm{BK}}(K, A)$ to be

$$\text{Ш}_{\mathrm{BK}}(K, A) = H^1_f(K, A)/\operatorname{im} H^1_f(K, V).$$

This is, unlike the usual Tate-Shafarevich group, known to be **always finite**. Rather a relation is that there is a surjection

$$\text{Ш}(E/K)[p^\infty] \twoheadrightarrow \text{Ш}_{\mathrm{BK}}(K, E[p^\infty]),$$

where the kernel is a maximal divisible subgroup, which is expected to be zero.

Conjecture 2.1 (Bloch-Kato conjecture. *) For V as above (i.e. geometric, regular, irreducible), then*

$$\mathrm{ord}_{s=0} L(V,s) = \dim_L H^1_f(K, V^\vee(1)).$$

Example 2.6. (1) For $V = \mathbb{Q}_p$, $L(V,s) = \zeta_K(s)$, and we know that $\mathrm{ord}_{s=0}\zeta_K(s) = \mathrm{rank}\,\mathcal{O}_K^\times$. On the other hand we know $\dim_{\mathbb{Q}_p} H^1_f(K, \mathbb{Q}_p(1)) = \dim_{\mathbb{Q}_p}\mathcal{O}_K^\times \otimes \mathbb{Q}_p$, so the Bloch-Kato conjecture is true in this case.

(2) For $V = \mathbb{Q}_p(-2m)$ for $2m > 0$, we know, as $L(V,s) = \zeta(s-2m)$, $\mathrm{ord}_{s=0} L(V,s) = 1$. Thus we expect $\dim_{\mathbb{Q}_p} H^1_f(\mathbb{Q}, \mathbb{Q}_p(2m+1)) = 1$. That it is ≥ 1 can be derived from simple global Euler characteristic calculation argument, and that it is 1 for all but finitely many m is a consequence of Iwasawa Main Conjecture. That it is always 1 is a deep result of Soulé.

(3) For $V = V_pE$, We know $\mathrm{ord}_{s=0} L(V,s) = \mathrm{ord}_{s=1} L(E,s) = r_{\mathrm{an}}(E/K)$. The other side is $\dim H^1_f(K, V^\vee(1)) = \dim H^1_f(K, V)$. Kummer theory gives $E(K)\otimes\mathbb{Q}_p \subset H^1_f(K,V)$, so the BSD is Bloch-Kato + finiteness of $Ш(E/K)[p^\infty]$.

Our first goal stated in this framework is the following.

Theorem 2.1 (Skinner-Urban)**.** *Suppose that K is an imaginary quadratic field, $p = v\overline{v}$ is split in K and V is polarized, which means $V^\vee(1) \cong V^c$ where c is a complex conjugation. Also assume that V is geometric, irreducible and associated with a cuspidal automorphic representation π of a Hermitian space $(K^{\dim V}, \Phi)$, such that $V|_{G_{K_v}}$ is regular, crystalline and $0, -1$ are not Hodge-Tate weights (which takes care of $V|_{G_{K_{\overline{v}}}}$ by complex conjugation + polarization). Then*

(1) $L(V,0) = 0$ *implies* $\dim H^1_f(K,V) =\geq 1$.

(2) *If* $\mathrm{ord}_{s=0} L(V,s)$ *is even and positive,* $\dim H^1_f(K,V) \geq 2$.

Remark 2.3. Excluding $0, -1$ as Hodge-Tate weights excludes $V = V_pE$, but includes all higher weight modular forms, or more generally RACSDC π of $\mathrm{GL}_{2n,\mathbb{Q}}$ with a regular weight hypothesis.

Eventually we will construct an extension of form $0 \longrightarrow L(1) \longrightarrow X \longrightarrow V \longrightarrow 0$. However our method will on the first hand gives an extension of form

$$\begin{pmatrix} \varepsilon & * & * \\ & 1 & * \\ & & \rho \end{pmatrix} \text{ or } \begin{pmatrix} 1 & * & * \\ & \varepsilon & * \\ & & \rho \end{pmatrix},$$

and we want an extension of form $\begin{pmatrix} \varepsilon & * \\ & \rho \end{pmatrix}$ which is only in the second form of extension. However as K has no unit of infinite order, $H^1_f(K, L(1)) = 0$, which means that an extension of form $\begin{pmatrix} \varepsilon & * & * \\ & 1 & * \\ & & \rho \end{pmatrix}$ is actually of form $\begin{pmatrix} \varepsilon & 0 & * \\ & 1 & * \\ & & \rho \end{pmatrix}$ which can be conjugated to be of the second form.

3. Automorphic forms on unitary groups

Let's consider the case of SL_2 first. A holomorphic modular form is a holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$, $\mathbb{H} = \{x+iy, y > 0\}$, such that for $\gamma \in \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ a congruence subgroup,

$f(\gamma(z)) = j(\gamma, z)^k f(z)$, where $j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = cz+d$. This could be also thought as a holomorphic section of a line bundle $\Gamma\backslash\mathbb{H} \times \mathbb{C} \longrightarrow \Gamma\backslash\mathbb{H}$ where Γ-action on $\mathbb{H} \times \mathbb{C}$ is defined by $\gamma \cdot (z, w) = (\gamma(z), j(\gamma, z)^k \omega)$; that it is a line bundle is because the factor of automorphy satisfies a cocycle condition $j(\gamma'\gamma, z) = j(\gamma', \gamma(z))j(\gamma, z)$. Indeed, as $\mathbb{H} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$, one can construct a smooth function $\varphi : \Gamma\backslash\mathrm{SL}_2(\mathbb{R}) \longrightarrow \mathbb{C}$ by $\varphi(g) = j(g, i)^{-k} f(g(i))$. This lies in $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}))$, with L^2-norm being Petersson norm (i.e. integration against the hyperbolic metric), and the Hilbert space has an obvious $\mathrm{SL}_2(\mathbb{R})$-action. Thus in this way we get a representation of $\mathrm{SL}_2(\mathbb{R})$.

Note that φ obtained this way is not merely just L^2, in fact it is smooth, and even more K-finite, where here $K = \mathrm{SO}_2(\mathbb{R})$. Indeed, for $u = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R})$, $\varphi(gu) = j(gu, i)^{-k} f(gu(i)) = j(u, i)^{-k} j(g, i)^{-k} f(g(i)) = j(u, i)^{-k}\varphi(g)$, and $j(u, i)^{-k} = (bi + a)^{-k} = e^{-ik\theta}$, so K-span of φ is a line.

3.1. **Unitary groups.** Let K be an imaginary quadratic field, and let us fix an embedding $K \hookrightarrow \mathbb{C}$. This is important as we would like to say something about signature of our unitary group. Let $\Phi \in M_d(K)$ be a skew-hermitian matrix, namely ${}^t\overline{\Phi} = -\Phi$. This defines a skew-hermitian pairing on K^d, via $\langle x, y\rangle = {}^t\overline{x}\Phi y$, as $\langle y, x\rangle = -\overline{\langle x, y\rangle}$. Assume that the pairing is nondegenerate (i.e. $\Phi \in \mathrm{GL}_d(K)$). Seeing at the archimedean place we can talk about the signature (a, b) where a is the number of eigenvalues in $i\mathbb{R}_{>0}$ and b is the number of eigenvalues in $i\mathbb{R}_{<0}$ (so that $a + b = d$). Suppose that $b \geq a$, then (K^d, Φ) is isomorphic to a copies of hyperbolic plane plus an anisotropic space of dimension $b - a$. For simplicity we can just consider

$$\Phi = \begin{pmatrix} & & I_a \\ & \theta I_{b-a} & \\ -I_a & & \end{pmatrix},$$

for $\theta \in K^\times$ totally imaginary with $-i\theta < 0$ (for a fixed choice of i).

Definition 3.1. *The unitary group $G = \mathrm{U}(\Phi)$ is defined by*

$$G(R) = \{g \in \mathrm{GL}_d(K \otimes_{\mathbb{Q}} R) \mid {}^t\overline{g}\Phi g = \Phi\},$$

for any $\mathbb{Q}$-algebra R.

This indeed is the right algebraic group, as $G(\mathbb{R}) \cong \mathrm{U}(a, b)$.

Remark 3.1. Usually $\mathrm{U}(a, b)$ is defined with a bilinear pairing corresponding to $\begin{pmatrix} I_a & \\ & -I_b \end{pmatrix}$, which is hermitian, not skew-hermitian. But one can just use i times this matrix to convert to skew-hermitian. The $\mathbb{Q}$-group defined with this matrix is also usually referred as $\mathrm{U}(a, b)$. For Φ of form $\begin{pmatrix} & & I_a \\ & \theta I_{b-a} & \\ -I_a & & \end{pmatrix}$, one has an explicit isomorphism $\mathrm{U}(\Phi) \xrightarrow{\sim, g \mapsto c^{-1}gc} \mathrm{U}(a, b)$, where $\mathrm{U}(a, b)$ is now defined by the bilinear pairing $\theta\begin{pmatrix} I_a & \\ & -I_b \end{pmatrix}$, and $c = \begin{pmatrix} \theta & & \overline{\theta} \\ & I_{b-a} & \\ I_a & & I_a \end{pmatrix}$.

Example 3.1. For Φ being just a single copy of hyperbolic plane and nothing else, then $\mathrm{U}(\Phi)(\mathbb{R}) = \mathrm{U}(1,1)$ is $\frac{\mathrm{SL}_2(\mathbb{R})\times\mathbb{C}_1^\times}{\pm 1}$ where $\mathbb{C}_1^\times$ means norm 1 elements in $\mathbb{C}$. To see unitary group as such quotient as an algebraic $\mathbb{Q}$-group we need to go to general similitude group as there is the issue of norms in the center. For $\mathrm{GU}(1,1)$, over $\mathbb{Q}$, it is the same as $(\mathrm{GL}_2 \times K^\times)' \subset \mathrm{GL}_2 \times K^\times$, the collection of elements (g,x) such that $\det g = N_{K/\mathbb{Q}}(x)^{-1}$.

This unitary group acts on a domain

$$\mathcal{D}_\Phi = \left\{ \begin{pmatrix} z \\ w \\ I_a \end{pmatrix} \mid z \in M_{a\times a}(\mathbb{C}), w \in M_{(b-a)\times a}(\mathbb{C}), i(z - {}^t\overline{z}) + i\,{}^t\overline{w}\theta w < 0 \right\},$$

via, for $g \in \mathrm{U}(\Phi)$, $x \in \mathcal{D}_\Phi$, $g \cdot x = \begin{pmatrix} \alpha\delta^{-1} \\ \beta\delta^{-1} \\ I_a \end{pmatrix}$ where $gx = \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix}$ is the usual matrix multiplication.

Example 3.2. If Φ is a hyperbolic plane, then $\mathcal{D}_\Phi = \mathcal{D}_{1,1}$ is $\mathbb{H}$, and the action is the usual action.

Similar to $i \in \mathbb{H}$, we can choose a basepoint $x_0 = \begin{pmatrix} iI_a \\ 0 \\ I_a \end{pmatrix} \in \mathcal{D}_\Phi$. The stabilizer of x_0, denoted as K_∞, is a maximal compact subgroup of $G(\mathbb{R})$, and $G(\mathbb{R})/K_\infty \cong \mathcal{D}_\Phi$. After the identification $G(\mathbb{R}) \cong \mathrm{U}(a,b)$, we have an identification $K_\infty \cong \mathrm{U}(a) \times \mathrm{U}(b)$.

We can define the automorphy factor $J : G(\mathbb{R}) \times \mathcal{D}_\Phi \longrightarrow K_\infty(\mathbb{C})$ so that

- $J(g,-)$ is a holomorphic function on $\mathcal{D}_\Phi$,
- $J(gg',x) = J(g,g'(x))J(g',x)$,
- $J(c,x_0) = c$ for all $c \in K_\infty$.

As we have done in the modular form case, we define holomorphic automorphic forms in three ways. Let W be a weight, i.e. a finite dimensional algebraic representation of K_∞. Let $\rho : K_\infty \longrightarrow \mathrm{GL}(W)$.

(1) A holomorphic automorphic form (**modular form**) of weight W and level Γ is a holomorphic function $f : D_\Phi \longrightarrow W$ such that $f(\gamma(x)) = \rho(J(\gamma,x))f(x)$ for all $\gamma \in \Gamma \subset G(\mathbb{R})$.

(2) From the above, a smooth automorphic form can also be defined as $\varphi : \Gamma\backslash G(\mathbb{R}) \longrightarrow W$ such that $\varphi(g) = \rho(J(g,x_0))^{-1}f(g(x_0))$. It inherits the K-finiteness property as follows. For $k \in K_\infty(\mathbb{R})$, we have

$$\varphi(gk) = \rho(J(gk,x_0))^{-1}F(gkx_0) = \rho(J(k,x_0))^{-1}\rho(J(g,x_0))^{-1}F(gx_0) = \rho(k)^{-1}\varphi(g),$$

so that the K-span of φ is a finite-dimensional vector subspace. Another way of saying this is that $W^\vee \longrightarrow C^\infty(\Gamma\backslash G(\mathbb{R}),\mathbb{C})$ is $K_\infty(\mathbb{R})$-equivariant, where the map is defined by $w \in W^\vee \mapsto w\circ\varphi : \Gamma\backslash G(\mathbb{R}) \longrightarrow \mathbb{C}$. The image of this map lies in a subspace of smooth automorphic forms $\mathscr{A}(\Gamma\backslash G(\mathbb{R})) \subset C^\infty(\Gamma\backslash G(\mathbb{R}))$, which is characterized by moderate growth condition and $U(\mathfrak{g})$-finiteness condition.

In our case, as the center is compact, $C^\infty(\Gamma\backslash G(\mathbb{R})) \subset L^2(\Gamma\backslash G(\mathbb{R}))$ and there we can use decomposition as $G(\mathbb{R})$-representations.

(3) One can realize these as admissible $(\mathfrak{g}, K_\infty)$-representations, which corresponds to smooth K_∞-finite vectors inside the Hilbert space representation corresponding to it. Here admissibility means for every irreducible $K_\infty(\mathbb{R})$-representation W, $\mathrm{Hom}_{K_\infty}(W, \pi)$ is finite.

Remark 3.2. (1) To adelically write an automorphic form, we need to use a strong approximation: in the case of SL_2 and $\Gamma(N) \subset \mathrm{SL}_2(\mathbb{Z})$, we can adelically extend to $K(N) = \{g \in \mathrm{SL}_2(\widehat{\mathbb{Z}}) \mid g \equiv 1 (\mathrm{mod}\, N)\}$ and then we have by strong approximation $\mathrm{SL}_2(\mathbb{A}) = \mathrm{SL}_2(\mathbb{Q})\,\mathrm{SL}_2(\mathbb{R})K(N)$. In this way, φ defined over $\Gamma\backslash G(\mathbb{R})$ extends to a smooth function over $\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})$. As the space of smooth functions on $\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})$ has a natural action of $\mathrm{SL}_2(\mathbb{A})$, we get an adelic representation (more precisely, a representation of $G(\mathbb{A}_f) \times (\mathfrak{g}, K_\infty)$). By the tensor product theorem, an automorphic irreducible adelic representation is a restricted tensor product of local representations, and then we can do p-adic theory and such.

(2) How do we see holomorphicity on $\mathscr{A}(\Gamma\backslash G(\mathbb{R}))$? Certainly we can define a smooth function, from $\varphi \in (\mathscr{A}(\Gamma\backslash G(\mathbb{R})) \otimes W)^{K_\infty}$, $F_\varphi(x) = \rho(J(g, x_0))\varphi(g)$, where $g \in G(\mathbb{R})$ is such that $g(x_0) = x$. Then, that F_φ is holomorphic is equivalent to that φ is annihilated by $U(\mathfrak{p}^-)$, where $\mathfrak{g}_\mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{k} \oplus \mathfrak{p}^-$ is the Cartan decomposition. In the case of GL_2, this is the usual holomorphicity (Cauchy-Riemann relation, or Maass-Shimura lowering operator).

(3) How do we know if such function exists in our automorphic representation? It turns out that it does exist when your representation at infinity is a holomorphic (limit of) discrete series. A holomorphic (limit of) discrete representation of $K_\infty(\mathbb{C})$ is a representation of form $U(\mathfrak{p}^+) \otimes W \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{k}\oplus\mathfrak{p}^-)} W$ for a finite dimensional irreducible representation W of $K_\infty(\mathbb{C})$ which happens to be irreducible. This holds under mild conditions, e.g. if, when W is seen as an irreducible representation of $K_\infty(\mathbb{C}) \cong \mathrm{GL}_a(\mathbb{C}) \times \mathrm{GL}_b(\mathbb{C})$, the highest weight with respect to the diagonal tori, $n_1 \geq n_2 \geq \cdots \geq n_a$ and $m_1 \geq m_2 \geq \cdots \geq m_b$, satisfies $n_a, m_b \geq \frac{a+b-1}{2}$. This really depends on how you pick torus/Borel/isomorphism of $K_\infty(\mathbb{C}) \cong \mathrm{GL}_a(\mathbb{C}) \times \mathrm{GL}_b(\mathbb{C})$, $\cdots$.

Remark 3.3. The factor of automorphy is naturally defined as follows. From the Cartan decomposition $\mathfrak{g}_\mathbb{C} \cong \mathfrak{p} \oplus \mathfrak{k}$ for $G = \mathrm{U}(a, b)$, with $\mathfrak{k} = (\mathrm{Lie}\, K_\infty)_\mathbb{C}$, and $G(\mathbb{R}) = KP$, we see that $K_\infty(\mathbb{C})P(\mathbb{C}) \subset G(\mathbb{C})$ contains $G(\mathbb{R})$, and the factor of automorphy is just the projection down to the $K_\infty(\mathbb{C})$-factor.

Here we become more precise. Along the veins of defining Shimura varieties, one can see $\mathcal{D}_\Phi$ as a set of $G(\mathbb{R})$-conjugacy classes of a Hodge cocharacter. In this case, such Hodge cocharacter can be given as

$$h : \mathbb{C}_1^\times \longrightarrow G(\mathbb{R}), z = u + iv \mapsto \begin{pmatrix} u & & -v \\ & \overline{z} & \\ v & & u \end{pmatrix},$$

where $G = \mathrm{U}(\Phi)$ as before, which corresponds to $\begin{pmatrix} zI_a & \\ & \overline{z}I_b \end{pmatrix}$ via the identification $G(\mathbb{R}) \xrightarrow{\sim} \mathrm{U}(a, b)(\mathbb{R})$ we discussed before. Under this, $\mathcal{D}_\Phi \cong G(\mathbb{R})/K_\infty$, where $K_\infty = \mathrm{Stab}_{G(\mathbb{R})}(h)$. Under the identification $G(\mathbb{R}) \cong \mathrm{U}(a, b)$, the stabilizer is $\mathrm{U}(a) \times \mathrm{U}(b)$. This sits inside the flag variety $\mathrm{U}(a, b)(\mathbb{C})/P_{a,b}(\mathbb{C})$ (here we embed via $[g] \mapsto [gc]$) where $P_{a,b}$ is the stabilizer of the vector subspace generated by the first a vectors (here we take the convention that $P_{a,b}$ is **lower**

triangular). Given an algebraic representation W of $P_{a,b}(\mathbb{C})$, we can form a local system $\mathcal{W} = (\mathrm{U}(a,b)(\mathbb{C}) \times W)/P_{a,b}(\mathbb{C})$ on the flag variety. As $\mathcal{D}_\Phi$ is simply connected, $\mathcal{W}|_{\mathcal{D}_\Phi}$ is trivial; we can take $\alpha : \mathrm{U}(a,b)(\mathbb{C}) \longrightarrow \mathrm{GL}(W)$, canonical up to $\mathrm{GL}(W)$-action on the whole, such that $\mathcal{W}|_{\mathcal{D}_\Phi} \cong \mathcal{D}_\Phi \times W$ can be given by $(g,w) \mapsto (x_g, \alpha(g)w)$. Then the automorphy factor is $j(\gamma, x_g) := \alpha(\gamma g)\alpha(g)^{-1}$, which is independent of the choice of α. This is defined such that under the same trivialization, $(\gamma' g, w)$ maps to $(x_{\gamma' g}, j(\gamma', x_g)\alpha(g)w)$.

Another way of seeing is that, consider the Cartan decomposition $\mathfrak{g}'_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}' \oplus \mathfrak{p}^-$, where all Lie algebras are complexified and $'$ means we are working instead with $\mathrm{U}(a,b)$, not $\mathrm{U}(\Phi)$. Then $j(g) = k'$ where $gc = p^+ k' p^-$ via the exponentiated Cartan decomposition. Here $P_{a,b} = K'_\infty(\mathbb{C})P^-(\mathbb{C})$. The canonical automorphy factor can be defined also as $j(g,x) = j(gg_x)j(g_x)^{-1} \in K'_\infty(\mathbb{C})$.

Let us go through a sanity check for $\mathrm{U}(1,1)$, and see that it recovers the theory of elliptic modular forms. The unitary group $G = \mathrm{U}(\Phi)$ for $\Phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is an algebraic group with identification

$$G(R) \cong \frac{\{(g,x) \in \mathrm{GL}_2(R) \times (K \otimes_{\mathbb{Q}} R)^\times \mid \det g = N(x) := x\overline{x}\}}{R^\times},$$

for $\mathbb{Q}$-algebras R. In this vein $G(\mathbb{R}) \cong \frac{\mathrm{SL}_2(\mathbb{R}) \times \mathbb{C}_1^\times}{\{\pm 1\}}$.

Our symmetric domain is $\mathcal{D}_\Phi = \{\begin{pmatrix} z \\ 1 \end{pmatrix} \mid z \in \mathbb{H}\}$. The action of $G(\mathbb{R})$ restricts to the usual action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$, and is acted trivially by $\mathbb{C}_1^\times$. Clearly $K_\infty(\mathbb{R})$ under the above identification is $K_\infty(\mathbb{R}) \cong \frac{\mathrm{SO}_2(\mathbb{R}) \times \mathbb{C}_1^\times}{\{\pm 1\}}$.

In our case $G(\mathbb{R}) \cong \mathrm{U}(1,1)(\mathbb{R})$ via $g \mapsto c^{-1}gc$ with $c = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. What is the automorphy factor as above in this case? Given $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, $gc = \begin{pmatrix} * & * \\ * & \gamma i + \delta \end{pmatrix}$. We only computed one entry because in the Cartan decomposition the bottom-right corner of $K_\infty(\mathbb{C})$-part carries over. Thus $j(g) = (\text{blah}, \gamma i + \delta)$. Thus $j(g,i) = (\frac{1}{\gamma i + \delta}, \gamma i + \delta)$ as $\mathrm{U}(1) = S^1$ and we have determinant 1 condition.

For an algebraic representation W of $K'_\infty(\mathbb{C}) \cong \mathrm{U}(1) \times \mathrm{U}(1)$ defined by $(u_1, u_2) \mapsto u_1^k u_2^\ell$, a holomorphic modular form of weight W is a function $f : \mathcal{D}_\Phi \longrightarrow W$ such that $f(\gamma(x)) = j(\gamma, x)f(x)$ for $\gamma \in \Gamma \subset G(\mathbb{R})$. The $\mathrm{SL}_2(\mathbb{R})$-part only sees $\ell - k$, but the additional $\mathbb{C}_1^\times$-action makes you to distinguish between different (k,ℓ)'s with same $\ell - k$. This is thus some information that is lost when you translate into classical holomorphic modular form languages (can do the inverse but then one needs to make choices).

Adelically we can proceed as follows. Consider a weight $2k$ newform of level $\Gamma_0(N)$, $f : \mathbb{H} \longrightarrow \mathbb{C}$. Then we can produce a smooth automorphic form $\varphi : \mathrm{GL}_2(\mathbb{A}) \longrightarrow \mathbb{C}$ such that $\varphi(g) = j(g_\infty, i)^{-2k} f(g_\infty i)$ where $g = \gamma g_\infty g_f$ with $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, $g_\infty \in \mathrm{GL}_2(\mathbb{R})^+$ and $g_f \in U_0(N) \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$. This is possible as the class group of $\mathbb{Q}$ is 1. As f is a newform, φ generates an irreducible automorphic cuspidal representation π.

Remark 3.4. We chose a specific action of center, so that $\varphi(rg) = r^{-2k}\varphi(g)$ for $r \in \mathbb{R}^\times$. We might as well have twisted by any power of absolute values of automorphy factor.

To see this as an automorphic representation of unitary groups, we consider the general unitary group $\mathrm{GU}(\Phi)$, which is isomorphic to $\frac{\mathrm{GL}_2 \times K^\times}{\mathbb{Q}^\times}$ as $\mathbb{Q}$-algebraic groups. Thus we have to add additional datum of character. To be more precise, we have to add $\psi : \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ such that $\psi|_{\mathbb{A}_\mathbb{Q}^\times} = \chi_\pi^{-1} = |-|^{2k}$. For example we can use $\psi = |-|_K^k = |N_{K/\mathbb{Q}}|^k$. This pair (π, ψ) defines an irreducible automorphic representation of $\mathrm{GU}(\mathbb{A})$. To truly work with unitary group automorphic representation, we can just choose $\sigma \subset (\pi, \psi)|_{\mathrm{U}(\Phi)(\mathbb{A})}$.

3.2. *L*-**functions for unitary groups.** In general we have a zoo of L-functions associated to an automorphic representation. Let G be a quasisplit reductive group over $\mathbb{Q}$. Then, we obtain a dual group $\widehat{G}$ over $\mathbb{C}$, equipped with an action by $W_\mathbb{Q}$. As G is quasisplit, the action factors through $G_\mathbb{Q}$, which further factors through a finite quotient. The L-group is defined as ${}^LG := \widehat{G} \rtimes W_\mathbb{Q}$. For any representation $r : {}^LG \longrightarrow \mathrm{GL}_m(\mathbb{C})$, for an irreducible automorphic representation π of $G(\mathbb{A})$, we expect to associate an L-function $L(\pi, s, r) = \prod_v L(\pi_v, s, r)$. At unramified places, the L-function $L(\pi_v, s, r)$ is the L-function of the Weil-Deligne representation $W_{\mathbb{Q}_v} \longrightarrow \mathrm{GL}_m(\mathbb{C})$ associated to π_v.

Example 3.3. For $G = \mathrm{GL}_{n,\mathbb{Q}}$, $\widehat{G} = \mathrm{GL}_n(\mathbb{C})$ and $W_\mathbb{Q}$ acts trivially. If π_ℓ is unramified, then it is principal series (supercuspidal representations are ramified), so that by the Satake isomorphism it corresponds to a semisimple conjugacy class $t_{\pi_\ell} \in \mathrm{GL}_n(\mathbb{C})$, say $\mathrm{diag}(\alpha_1, \cdots, \alpha_n)$, so that $\pi_\ell = \mathrm{Ind}(\chi)$ (normalized induction) where $\chi(\ell e_{i,i}) = \alpha_i$. For the standard representation r_{std} of $\mathrm{GL}_n(\mathbb{C})$, $L(\pi_\ell, s, r_{\mathrm{std}}) = \det(1 - \ell^{-s} t_{\pi_\ell})^{-1} = \prod_{i=1}^n (1 - \alpha_i \ell^{-s})^{-1}$, the usual local L-factor one sees in a classical setting.

We will always refer to $L(\pi, s, r_{\mathrm{std}})$ when talking about L-function of a cuspidal automorphic representation if there is no other indication on r.

3.3. **Galois representation associated to a cuspidal automorphic representation of** $\mathrm{U}(\Phi)$**.** Let $G = \mathrm{U}(\Phi) \cong \mathrm{U}(a, b)$, with $a \le b$. Let π be a cuspidal automorphic representation of $G(\mathbb{A})$. We can freely go back and forth between the whole L^2-space, its subspace of smooth vectors, and its subspace of K_∞-finite vectors (i.e. π also can be thought as a representation of $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$). Let $\pi \cong \pi_\infty \otimes \pi_f \cong \otimes'_{\ell \neq \infty} \pi_\ell$, where π_ℓ is an irreducible admissible representation of $G(\mathbb{Q}_\ell)$. We assume that π_∞ is a holomorphic discrete series. Recall, if we identify $K_\infty \cong \mathrm{U}(a) \times \mathrm{U}(b)$ and $K_\infty(\mathbb{C}) \cong \mathrm{GL}_a(\mathbb{C}) \times \mathrm{GL}_b(\mathbb{C})$ (dependent upon a choice of $K \hookrightarrow \mathbb{C}$; recall that by definition

$$\mathrm{U}(m)(\mathbb{C}) = \{g \in \mathrm{GL}_m(K \otimes_\mathbb{Q} \mathbb{C}) \mid g^t \overline{g} = I_m\},$$

and as $K \otimes_\mathbb{Q} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$, by projecting down to $K \otimes_\mathbb{Q} \mathbb{C} \longrightarrow \mathbb{C}$ coming from the fixed choice of $K \hookrightarrow \mathbb{C}$, we get an isomorphism $\mathrm{U}(m)(\mathbb{C}) \cong \mathrm{GL}_m(\mathbb{C})$. Choosing the other embedding gives an involution by transpose-inverse, giving the dual of the representation), then π_∞ being a holomorphic discrete series is the same as having Blattner parameter $(c_{b+1} \ge \cdots \ge c_d; c_1 \ge \cdots \ge c_b)$, with $c_b - c_{b+1} \ge d = a + b$. Then $\pi_\infty \cong \mathcal{U}(\mathfrak{p}^+) \otimes_\mathbb{C} W^\vee$ where W is the irreducible reprsentation of $K_\infty(\mathbb{C})$ with highest weight $(c_{b+1} \ge \cdots \ge c_d; c_1 \ge \cdots \ge c_b)$.

The L-group of G is of form $\widehat{G} \rtimes \mathrm{Gal}(K/\mathbb{Q})$, as $G_K \cong \mathrm{GL}_{d,K}$, and the involution on $\mathrm{GL}_d(\mathbb{C})$ is given by $g \mapsto \Phi_d^{-1}{}^t g^{-1} \Phi_d$ where $\Phi_d = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ \cdots & & & & \\ & & & & \end{pmatrix}$. **Check this.**

We basically only know local Langlands for GL_n, so it is desirable to utilize base change, which is an instance of Langlands functoriality. Let $H = \mathrm{Res}_{K/\mathbb{Q}} \mathrm{GL}_{d,K}$. Then ${}^L H = (\widehat{G} \times \widehat{G}) \rtimes \mathrm{Gal}(K/\mathbb{Q})$ such that $c(g_1, g_2) = (cg_2, cg_1)$ where c is the nontrivial element of $\mathrm{Gal}(K/\mathbb{Q})$. The diagonal embedding $\widehat{G} \longrightarrow \widehat{G} \times \widehat{G}$ respects Galois action and gives an L-homomorphism ${}^L G \longrightarrow {}^L H$. Thus, for an automorphic representation π of $G(\mathbb{A})$, there should be a base change $\mathrm{BC}(\pi)$, an automorphic representation of $H(\mathbb{A})$ that satisfies the local-global compatibility $\mathrm{BC}(\pi_\ell) = \mathrm{BC}(\pi)_\ell$. As we know local Langlands correspondence for $H(\mathbb{Q}_\ell)$, we would like to know if these local representations gather up to an automorphic representation. By the work of Langlands, certainly a weak base change exists, where weak base change means an automorphic representation matching up with local representation for all but finitely many places exists. By the multiplicity one, then this should be equal to the strong base change if it exists.

Now let's just suppose that $\mathrm{BC}(\pi)$ exists as a cuspidal automorphic representation. Note that this is not always the case (when π is only tempered and stable). The standard representation for ${}^L H$ is

$$r_{\mathrm{std}} : {}^L H \longrightarrow \mathrm{GL}_{2d}(\mathbb{C}), (g_1, g_2) \rtimes 1 \mapsto \begin{pmatrix} g_1 & \\ & cg_2 \end{pmatrix}, 1 \rtimes c \mapsto \begin{pmatrix} & 1 \\ 1 & \end{pmatrix},$$

and by the formalism of local-global compatibility, we can define local L-factors from local Weil-Deligne representation, namely for the Weil-Deligne representation $(\rho : G_{\mathbb{Q}_\ell} \longrightarrow \mathrm{Aut}(V), N)$ corresponding to π_ℓ, we define

$$L(\pi_\ell, s, r_{\mathrm{std}}) = \det(1 - \ell^{-s} \mathrm{Frob}_\ell \mid (V^N)^{I_\ell})^{-1}.$$

Then $L(\pi, s, r_{\mathrm{std}}) := \prod_\ell L(\pi_\ell, s, r_{\mathrm{std}})$ is formally equal to $L(\mathrm{BC}(\pi), s, r_{\mathrm{std}})$, and as we can see $\mathrm{BC}(\pi)$ as an automorphic representation of $\mathrm{GL}_d(\mathbb{A}_K) = H(\mathbb{A})$, we know that it has nice analytic properties (Ramanujan conjecture for cohomological self-dual cuspidal blah blah... of GL_d over CM field is true).

Remark 3.5. We can certainly establish partial L-function unconditionally, and nice analytic property can also be deduced from Rankin-Selberg integral representation of L-functions.

Now we can talk about the associated Galois reprsentation.

Theorem 3.1. *Fix $\overline{\mathbb{Q}}_p \cong \mathbb{C}$. For $\pi = \otimes' \pi_\ell$ with $\pi_\infty = U(\mathfrak{p}^+) \otimes W^\vee$ holomorphic discrete series, there exists a continuous semisimple homomorphism $\rho : G_K \longrightarrow \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$, such that the following conditions are satisfied.*

- *For $w \nmid p$, $\mathrm{WD}(\rho|_{G_{K_w}})^{\mathrm{Frob}\text{-}ss} \cong \mathrm{rec}(\mathrm{BC}(\pi)_w^\vee \otimes |\cdot|^{\frac{1-d}{2}})$, where rec is the local Langlands correspondence normalized as e.g. in Harris-Taylor.*
- *If $v \mid p$, then $\rho_\pi|_{G_{K_v}}$ is potentially semistable.*

- *If p is unramified, π_p is unramified, then for all $v \mid p$, $\rho_\pi|_{G_{K_v}}$ is crystalline, and* $\mathrm{WD}(\rho|_{G_{K_v}}) = D_{\mathrm{cris},v}(\rho_{\pi,v}) \cong \mathrm{rec}(\mathrm{BC}(\pi)_v^\vee \otimes |\cdot|^{\frac{1-d}{2}})$.
- *The place v coming from $K \hookrightarrow \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ has regular Hodge-Tate weights (i.e. all Hodge-Tate weights are distinct)* $(\kappa_1 < \kappa_2 < \cdots < \kappa_d) = (c_d + b < c_{d-1} + b + 1 < \cdots < c_{b+1} + a + b - 1 < c_b < c_{b-1} + 1 < \cdots < c_1 + b - 1)$.

Example 3.4. Under the identification of classical modular forms with $\mathrm{U}(1,1)$-automorphic forms, what is $L(\sigma, s, \mathrm{std})$ and $\rho_\sigma : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ for $\sigma \in (\pi, \psi)|_{G(\mathbb{A})}$? The answers are

$$L(\sigma, s, \mathrm{std}) = L(\mathrm{BC}(\pi) \otimes |-|_K^k, s) = L(f, s + k - 1/2)L(f \otimes \eta_K, s + k - 1/2),$$
$$L(\rho_s, s) = L(\mathrm{BC}(\sigma)^\vee s - 1/2) = L(V_f(k-1), s).$$

To convince ourselves let's calculate Satake parameters. At unramified ℓ, π_ℓ is an unramified principal series

$$\pi(\mu_1, \mu_2) = \{f : \mathrm{GL}_2(\mathbb{Q}_\ell) \longrightarrow \mathbb{C} \text{ smooth} \mid f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \mu_1(a)\mu_2(d)|\frac{a}{d}|^{1/2} f(g)\}.$$

Then $(g, x) \in \mathrm{GL}_2(\mathbb{Q}_\ell) \times (K \otimes \mathbb{Q}_\ell)^\times$ acts on $f \in \pi(\mu_1, \mu_2)$ via $\psi(x^{-1})gf$. Suppose ℓ is split in K. Then $\mathrm{U}(\mathbb{Q}_\ell) \subset \mathrm{GL}_2(\mathbb{Q}_\ell) \times \mathrm{GL}_2(\mathbb{Q}_\ell)$ which is identified with $\mathrm{GL}_2(\mathbb{Q}_\ell)$ via the projection to the first factor.

We want to see the action of a diagonal torus $\begin{pmatrix} x & \\ & \overline{x}^{-1} \end{pmatrix}$ where $x = \mathrm{diag}(a, d^{-1})$ to know to which extent is there a character twist. The action of it is $\mu_1(\frac{a}{d})|x\overline{x}|^{1/2}|\overline{x}|_K^k$, which is $\mu_1(a)\mu_1(d)^{-1}|a/d|^{k+1/2}$. This is the same as sitting in the unramified principal series $\pi_\ell(\mu_1|-|^k, \mu_1^{-1}|-|^{-k}) = \pi_\ell(\mu_1, \mu_2) \otimes |-|^k$ as $\mu_1^{-1} = \mu_2|-|^{2k}$.

4. Eisenstein series

Now it is time to study Eisenstein series, as L-functions arise as constant terms of Eisenstein series. Let $\Phi := \begin{pmatrix} & & I_a \\ & \theta & \\ -I_a & & \end{pmatrix}$ $\Phi' := \begin{pmatrix} & & I_{a+1} \\ & \theta & \\ -I_{a+1} & & \end{pmatrix}$, and $G = \mathrm{U}(\Phi)$ and $H = \mathrm{U}(\Phi')$. Let $a + b = d$. Let V be the $d+2$-dimensional K-vector space equipped with a skew-hermitian form Φ', and let $L \subset V$ be an isotropic line generated by the $(a+1)$-st vector (one of the two "appended" by going from G up to H). Then $P = \mathrm{Stab}_H(L)$ has Levi M of form

$$\begin{pmatrix} \mathrm{Mat}_{a\times a} & & \mathrm{Mat}_{a\times b} & \\ & \overline{t}^{-1} & & \\ \mathrm{Mat}_{b\times a} & & \mathrm{Mat}_{b\times b} & \\ & & & t \end{pmatrix},$$

so that it is isomorphic to $G \times \mathrm{Res}_{K/\mathbb{Q}} K^\times$.

Remark 4.1. For any isotropic subspace of V, the Levi of the parabolic is of form (unitary) $\times\, \mathrm{GL}_*$. The reason why we do not deal with higher dimensional GL is that Eisenstein series obtained in such way is very rarely holomorphic, unless you take something very degenerate for GL_*-automorphic form.

Take (π, V_π) an automorphic representation of $G(\mathbb{A})$ and a Hecke character χ of $\mathbb{A}_K^\times$. This gives rise to an automorphic representation (ρ, V_ρ) of $M(\mathbb{A})$. Take $\rho = \otimes'_v \rho_v$; here for ρ_∞ one rather takes smooth representation (as opposed to $(\mathfrak{g}, K_\infty)$-representation) to make things easier. Then the normalized induction is a representation

$$I(\rho) = \{f : H(\mathbb{A}) \longrightarrow V_\pi \text{ smooth} \mid f(mng) = \rho(m)f(g) \forall mn \in P(\mathbb{A})\}.$$

Let δ_P be the modulus character of P, which is determined by the adjoint action of M on N. It turns out that $\delta_P(m(g,t)n) = |t\bar{t}|^{-(d+1)}$; as g is unitary it shouldn't show up after all.

Example 4.1. (1) For $\mathrm{U}(1,1)$ and the Siegel parabolic P, we see that the Levi is just $\begin{pmatrix} \bar{t}^{-1} & \\ & t \end{pmatrix}$ and the unipotent radical is $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$. Then we see that the modulus character is indeed $|t\bar{t}|^{-1}$ as

$$\begin{pmatrix} \bar{t}^{-1} & \\ & t \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{t} & \\ & t^{-1} \end{pmatrix} = \begin{pmatrix} 1 & m/t\bar{t} \\ & 1 \end{pmatrix}.$$

(2) For GL_2 and upper triangular Borel, we get the usual formula $\delta_B = |\frac{a}{d}|$.

Now take $f \in I(\rho)$, but thought as a smooth $\mathbb{C}$-valued function $G(\mathbb{A}) \longrightarrow \mathbb{C}$. Then it is at least left $P(\mathbb{Q})$-invariant, so that we can create a left $G(\mathbb{Q})$-invariant thing

$$E(f; s, h) = \sum_{\gamma \in P(\mathbb{Q}) \backslash H(\mathbb{Q})} f(\gamma h) \delta_P(h)^{s+1/2},$$

where $\delta_P(h)$ makes sense as $H(\mathbb{A}) = P(\mathbb{A}) K_\infty H(\widehat{\mathbb{Z}})$. This sits inside $I(\rho, s) := I(\rho \times \delta_P^{s+1/2})$, whenever makes sense.

Theorem 4.1 (Langlands)**.** *$E(f; s, h)$ converges absolutely for s, h in any compact region, with $\mathrm{Re}(s) \gg 0$. In this region, $E(f; s, h)$ is holomorphic in s.*

We are interested in whether a smooth automorphic form $E(f; s, h)$ is holomorphic as a function on the symmetric space ("holomorphic in h"). We know that such property can be characterized by vanishing by some differential operators (namely $U(\mathfrak{p}^-)$). So, the question is, for $X \in U(\mathfrak{g})$,

$$X * E(f; s, h) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} (X * (f \otimes \delta_P^{s+1/2}))?$$

Here the RHS is also something analytically continued (it's also an Eisenstein series; each summand is also a section in $I(\rho, s)$). This does not necessarily always hold, for example nonholomorphic weight 2 Eisenstein series. The order of non-holomorphicity is actually related to (non)vanishing of central L-values.

We specialize further to our situation. Let π be a cuspidal automorphic representation of $G(\mathbb{A})$. Let $\pi = \otimes'_v \pi_v$, with $\pi_\infty = \mathcal{D}_W^+$ a holomorphic discrete series, with W an irreducible algebraic representation of $K_\infty(\mathbb{C}) \cong \mathrm{GL}_a(\mathbb{C}) \times \mathrm{GL}_b(\mathbb{C})$ of highest weight $(c_{b+1} \geq \cdots \geq c_d; c_1 \geq \cdots \geq c_b)$ with $c_b - c_{b+1} \geq d$. Let $\chi = \otimes \chi_v$ be an algebraic Hecke character, where algebraicity means $\chi_\infty : (K \otimes \mathbb{R})^\times \longrightarrow \mathbb{C}^\times$, which can be thought as $\mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ upon a choice of $K \hookrightarrow \mathbb{C}$, being $\chi_\infty(z) = z^n \bar{z}^m$ for $n, m \in \mathbb{Z}$, $n + m \equiv d (\mathrm{mod}\, 2)$. Let $\kappa = \frac{n-m}{2}$, $\kappa' = \frac{n+m}{2}$. We want ρ is not only just an automorphic form of $H(\mathbb{A})$ but a **holomorphic discrete series**. The condition for such thing to happen is

$$c_b \geq \kappa + \frac{d}{2} + 1 \geq \kappa - (\frac{d}{2} + 1) \geq c_{b+1};$$

note that the central gap is now required to be at least $d+2$. Let us assume this condition throughout the section.

Let K'_∞ be the maximal compact subgroup of $H(\mathbb{R})$, chosen appropriately so that under the usual identification $K'_\infty(\mathbb{C}) \cong \mathrm{GL}_{a+1}(\mathbb{C}) \times \mathrm{GL}_{b+1}(\mathbb{C})$ and compatible with $K_\infty(\mathbb{C}) \cong \mathrm{GL}_a(\mathbb{C}) \times \mathrm{GL}_b(\mathbb{C})$. Let W' be the irreducible algeraic representation with highest weight $(\kappa - (\frac{d}{2} + 1) \geq c_{b+1} \geq \cdots \geq c_d; c_1 \geq \cdots \geq c_b \geq \kappa + \frac{d}{2} + 1)$. By the way we picked K'_∞ and because of branching laws for unitary groups, W occurs with multiplicity one in $W'|_{K_\infty(\mathbb{C})}$. Thus $(\rho_\infty \otimes W')^{K_\infty(\mathbb{C}) \times K^\times(\mathbb{R})} = (\pi_\infty \otimes W)^{K_\infty(\mathbb{C})}$, which is 1-dimensional by the theory of minimal K-types. Thus, by the Frobenius reciprocity, $(I(\rho_\infty) \otimes W')^{K'_\infty(\mathbb{C})}$ is also 1-dimensional. Here, $I(\rho_\infty)$ means the local representation of $I(\rho)$ at ∞ coming from the tensor product theorem. Pick Ψ_∞ inside that line. Similarly one can find $\Psi_f \in I(\rho_f)$ such that $\Psi = \Psi_\infty \otimes \Psi_f \in (I(\rho) \otimes W')^{K'_\infty(\mathbb{C})}$. We can also choose $\psi \in (\pi \otimes W)^{K_\infty(\mathbb{C})}$ corresponding to Ψ under the identification $(\rho_\infty \otimes W')^{K_\infty(\mathbb{C}) \times K^\times(\mathbb{R})} = (\pi_\infty \otimes W)^{K_\infty(\mathbb{C})}$. Note that by general theory, ψ, a highest weight vector of the minimal K-type of holomorphic discrete series, defines a holomorphic modular form, in a sense that the usual associated smooth function $f_h : \mathcal{D}_\Phi \longrightarrow \mathbb{C}$ via $f_h(g_\infty x_0) = j_G(g_\infty, x_0)\psi_h(g_\infty)$, where ψ_h is the right $h \in H(\mathbb{A})$-translate of the vector, is holomorphic.

Similarly we can also define a function on $\mathcal{D}_{\Phi'}$,

$$F_h(s, z) = j_H(h_\infty, x'_0)\Psi(h_\infty h)\delta_P(h_\infty h)^{s+1/2},$$

where $h_\infty \in H(\mathbb{R})$ such that $h_\infty(x'_0) = z$.[1]

What is the relation between F_h and f_h? As $h_\infty = m(g_\infty, t)n$, it turns out that

$$F_h(s, z) = |t\bar{t}|^{1/2+\kappa'-s(d+1)} f_h(r(z)),$$

where $r : \mathcal{D}_{\Phi'} \longrightarrow \mathcal{D}_\Phi, \begin{pmatrix} U \\ W \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} U' \\ W' \\ 1 \end{pmatrix}$ is the holomorphic projection, with $U = \begin{pmatrix} U' & * \\ * & * \end{pmatrix}$, $W = \begin{pmatrix} W' & * \\ * & * \end{pmatrix}$. Thus F_h is vividly holomorphic **if and only if** the exponent of $|t\bar{t}|$ is zero.

4.1. **Constant terms of Eisenstein series.** We first recall what a constant term is for an automorphic form. Given a smooth function $\varphi : G(\mathbb{Q})\backslash G(\mathbb{A}) \longrightarrow \mathbb{C}$, a constant term along a parabolic $P \subset G$ is $\varphi_P(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)dn$.

Remark 4.2. (1) Note that $N(\mathbb{Q})\backslash N(\mathbb{A})$ is always compact.

(2) Checking if φ vanishes along all cusps can be done by checking only maximal parabolics, because $P' \subset P$ implies $N' \supset N$, which is also a normal subgroup, so that $\varphi_P = 0$ implies $\varphi_{P'} = 0$. If it happens we say φ is a **cusp form**.

[1]For this part I thank Congling Qiu for some helpful discussions.

(3) Suppose N is abelian, e.g. $P \subset \mathrm{GL}_2$ is the standard upper triangular Borel. For any choice of character ψ of $N(\mathbb{Q})\backslash N(\mathbb{A})$ (which is some additive character; for example if $N \cong \mathbb{G}_a$ then any character is of form $\psi(x) = e_{\mathbb{A}}(\xi x)$ for $\xi \in \mathbb{Q}$, where $e_{\mathbb{A}} = \prod_{v \le \infty} e_v$ is the product of standard "exponentials"), we can calculate the "Fourier coefficient"

$$\varphi_\psi(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi(n)}dn,$$

which yields a Fourier expansion $\varphi(g) = \sum_\psi \varphi_\psi(g)$. Then φ_P is really the "constant term of the Fourier expansion."

Let's see what this tells you when $G = \mathrm{GL}_2$. Choose Dirichlet characters χ, ψ, considered as finite order characters of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$. Consider the space

$$I_s(\chi,\psi) = \{f : \mathrm{GL}_2(\mathbb{A}) \longrightarrow \mathbb{C} \mid f\left(\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} g\right) = \chi(a)\psi(d)|\frac{a}{d}|^{s+1/2} f(g)\},$$

where we can think this as family of representations by grouping sections of each representation $I_s(\chi,\psi)$ into **flat sections**, where $f \in I_{-1/2}(\chi,\psi) \mapsto f_s \in I_s(\chi,\psi)$ is defined by $f_s(g) = f(g)\delta(g)^{s+1/2}$ where δ is the extension of modulus character of B via the decomposition $\mathrm{GL}_2(\mathbb{A}) = B(\mathbb{A})\,\mathrm{SO}_2(\mathbb{R})\,\mathrm{GL}_2(\widehat{\mathbb{Z}})$. For $f \in I(\chi,\psi) := I_{-1/2}(\chi,\psi)$, we define

$$E(f;s,g) = \sum_{\gamma \in B(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} f_s(\gamma g),$$

which converges absolutely for $\mathrm{Re}(s) > 1/2$, and is holomorphic in s in the region.

What is $E_B(f;s,g)$? Note that we have a Bruhat decomposition $\mathrm{GL}_2(\mathbb{Q}) = B(\mathbb{Q}) \coprod B(\mathbb{Q})\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} N(\mathbb{Q})$, so that $B(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q}) = \{1\} \coprod \left\{\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} N(\mathbb{Q})\right\}$. Thus

$$\begin{aligned} E_B(f;s,g) &= \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \sum_{\gamma \in B(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} f_s(\gamma n g)dn \\ &= \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f_s(ng)dn + \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \sum_{n \in N(\mathbb{Q})} f_s\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} mng\right) dn \\ &= f_s(g) + \int_{N(\mathbb{A})} f_s\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} ng\right) dn, \end{aligned}$$

and the latter integral has a hope of factoring into product of local integrals.

Suppose $\chi = \otimes\chi_v$ and $\psi = \otimes\psi_v$, then we can similarly define local versions of $I_s(\chi,\psi)$. Firstly, do we have $I(\chi,\psi) \cong \otimes'_v I(\chi_v,\psi_v)$?

- If both χ_v, ψ_v are unramified (this is the case for a.e. v), then $I(\chi_v,\psi_v)^{\mathrm{GL}_2(\mathbb{Z}_\ell)} = \mathbb{C}f_\ell^0$ where $f_\ell^0(\mathrm{GL}_2(\mathbb{Z}_\ell)) = 1$. This really defines an element in the space as $\mathrm{GL}_2(\mathbb{Q}_\ell) = B(\mathbb{Q}_\ell)\,\mathrm{GL}_2(\mathbb{Z}_\ell)$. Thus we can make sense of the restricted tensor product using these spherical vectors.
- Then, any function $f \in I(\chi,\psi)$ is smooth, so that it is trivial on $\prod_{\ell \notin S} \mathrm{GL}_2(\mathbb{Z}_\ell)$ for some finite S. Thus it sits inside $\otimes' I(\chi_v,\psi_v)$. Such S should contain ∞ and all ramifying places

of χ, ψ, and then $f = f_S f^{S,0}$ where $f^{S,0} = \otimes_{\ell \notin S} f^0_\ell$ and $f_S \in \otimes_{v \in S} I(\chi_v, \psi_v)$, a function on $\prod_{v\in S} \mathrm{GL}_2(\mathcal{O}_v)$.

Then, we can express the integral as

$$\begin{aligned}\int_{N(\mathbb{A})} f_s\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} ng\right) dn &= \int_{N(\mathbb{Q}_S)} f_{S,s}\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} ng\right) dn \int_{N(\mathbb{A}^S)} f_s^{S,0}\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} ng\right) dn \\ &= \int_{N(\mathbb{Q}_S)} f_{S,s}\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} ng\right) dn \prod_{\ell \notin S} \int_{N(\mathbb{Q}_\ell)} f^0_{\ell,s}\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} ng_\ell\right) dn,\end{aligned}$$

whenever the product makes sense.

Now what is $\int_{N(\mathbb{Q}_\ell)} f^0_{\ell,s}\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} ng\right) dn =: M(f_s)(g)$?

$$\begin{aligned}M(f_s)(g) &= M(f_s)\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & \\ & d \end{pmatrix} k\right) \\ &= M(f_s)\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & \\ & d \end{pmatrix}\right) \\ &= \int_{\mathbb{Q}_\ell} f^0_{\ell,s}\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & \\ & d \end{pmatrix}\right) dm \\ &= \int_{\mathbb{Q}_\ell} f^0_{\ell,s}(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & \\ & d \end{pmatrix}) dm \\ &= \int_{\mathbb{Q}_\ell} f^0_{\ell,s}\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} a & \\ & d \end{pmatrix}\begin{pmatrix} 1 & \frac{d}{a} m \\ 0 & 1 \end{pmatrix}\right) dm \\ &= \int_{\mathbb{Q}_\ell} f^0_{\ell,s}\left(\begin{pmatrix} d & \\ & a \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\right) |\frac{a}{d}| dm \\ &= \int_{\mathbb{Q}_\ell} \chi_\ell(d)\psi_\ell(a)|\frac{d}{a}|^{s+1/2} f^0_{\ell,s}\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\right) |\frac{a}{d}| dm.\end{aligned}$$

Thus we know that $M(f^0_{\ell,s}) \in I_{-s}(\psi_\ell, \chi_\ell)^{\mathrm{GL}_2(\mathbb{Z}_\ell)}$. Thus, $M(f^0_{\ell,s}) = c_\ell(s)\widetilde{f^0_{\ell,-s}}$, where $\widetilde{f^0_{\ell,-s}}$ is the spherical vector in $I_{-s}(\psi_\ell, \chi_\ell)$. Here $c_\ell(s) = M(f^0_{\ell,s})(1)$, so we need to calculate

$$c_\ell(s) = \int_{\mathbb{Q}_\ell} f^0_{\ell,s}\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right) dn.$$

We decompose $\mathbb{Q}_\ell = \mathbb{Z}_\ell \bigsqcup \coprod_{r=1}^{\infty} \ell^{-r}\mathbb{Z}_\ell^\times$, so

$$c_\ell(s) = \int_{\mathbb{Z}_\ell} f^0_{\ell,s}\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right) dn + \sum_{r=1}^{\infty} \int_{\mathbb{Z}_\ell^\times} f^0_{\ell,s}\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & \ell^{-r}u \\ 0 & 1 \end{pmatrix}\right) \ell^r d^+u,$$

where we use additive measure of $\mathbb{Z}_\ell$ restricted to $\mathbb{Z}_\ell^\times$. The first integral is just 1. The integrand in the r-th summand, using the identity

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & \ell^{-r}u \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ -\ell^{-r}u & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} -\ell^r u^{-1} & 1 \\ & -\ell^{-r}u \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -\ell^r u^{-1} \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix},$$

which is in $\begin{pmatrix} -\ell^r u^{-1} & 1 \\ & -\ell^{-r} u \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_\ell)$, is

$$f^0_{\ell,s}\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & -\ell^{-r}u \\ & 1 \end{pmatrix}\right) = f^0_{\ell,s}\left(\begin{pmatrix} -\ell^r u^{-1} & 1 \\ & -\ell^{-r}u \end{pmatrix}\right) = \chi_\ell/\psi_\ell(\ell^r)|\ell^{2r}|^{s+1/2} = \chi_\ell/\psi_\ell(\ell^r)\ell^{-2rs-r}.$$

Thus,

$$c_\ell(s) = 1 + \sum_{r=1}^{\infty}\int_{\mathbb{Z}_\ell^\times} \chi_\ell/\psi_\ell(\ell^r)\ell^{-2rs}\,du = 1 + \sum_{r=1}^{\infty}\chi_\ell/\psi_\ell(\ell^r)\ell^{-2rs}\operatorname{vol}(\mathbb{Z}_\ell^\times) = 1 + \frac{\chi_\ell/\psi_\ell(\ell)\ell^{-2s}(1-1/\ell)}{1-\chi_\ell/\psi_\ell(\ell)\ell^{-2s}},$$

which can be rewritten as $\frac{1-\chi_\ell/\psi_\ell(\ell)\ell^{-2s-1}}{1-\chi_\ell/\psi_\ell(\ell)\ell^{-2s}} = \frac{L_\ell(\chi/\psi,2s)}{L_\ell(\chi/\psi,2s+1)}$. Thus,

$$\prod_{s\notin S} M(f^0_{\ell,s}) = \frac{L^S(\chi/\psi,2s)}{L^S(\chi/\psi,2s+1)}\widetilde{f^{S,0}_{-s}},$$

which is absolutely convergent for $\mathrm{Re}(s) > 1/2$.

What we have shown thus is that

$$E_B(f;s,g) = f_s + \frac{L^S(\chi/\psi,2s)}{L^S(\chi/\psi,2s+1)}M(f_{S,s})\widetilde{f^{S,0}_{-s}}.$$

From Langlands' general theory, we know $E(f;s,g)$ is meromorphic in s, so that $E_B(f;s,g)$ is meromorphic in s (integral over a compact region). We also know from general theory that local intertwining operator is meromorphic in s, as well as holomorphicity of $f_s(g)$. Thus, we know that $\frac{L^S(\chi/\psi,2s)}{L^S(\chi/\psi,2s+1)}$ is meromorphic in s. This enables you to meromorphically continue $L^S(\chi/\psi,s)$ to the left by 1, thus giving meromorphic continuation of $L^S(\chi/\psi,s)$ to the whole plane.

Remark 4.3. A refinement of this idea is by Shahidi (Langlands-Shahidi method), using a non-constant Fourier coefficient of Eisenstein series.

What we mean by meromorphic continuation of intertwining operator? Locally at a finite place v, $M : I_s(\chi_v,\psi_v) \longrightarrow I_{-s}(\psi_v,\chi_v)$ if well-defined. A meromorphic continuation of it is a meromorphic continuation after you unwind back both sides to the unnormalized induction. Namely, a meromorphic continuation is a rule of meromorphically assigning for any open compact $U \subset \mathrm{GL}_2(\mathbb{Z}_\ell)$ and $s \in \mathbb{C}$ a homomorphism $\mathrm{Hom}_{\mathbb{C}}(I(\chi_\ell,\psi_\ell)^U, I(\psi_\ell,\chi_\ell)^U)$, extending $\delta^{s-1/2}M(\cdot_s)$ (note that we have been using $I(\cdot,\cdot)$ without subscript for unnormalized induction). This makes sense in particular because both representations are smooth admissible representations so that the U-fixed spaces are finite-dimensional. At infinity, the same reasoning can be applied using ρ-isotypic spaces of K_∞-finite vectors (also using admissibility as $(\mathfrak{g},K_\infty)$-representations).

4.2. **Choice of sections:** GL_2**.** Now we try to choose sections for bad places. We start with Archimedean place. Let $k > 0$ be an integer such that $\chi_\infty\psi_\infty = \mathrm{sgn}^k$. Then we can pick a section $f_k \in I(\chi_\infty,\psi_\infty)$ defined by

$$f_k\left(r\begin{pmatrix} y & * \\ & y^{-1}\end{pmatrix}\begin{pmatrix} a & -b \\ b & a\end{pmatrix}\right) = \mathrm{sgn}(y)^k(a+ib)^k,$$

so that

$$f_{k,s}\left(r\begin{pmatrix} y & * \\ & y^{-1}\end{pmatrix}\begin{pmatrix} a & -b \\ b & a\end{pmatrix}\right) = \operatorname{sgn}(y)^k|y^2|^{s+1/2}(a+ib)^k.$$

In terms of automorphy factor, this can be rewritten as

$$f_{k,s}(g_\infty) = \det(g_\infty)^{k/2}j(g_\infty, i)^{-k}|j(g_\infty, i)|^{k-2s-1}.$$

For a choice of section $f = f_k \otimes f^\infty$,

$$E(f_s, g_\infty g_f) = \sum_{\gamma\in B(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} \det(\gamma g_\infty)^{k/2}j(\gamma g_\infty, i)^{-k}|j(\gamma g_\infty, i)|^{k-2s-1}f^\infty(\gamma g_f),$$

where $g_\infty \in \mathrm{GL}_2(\mathbb{R})^+$. In terms of $g_\infty = r\begin{pmatrix} y & * \\ & y^{-1}\end{pmatrix}\begin{pmatrix} a & -b \\ b & a\end{pmatrix}$, $g_\infty(i) =* + iy^2$, so that, after suitable normalization,

$$\det(g_\infty)^{-k/2}j(g_\infty, i)^k E(f_s, g_\infty g_f) = \sum_{\gamma\in B(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} j(\gamma, z)^{-k}|y(z)|^{s-\frac{k-1}{2}}f^\infty(\gamma g_f),$$

where $z = g_\infty(i)$ and $y(z)$ means the y-coordinate of z. Thus this is holomorphic in z (when it makes sense) if $s = s_k = \frac{k-1}{2}$.

This choice of section is a section of the minimal K_∞-type of D_k, holomorphic discrete series of weight k, which happens to sit inside $I_{s_k}(\chi_\infty, \psi_\infty)$.

Remark 4.4. Note that as $K_\infty = \mathrm{O}_2(\mathbb{R})$, D_k contains holomorphic and antiholomorphic vectors, $D_k = D_k^+ \oplus D_k^-$.

Now a general theory of Langlands quotient tells us that any intertwining operator on a proper subrepresentation is zero. Thus, $M_\infty(f_{k,s}) = 0$! Therefore, for a choice of adelic section $f = f_k \otimes f_{S^\infty} \otimes f^{S,0}$, we have

$$E_B(f_s, g) = f_s + \frac{L^S(\chi/\psi, 2s)}{L^S(\chi/\psi, 2s+1)}M_\infty(f_{k,s}) \otimes M(f_{S^\infty,s}) \otimes \widetilde{f^{S,0}_{-s}}.$$

Thus, as long as we don't obtain any pole from partial L-functions, $E_B(f_{s_k}, g) = f_{s_k}$, and by the theory of constant terms, $E(f_{s_k}, g)$ is holomorphic. For example, if $\chi/\psi \neq 1$ or $k > 2$, $\frac{L^S(\chi/\psi,k-1)}{L^S(\chi/\psi,k)}$ does not give you a pole.

On the other hand, if $\chi = \psi$ and $k = 2$, then $s_k = 1/2$, and $\zeta(2s)/\zeta(2s+1)$ has a pole at $s = s_k$, so that the resulting Eisenstein series is not necessarily holomorphic; this is why there is **no** holomorphic Eisenstein series of weight 2 and level 1. On the other hand, we know that there is a holomorphic Eisenstein series of weight 2 and level q, for a prime q. This is because we can choose a q-section to be a critical stabilization of the spherical vector, i.e. $f_q = f^0_{\frac{1}{2}}(g) - f^0_{\frac{1}{2}}\left(g\begin{pmatrix} 1 & \\ & q\end{pmatrix}\right)$. This will result in a vanishing intertwining operator at q, because $M(f_q) = f^0_{-1/2}(g) - f^0_{-1/2}\left(g\begin{pmatrix} 1 & \\ & q\end{pmatrix}\right) = 0$ (which you need to check only for representatives of $B(\mathbb{Q}_q)\backslash \mathrm{GL}_2(\mathbb{Q}_q)/\left\{\begin{pmatrix} * & * \\ q* & *\end{pmatrix}\right\}$, which is just W).

Remark 4.5. Another reason you want holomorphicity is as follows. Although $I_s(\chi, \psi)$ is generically irreducible, $I_{1/2}(\chi, \chi)$ is not; it fits into an exact sequence

$$0 \longrightarrow \mathrm{St} \otimes \chi \longrightarrow I_{1/2}(\chi, \chi) \longrightarrow \text{character} \longrightarrow 0,$$

so that a choice $f_q \in \mathrm{St} \otimes \chi$. This implies that, on the Galois side, any q-adic deformation of the weight 2 critical Eisenstein series is locally Steinberg, so that it has monodromy at q (in particular ramified). Thus this does not necessarily sit inside the Bloch-Kato f-Selmer group (but rather somewhere else with different condition).

Now we sketch a generalization of this, for $G = \mathrm{U}(\Phi_{a,b}) \subset H = \mathrm{U}(\Phi_{a+1,b+1})$. Thinking $\Phi_{a+1,b+1} \sim \begin{pmatrix} & & 1 \\ & \Phi' & \\ -1 & & \end{pmatrix}$, the stabilizer $P = MN$ of the anisotropic line spanned by the last vector is of form

$$P = \begin{pmatrix} \overline{x}^{-1} & \cdots_1 & * \\ & g & \cdots_2 \\ & & x \end{pmatrix},$$

where $\cdots_2 = -{}^t\overline{\cdots_1}$, $g \in G$, $x \in K^\times$ and $* \in \mathbb{Q}$, so that $M \cong G \times K^\times$. As the action of M on $\cdots_1$ is the standard representation, the analogous computation in this generality will yield a standard L-function (general classification is done by Langlands, Shahidi).

Remark 4.6. The combinatorics is easy in this case because P is a maximal parabolic and we are taking a cusp form on the Levi so that all but at most two constant terms vanish.

4.3. **Choice of sections: General cases.** Let $H = \mathrm{U}(\Phi_{a,b})$ and $G = \mathrm{U}(\Phi_{a+1,b+1})$, $b \geq a$, and pick $P \subset G$ the stabilizer of the isotropic line generated by the $(a+1)$-st vector. It has a Levi decomposition $P = MN$ where $M \cong H \times \mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m$. Let π be a cuspidal automorphic representation of $H(\mathbb{A})$ and χ be a character of $K^\times \backslash \mathbb{A}_K^\times$. This gives a representation ρ of $M(\mathbb{A})$, and by the usual extension process we can consider the (unnormalized) induced representation $I(\rho)$ of $G(\mathbb{A})$. Given $\Phi \in I(\rho)$, the flat section $\Phi_s(g) = \Phi(g)\delta_P^{s+1/2}(g) \in I_s(\rho)$ can be assembled into an Eisenstein series $E(\Phi, s, g) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} \Phi_s(\gamma g)$.

Now let $\pi = \pi_\infty \otimes \pi_f$ be such that $\pi_\infty = \mathcal{D}_W^+$ be a holomorphic discrete series of weight W, where W is an irreducible algebraic representation of $K_\infty(\mathbb{C}) \cong \mathrm{GL}_a(\mathbb{C}) \times \mathrm{GL}_b(\mathbb{C})$. In this section we choose a convention where the highest weight of W is regarded as $(c_{b+1} \geq \cdots \geq c_d; c_1 \geq \cdots \geq c_b)$ such that $c_b - c_{b+1} \geq d$. We also require χ to be algebraic so that $\chi_\infty(z) = z^n \overline{z}^m$, for $n, m \in \mathbb{Z}$, with $n + m \equiv d (\mathrm{mod}\, 2)$, $a + b = d$. We would also like $I_0(\rho)$ to have holomorphic discrete series as ∞-type, so we actually have to have an extra regularity that $c_b - c_{b+1} \geq d + 2$. We will normalize so that

$$c_b \geq \kappa' + \frac{d}{2} + 1 \geq \kappa' - \frac{d}{2} - 1 \geq c_{b+1},$$

where $\kappa' = \frac{n+m}{2}$ and $\kappa = \frac{n-m}{2}$. Indeed, if we let W' be the algebraic representation of $K'_\infty \cong \mathrm{GL}_{a+1}(\mathbb{C}) \times \mathrm{GL}_{b+1}(\mathbb{C})$ whose highest weight is $(\kappa' - \frac{d}{2} - 1 \geq c_{b+1} \geq \cdots \geq c_d, c_1 \geq \cdots \geq c_b \geq \kappa' + \frac{d}{2} + 1)$, then we have

$$\dim_{\mathbb{C}}(\pi_\infty \otimes W)^{K_\infty} = \dim_{\mathbb{C}}(I(\rho_\infty) \otimes W')^{K'_\infty} = 1,$$

(can take any $I_s(\rho_\infty)$ because modulus character is trivial on K'_∞). We pick a nonzero vector $\Phi_\infty \in (I(\rho_\infty) \otimes W')^{K'_\infty}$. As before, we will try to pick a good section $\Phi = \Phi_\infty \otimes \Phi_f \in I(\rho) \otimes W'$, where $\Phi_f = \Phi_S \otimes \Phi^{S,0}$, where S is a finite set of finite primes, containing all bad primes. Here bad primes are those over which either K is ramified, $\Phi_{a,b}$ is ramified, π is ramified or χ is ramified.

A choice of spherical section is standard. For $\ell \notin S$, $I(\rho_\ell)^{G(\mathbb{Z}_\ell)}$ is 1-dimensional, so that we can choose a basis Φ^0_ℓ. Accordingly we choose a spherical vector ϕ^0_ℓ for π_ℓ such that $\Phi^0_\ell(1) = \phi^0_\ell$.

Remark 4.7. In this case the hyperspecial maximal group $G(\mathbb{Z}_\ell)$ is easy to describe, namely $G(\mathbb{Z}_\ell) = \{g \in \mathrm{Aut}(L_\ell) \mid \forall x, y \in L_\ell, \langle gx, gy\rangle = \langle x, y\rangle\}$, where $L_\ell = (\mathcal{O}_K \otimes \mathbb{Z}_\ell)^{d+2}$ and $\langle , \rangle : L_\ell \times L_\ell \longrightarrow (\mathcal{O}_K \otimes \mathbb{Z}_\ell)$ is the bilinear pairing defined by $\Phi_{a+1,b+1}$.

Then, upon a choice of Φ_S, $E(\Phi, s, g)$ is a W'-valued automorphic form, which has meromorphic continuation and we can extract information about analytic behavior from that of constant term. The exact point of interest is

$$s_0 = \frac{\kappa' + 1/2}{d+1} = \frac{m+n+1}{2(d+1)},$$

because $\Phi_s(g_\infty g_f)$ becomes a holomorphic function in z (the variable varying over the symmetric space) precisely when $s = s_0$ (done before in the calculation of automorphy factors).

We know that in this case the only nonvanishing constant term of $E(\Phi, s, g)$ is over P (up to G-conjugates), as π is cuspidal. Note that $G = P \coprod PwP$ where

Check: the shape is wrong.

$$P = \left\{ \begin{pmatrix} A & * & B & * \\ & \bar{t}^{-1} & * & * \\ C & & D & * \\ & & & t \end{pmatrix} \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in H, t \in K^\times \right\},$$

and

$$w = \begin{pmatrix} 1_a & & & \\ & & & 1 \\ & & 1_b & \\ & -1 & & \end{pmatrix}.$$

Thus $G = P \coprod PwN$, so that

$$E(\Phi, s, g) = \Phi_s(g) + \sum_{n \in N(\mathbb{Q})} \Phi_s(wng),$$

and

$$E_P(\Phi, s, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\Phi, s, ng)dn = \Phi_s(g) + \int_{N(\mathbb{A})} \Phi_s(wng)dn.$$

Where does this intertwining integral lie in? For $\Phi_v \in I(\rho_v)$, $\int_{N(\mathbb{Q}_v)} \Phi_{v,s}(wng)dn \in I_{-s}(\rho^\vee)$, where $\rho^\vee$ is formed by (π, χ^{-c}). This is because

$$\begin{aligned}
\int_{N(\mathbb{Q}_v)} \Phi_s(wnm(h,t)n'g)dn &= \int_{N(\mathbb{Q}_v)} \Phi_s(wm(h,t)nn'g)\delta_P(m(h,t))dn \\
&= \int_{N(\mathbb{Q}_v)} \Phi_s(m(h,\bar{t}^{-1})wnn'g)\delta_P(m(h,t))dn \\
&= \int_{N(\mathbb{Q}_v)} \Phi_s(m(h,\bar{t}^{-1})wng)\delta_P(m(h,t))dn \\
&= \int_{N(\mathbb{Q}_v)} \pi(h)\chi(\bar{t}^{-1})\delta(m(h,\bar{t}^{-1}))\Phi_s(wng)\delta_P(m(h,t))dn \\
&= \int_{N(\mathbb{Q}_v)} \pi(h)\chi^{-c}(t)\delta(m(h,t))^{-s+1/2}\Phi(wng)dn.
\end{aligned}$$

We want to compute the global intertwining operator as a product of local intertwining operators. At unramified places, $M(\Phi_\ell^0, s) = c_\ell(s)\Phi_{\ell,-s}^{\vee,0}$ by the sphericity.

Proposition 4.1 (Gindikin-Karpelevich formula)**.**

$$c_\ell(s) = \frac{L_\ell(\pi_\ell, \chi_\ell^{-1}, (d+1)s)}{L_\ell(\pi_\ell, \chi_\ell^{-1}, (d+1)s+1)} \frac{L_\ell(\chi'_\ell, 2(d+1)s)}{L_\ell(\chi'_\ell, 2(d+1)s+1)},$$

where $L_\ell(\pi_\ell, \chi_\ell^{-1}, s) = L(\mathrm{BC}(\phi_\ell) \otimes \chi_\ell^{-1}, s)$, and $\chi'_\ell = \chi_\ell^{-1}|_{\mathbb{Q}_\ell^\times}\eta_{K_\ell}^d$ where η_{K_ℓ} is the quadratic character and $\mathbb{Q}_\ell^\times \subset (K \otimes \mathbb{Q}_\ell)^\times$.

I guess you can see this calculation in Langlands' Euler products, Shahidi (which reference?), Lapid-Rallis, $\cdots$.

Thus,

$$E_P(\Phi, s, g) = \Phi_s(g) + \frac{L^S(\pi, \chi^{-1}, (d+1)s)}{L^S(\pi, \chi^{-1}, (d+1)s+1)} \frac{L^S(\chi', 2(d+1)s)}{L^S(\chi', 2(d+1)s+1)} M(\Phi_\infty, s) \otimes M(\Phi^S, s) \otimes \Phi_s^{v,0,S},$$

for s in the region of convergence. We should then ask ourselves, is this holomorphic at $s = s_0$? If so, is it holomorphic in z at $s = s_0$? Recall that the relevant function on the symmetric space can be defined as

$$E(z, g_f) = J_G(g_\infty, x_0)E(\Phi, s_0, g_\infty g_f).$$

We assume that $\mathrm{BC}(\pi)$ is cuspidal (requires that π lies in the stable spectrum). Then, the analysis of $c^S(s)|_{s=s_0}$ is as follows.

(1) The cuspidality of $\mathrm{BC}(\pi)$ implies that $\frac{L^S(\mathrm{BC}(\pi)\otimes\chi^{-1},(d+1)s)}{L^S(\mathrm{BC}(\pi)\otimes\chi^{-1},(d+1)s+1)}$ is holomorphic at $s = s_0$. This is because the numerator is the central critical value whereas the denominator is at 1 plus the center, which is actually in the region of absolute convergence.
   - Why? Note that $\chi = |-|^{\frac{n+m}{2}} \times$ (unitary character), and $\mathrm{BC}(\pi)$ is unitary; at least the central character is unitary because $\chi^c_{\mathrm{BC}(\pi)} \cong \chi^\vee_{\mathrm{BC}(\pi)}$, which means $\chi_{\mathrm{BC}(\pi)}|_{\mathbb{A}_\mathbb{Q}^\times}$ is quadratic (in particular of finite order), so that $\chi_{\mathrm{BC}(\pi),\infty} = \left(\frac{z}{\bar{z}}\right)^k$ for some $k \in \mathbb{Z}$. Thus, $(d+1)s_0$ is indeed the central critical value, 1/2.

- What about 3/2? This is in the region of absolute convergence because unitary (=unitarizable, not to be confused with being an automorphic form of unitary group; $\mathrm{BC}(\pi)$ is an automorphic form of $\mathrm{GL}_n(\mathbb{A}_K)$) cuspidal automorphic representation has to be tempered because of Ramanujan's conjecture. This can be known by either knowing base change operation better or showing local-global compatibility for Galois representations showing up in the cohomology of Shimura varieties. Indeed, cohomological automorphic representations contribute to only the middle degree of the associated unitary Shimura variety Sh_H, but we do not know the local-global compatibility to translate the purity of cohomology into temperedness (as well as $H^{ab}(\mathrm{Sh}_H, \text{coeff.})[\mathrm{BC}(\pi)_f]^{\mathrm{ss}} \cong ((\Lambda^a \rho_{\mathrm{BC}(\pi)})^{\mathrm{ss}})^{m_{\mathrm{BC}(\pi)}}$, so extra work is needed).

(2) Note that $\chi'_\infty|_{\mathbb{R}_{>0}}(r) = r^{-(n+m)}$, so that $\chi_0 := \chi'|-|^{n+m}$ is a finite order Dirichlet character. Thus, $\frac{L^S(\chi',2(d+1)s_0)}{L^S(\chi',2(d+1)s_0+1)} = \frac{L^S(\chi_0,1)}{L^S(\chi_0,2)}$, whose denominator is in the region of absolute convergence and the numerator has a pole only when $\chi_0 = 1$, i.e. $\chi' = |-|^{-(n+m)}$.

What about $M(\Phi_S, s)$? Note that a local intertwining operator has a meromorphic continuation, so it makes sense to talk about $s = s_0$. Also, by the temperedness assumption of $\mathrm{BC}(\pi)_f$, we know that it is holomorphic in s at $s = s_0$ (Harish-Chandra). Thus we know that it is holomorphic and does not contribute to poles (finite product of meromorphic continuations is a meromorphic continuation of the finite product). Note that it may contribute to a zero.

Finally, $M(\Phi_\infty, s)$ also has meromorphic continuation and is holomorphic by the same reason, but as $D^+_{W'} \subset I_{s_0}(\rho_\infty)$ is a proper subrepresentation, the intertwining operator has to vanish at $\Phi_\infty \in D^+_{W'}$.

Combining these, we have shown the following.

Theorem 4.2. *Let π be as above (in particular, $\mathrm{BC}(\pi)$ is cuspidal and tempered). Then,*

(1) *$M(\Phi, s)$ is defined and holomorphic in s at $s = s_0$. Thus, $E(\Phi, s, g)$ is holomorphic in s at $s = s_0$.*

(2) *$M(\Phi, s_0) = 0$, except possibly when $\chi' = |-|^{-2\kappa'}$ and $L(\mathrm{BC}(\pi), \frac{1-m-n}{2}) \neq 0$. If so, $E_P(\Phi, s_0, g) = \Phi_{s_0}(g)$.*

As we know that $J_G(g_\infty, x_0)\Phi_{s_0}(g_\infty g_f)$ is a holomorphic function in z as a function on $(z, g_f) \in \mathcal{D}_{\Phi_{a,b}} \times G(\mathbb{A}_f)$ ($g_\infty x_\infty = z$), we know that $E(z, g_f) = J_G(g_\infty, x_0)E(\Phi, s_0, g_\infty g_f)$ is also a holomorphic function in z, if the second condition holds.

Example 4.2. (1) If $a = b = 0$, we recover the classical Eisenstein series on SL_2.

(2) If $a = b = 1$ with f a weight $2k$ newform with trivial Nebentypus, giving rise to a cuspidal automorphic representation τ of GL_2, with $\chi_\tau = |-|^{-2k}$, pairing with $\psi = |-|_K^{-k}$ gives a cuspidal automorphic representation π of $H = \mathrm{U}(\Phi_{1,1})$. Then, $L(\pi, s) = L(f_{/K}, s - \frac{1}{2} + k)$. Then, $\pi_\infty \cong D^+_{(-k;k)}$, so that after choosing χ to be the trivial character (i.e. $n = m = 0$), then the extra requirement is that

$$k \geq 2 \geq -2 \geq -k,$$

or $2k \geq 4$; in particular we do not cover weight 2 modular forms.

In any case, $L(\mathrm{BC}(\pi), \chi^{-1}, 1/2) = L(f,k)L(f \otimes \eta_K, k)$ where η_K is the quadratic character associated to $K/\mathbb{Q}$. What we have shown says that if $L(f,k) = 0$ then $E(z, g_f)$ is holomorphic as a function of $z \in \mathcal{D}_{\Phi_{2,2}}$.

Remark 4.8. We can ensure a choice of Φ_S so that $M(\Phi_S, s_0)$ is nonzero. This requires some work. At least you expect this by reverse-engineering our argument starting from Bloch-Kato conjecture.

4.4. **Galois representation associated to Eisenstein series.** Let π be a cuspidal automorphic representation of $H = \mathrm{U}(\Phi_{a,b})$ and let χ be a Hecke character of $K^\times$. We had several restrictions, most notably π_∞ is some discrete series representation. Then we produced ρ a representation of $M(\mathbb{A})$, where M is the Levi of the Klingen parabolic of $G = \mathrm{U}(\Phi_{a+1,b+1})$. We studied Eisenstein series $E(\Phi, g, s)$, where $\Phi \in I(\rho)$, Φ_∞ being in also the appropriate discrete series. Let Π be any irreducible subquotient of $(\mathfrak{g}, K) \times G(\mathbb{A}_f)$-representation generated by $E(\Phi, g, s_0)$'s for such Φ inside the space of automorphic forms of G. Our objective is to show that there is a Galois representation R associated to Π, which means the standard (partial) L-functions are the same, such that $R = V \oplus \varepsilon \oplus 1$ for some self-dual automorphic V. Here ε is the cyclotomic character.

We know $\Pi \cong \otimes_{\ell \le \infty} \Pi_\ell$ where at almost all places (say $\ell \notin S$) Π_ℓ is unramified, and Π_ℓ is an irreducible quotient of the subrepresentation of $I_{s_0}(\rho_\ell)$ generated by spherical vector Φ^0_{ℓ,s_0}. Thus, the Langlands parameter of Π_ℓ can be read off from $I_{s_0}(\rho_\ell)$. To be more precise, first note that Π_ℓ for $\ell \notin S$ sits inside $\pi_\ell \hookrightarrow \mathrm{Ind}_{B_{\mathbb{Q}_\ell}}^{G_{\mathbb{Q}_\ell}} \psi_\ell$ for some unramified character ψ_ℓ of $T_{\mathbb{Q}_\ell} \subset B_{\mathbb{Q}_\ell}$; B and T are what you think. This is a general technique using the so-called **Jacquet functor** (this is just taking N-coinvariants for a unipotent radical N; this gives a way of associating an M-representation from a G-representation, for a parabolic $P = MN \subset G$) and that the Levi of B is T (if V is the space corresponding to π_ℓ then $V^{G(\mathbb{Z}_\ell)} \longrightarrow V_N^{T(\mathbb{Z}_\ell)}$ and the target is just an unramified character).

To calculate L-factors, we study the Langlands parameter. In this case a Langlands parameter is given by

$$\phi_\ell \ : \ W_{\mathbb{Q}_\ell} \longrightarrow {}^L G = \widehat{G} \rtimes W_{K/\mathbb{Q}}, \mathrm{Frob}_\ell \mapsto (t_\ell, \mathrm{Frob}_\ell),$$

where $\widehat{G} = \mathrm{GL}_{d+2}$ and $c \in \mathrm{Gal}(K/\mathbb{Q}) - \{1\}$ acts as $g \mapsto \Phi^{-1}{}^t g^{-1}\Phi$, $\Phi = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \cdots & & & \end{pmatrix}$.

From this, the standard local L-factor can be read off as follows. As the standard representation of ${}^L G$ in this case is $r_{\mathrm{std}} \ : \ {}^L G \longrightarrow \mathrm{GL}_{2(d+2)}(\mathbb{C})$, sending $(g, 1) \mapsto \begin{pmatrix} g & \\ & \Phi^{-1}{}^t g^{-1}\Phi \end{pmatrix}$ and $(1, c) \mapsto \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, the standard local L-factor is

$$L(\phi_\ell, s, r_{\mathrm{std}}) = \det(1 - \ell^{-s} r_{\mathrm{std}}(t_\ell, \mathrm{Frob}_\ell))^{-1}.$$

We from now on deliberately ignore inert ℓ's, although you have to check yourself ("okay because of Chebotarev density"). If ℓ splits, $G(\mathbb{Q}_\ell) \cong \mathrm{GL}_{d+2}(\mathbb{Q}_\ell)$, depending on a choice of $v \mid \ell$ in K, such that $B_{\mathbb{Q}_\ell}$ becomes upper triangular Borel and $T_{\mathbb{Q}_\ell}$ is the diagonal torus. Then t_ℓ is given by $(\psi_1, \cdots, \psi_{d+2})$, where $\psi_i = \psi(\mathrm{diag}(1, \cdots, \ell, \cdots, 1))$ where ℓ appears in the i-th entry. Thus, in this case, the standard local L-factor is

$$\prod_{i=1}^{d+2} (1 - \ell^{-s}\psi_i)^{-1}(1 - \ell^{-s}\psi_i^{-1})^{-1} = L(\mathrm{BC}(\Pi_\ell), s).$$

The above applies both to π_ℓ and Π_ℓ. To relate these, we have to understand the relation between ψ_ℓ's for Π_ℓ and π_ℓ. From now on until the end of this subsection, we put H in the subscript for everything related to H and do not put anything for everything related to G. Then, from $\pi_\ell \hookrightarrow \mathrm{Ind}_{B_{H,\mathbb{Q}_\ell}}^{H_{\mathbb{Q}_\ell}} \psi_{H,\ell}$, we have

$$I_{s_0}(\rho_\ell) \hookrightarrow \mathrm{Ind}_{P_{\mathbb{Q}_\ell}}^{G_{\mathbb{Q}_\ell}} (\mathrm{Ind}_{B_{H,\mathbb{Q}_\ell}}^{H_{\mathbb{Q}_\ell}} \psi_{H,\ell} \otimes \chi_\ell \delta_P^{s_0}) = \mathrm{Ind}_{B_{\mathbb{Q}_\ell}}^{G_{\mathbb{Q}_\ell}} (\psi_{H,\ell} \otimes \chi_\ell \delta_P^{s_0}).$$

Let $\chi = (\chi_1, \chi_2)$, coming from $K_{\mathbb{Q}_\ell}^\times \cong \mathbb{Q}_\ell^\times \times \mathbb{Q}_\ell^\times$. As the Levi M of the Klingen parabolic P can be written in a form

$$\left\{ \begin{pmatrix} \bar{t}^{-1} & & \\ & h & \\ & & t \end{pmatrix}, h \in H, t \in K^\times \right\},$$

we see that $B \cap M = B_H \times K^\times$, $T \cap M = T_H \times K^\times$; you know what these notations mean. Then $K_{\mathbb{Q}_\ell}^\times \subset \mathrm{GL}_{d+2}$ sits as $(a, b) \in \mathbb{Q}_\ell^\times \times \mathbb{Q}_\ell^\times \cong K_{\mathbb{Q}_\ell}^\times \mapsto \begin{pmatrix} b^{-1} & & \\ & 1_d & \\ & & a \end{pmatrix}$. Thus, a diagonal character ψ_ℓ for Π_ℓ sends $\begin{pmatrix} t & & \\ & t_H & \\ & & t' \end{pmatrix}$ to $\psi_H(t_H)\chi_1(t')\chi_2^{-1}(t)|t'/t|^{(d+1)s_0}$. In other words,

$$\psi_\ell = (\chi_2^{-1}|t|^{-(d+1)s_0}, \psi_H, \chi_1|t|^{(d+1)s_0}).$$

Thus, $L(\Pi_\ell, s, r_{\mathrm{std}})$ is

$$L(\pi_\ell, s, r_{\mathrm{std}})(1 - \chi_2^{-1}(\ell)\ell^{(d+1)s_0 - s})^{-1}(1 - \chi_1(\ell)\ell^{-(d+1)s_0 - s})^{-1}(1 - \chi_2(\ell)\ell^{-(d+1)s_0 - s})^{-1}(1 - \chi_1^{-1}(\ell)\ell^{(d+1)s_0 - s})^{-1},$$

which groups into

$$L(\Pi_\ell, s, r_{\mathrm{std}}) = L(\pi_\ell, s, r_{\mathrm{std}})L(\chi_\ell^c, s + (d+1)s_0)L(\chi_\ell^{-1}, s - (d+1)s_0),$$

yielding

$$L^S(\Pi, s, r_{\mathrm{std}}) = L^S(\pi, s, r_{\mathrm{std}})L^S(\chi^c, s + (d+1)s_0)L^S(\chi^{-1}, s - (d+1)s_0).$$

Remark 4.9. Note that in the split case there is little difference between χ_ℓ and χ_ℓ^c. Thus to really pin down the above global expression correctly, one needs to calculate at inert primes too.

Now we can see that there is a Galois representation $R : G_K \longrightarrow \mathrm{GL}_{d+2}(\overline{\mathbb{Q}}_p)$ such that $L^S(R, s) = L^S(\mathrm{BC}(\Pi)^\vee, s + \frac{d+2}{2} - \frac{1}{2})$; indeed the latter value is just

$$L^S(\mathrm{BC}(\pi)^\vee, s + \frac{d+2}{2} - \frac{1}{2})L^S(\chi^{-c}, s - (d+1)s_0 + \frac{d}{2} - \frac{1}{2})L^S(\chi, s + (d+1)s_0 + \frac{d}{2} - \frac{1}{2}),$$

or

$$L^S(\rho_\pi(1) \oplus \rho_\chi^{-c}\varepsilon^{\frac{d-(m+n)}{2}} \oplus \rho_\chi^c \varepsilon^{\frac{d+(m+n)}{2}+1}, s),$$

where the half integers in the exponent are actually integers because of the condition we had, $d \equiv m + n (\mathrm{mod}\, 2)$. Thus there really is a Galois representation R,

$$\begin{pmatrix} \rho_\pi(1) & & \\ & \rho_\chi^{-c}\varepsilon^{\frac{d-(m+n)}{2}} & \\ & & \rho_\chi \varepsilon^{\frac{d+(m+n)}{2}+1} \end{pmatrix},$$

whose partial L-function is the L-function of the Eisenstein series. This can be simplified: as $\chi' := \chi^{-1}|_{\mathbb{A}_\mathbb{Q}^\times} = |-|^{n+m}$, we see that $|-|_K^{n+m} = \mathrm{BC}(\chi') = \chi^{-1}\chi^{-c}$, so that $\rho_\chi^{-1}\rho_\chi^{-c} = \varepsilon^{n+m}$. Thus, R can be rewritten as

$$\begin{pmatrix} \rho_\pi(1) & & \\ & \rho_\chi^{-c}\varepsilon^{\frac{d-(m+n)}{2}} & \\ & & \rho_\chi^{-c} \varepsilon^{\frac{d-(m+n)}{2}+1} \end{pmatrix}.$$

Twisting by $\rho_\chi^c \varepsilon^{\frac{m+n-d}{2}}$, we get

$$\begin{pmatrix} \rho_\pi(1) \otimes \rho_\chi^c \varepsilon^{\frac{m+n-d}{2}} & & \\ & 1 & \\ & & \varepsilon \end{pmatrix}.$$

We expect that $H_f^1(K, V^\vee(1)) \neq 0$ iff $L(V, 0) \neq 0$, where $V = \rho_\pi(1) \otimes \rho_\chi^c \varepsilon^{\frac{m+n-d}{2}}$; note that indeed

$$\begin{aligned} L(V, 0) &= L(\rho_\pi(1) \otimes \rho_\chi^c \varepsilon^{\frac{m+n-d}{2}}, 0) \\ &= L(\mathrm{BC}(\pi)^\vee \otimes \chi^c, \frac{m+n+1}{2}) \\ &= L(\mathrm{BC}(\pi) \otimes \chi^{-1}, (d+1)s_0), \end{aligned}$$

by the functional equation, self-duality of π and $|-|_K^{n+m} = \chi^{-1}\chi^{-c}$.

Thus, to prove the desired instance of Bloch-Kato conjecture for a polarized Galois representation V, a general strategy is to find π, χ such that $V = \rho_\pi(1) \otimes \rho_\chi^c \varepsilon^{\frac{n+m-d}{2}}$, so that $L(V, 0) = 0$ implies $E(z, s_0)$ is holomorphic for the Eisenstein series assocaited to (π, χ). We want to deform the Eisenstein series to deform $R = V \oplus \varepsilon \oplus 1$. Note that deforming R is not straightforward, as one needs a nontrivial element in the Bloch-Kato Selmer group to argue that there is a deformation just from the general deformation theory argument using tangent space calculations.

5. p-adic deformation of automorphic forms

5.1. **p-adic deformation of modular forms.** In the classical situation of modular forms, a holomorphic Eisenstein series E_{2k} has two p-stabilizations (i.e. U_p-eigenforms of level $\Gamma_0(p)$ in the same representation space), the ordinary stabilization $E_{2k}^{\mathrm{ord}} = E(z) - p^{2k-1}E(pz)$ (slope 0) and the critical stabilization $E_{2k}^{\mathrm{crit}} = E(z) - E(pz)$ (slope $2k-1$). Any p-adic deformation, if it is p-adic analytic in a reasonable sense, must have a locally constant slope, so a p-adic deformation of E_{2k}^{ord} (E_{2k}^{crit}, resp.) must have constant slope 0 ($2k-1$, resp.) in an appropriate neighborhood of the weight space. Note however that in general an ordinary modular form has only one ordinary (=slope 0) deformation (Hida theory), and we already know one: the family of ordinary Eisenstein series. Namely, as the q-expansion of E_{2k}^{ord} is

$$E_{2k}^{\mathrm{ord}} = \frac{1}{2}\zeta(1-2k)(1-p^{2k-1}) + \sum_{n=1}^{\infty}\sum_{d\mid n,(p,d)=1} d^{2k-1}q^n,$$

we can naturally p-adically interpolate this q-expansion into

$$\mathbb{E}_{k_0,s} = \frac{1}{2}\zeta_p(-s) + \sum_{n=1}^{\infty}\sum_{d\mid n,(p,d)=1} \omega(d)^{2k_0-1}\langle d\rangle^s q^n,$$

for each residue disc $2k_0 - 1 \equiv 2k - 1(\mathrm{mod}(p-1))$, and $d = \omega(d)\langle d\rangle$.

In terms of Galois representations, this is less interesting. On the other hand, a deformation of critical Eisenstein series, if exists, will produce a generically cuspidal deformation, as an Eisenstein series of weight k' only can have either 0 or $k'-1$ as its slope. Indeed $E_k^{\mathrm{crit}}(2s\mathbb{E}_{0,s})$ ($2s$ is multiplied to clear out the simple pole) is a family of modular forms (not necessarily U_p-eigenforms), and we can use spectral theory of U_p-operator to decompose this into eigenfamilies. This approach is the original approach of Coleman but also quite ad hoc.

We take a topological approach to modular forms (as opposed to algebro-geometric approach) to make things more natural. Note that a holomorphic modular form can be seen as a meromorphic differential form on a closed modular curve. Namely, $M_k(\Gamma_1(N)) \hookrightarrow H^1(Y_1(N), \mathbb{L}_{k-2}) = H^1(\Gamma_1(N), \mathrm{Sym}^{k-2} V)$ where $V \cong \mathbb{C}^2$ is the standard representation of $\mathrm{GL}_2(\mathbb{R})$ and $\mathbb{L}_{k-2}$ is the vector bundle on $Y_1(N)$ induced from $\mathrm{Sym}^{k-2} V$; the equality is because $Y_1(N) = \mathbb{H}/\Gamma_1(N)$. Now note that by de Rham isomorphism, giving a differential on $Y_1(N)$ with values in $\mathbb{L}_{k-2}$ is the same as giving an infinitesimal differential at the origin (i.e. a linear map on the tangent space of the origin) tensored with a function on $\Gamma_1(N)\backslash \mathrm{GL}_2(\mathbb{R})^+$, in a compatible way. Namely,

$$H^1(Y_1(N), \mathbb{L}_{k-2}) \cong H^1\left((C^\infty(\Gamma_1(N)\backslash \mathrm{GL}_2(\mathbb{R})^+) \otimes \Lambda^i(\mathfrak{gl}_2/(\mathfrak{z}\cdot\mathfrak{so}_2))^\vee \otimes \mathrm{Sym}^{k-2} V)^{\mathbb{R}^\times \mathrm{SO}_2(\mathbb{R})}\right),$$

which is just the $(\mathfrak{g}, K)$-cohomology computation in this setting. A theorem of Borel (and Franke more generally) says that we can use the space of automorphic forms instead of the space of C^∞-functions.

Let $B = TN$ be the upper-triangular Borel of GL_2 and B^{op} be the lower-triangular Borel. Then consider $\mathcal{O}(N^{\mathrm{op}}\backslash \mathrm{GL}_2)$ (algebro-geometric functions), which has an action by T as $N^{\mathrm{op}}\backslash \mathrm{GL}_2 \longrightarrow B^{\mathrm{op}}\backslash \mathrm{GL}_2$ is a T-torsor. Given $\kappa = (k_1 \geq k_2)$, let

$$W_\kappa = \mathcal{O}(N^{\mathrm{op}}\backslash \mathrm{GL}_2)[\kappa] = \{f \in \mathcal{O}(N^{\mathrm{op}}\backslash \mathrm{GL}_2) \mid f\left(\begin{pmatrix} t_1 & \\ & t_2\end{pmatrix} g\right) = t_1^{k_1}t_2^{k_2}f(g)\}.$$

By just calculating (really a function on $N^{\mathrm{op}}\backslash \mathrm{GL}_2$ is a function of $a, b, \det g$, for $g = \begin{pmatrix} a & b \\ * & * \end{pmatrix}$), we see that this is, as an algebraic representation of GL_2, $\mathrm{Sym}^{k_1-k_2} \otimes \det^{k_2}$. Thus we see that $\mathcal{O}(N^{\mathrm{op}}\backslash \mathrm{GL}_2)$ contains all the algebraic representations of GL_2.

As $I_m = \left\{\begin{pmatrix} a & b \\ p^m c & d\end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p)\right\} = N^{\mathrm{op}}(p^m\mathbb{Z}_p)T(\mathbb{Z}_p)N(\mathbb{Z}_p)$ is dense in $\mathrm{GL}_2(\mathbb{Q}_p)$, we see that $W_\kappa(\mathbb{Q}_p)$ embeds into a space of functions $f : I_m \longrightarrow \mathbb{Q}_p$ such that $f(n'tn) = \kappa(t)f(n)$, where $n' \in N^{\mathrm{op}}(p^m\mathbb{Z}_p)$, $t \in T(\mathbb{Z}_p)$ and $n \in N(\mathbb{Z}_p)$. Thus the data of f is completely determined by its restriction to $N(\mathbb{Z}_p) \cong \mathbb{Z}_p$. Functions in W_κ are sent to f's which are polynomials on $N(\mathbb{Z}_p) = \mathbb{Z}_p$ of degree $\leq k_1 - k_2$, and we can imagine ourselves of enlarging the space appropriate for p-adic interpolation by considering locally analytic functions (of some fixed convergence radius) on $N(\mathbb{Z}_p) = \mathbb{Z}_p$. Also we can use a p-adic weight $\kappa \in \mathfrak{X}(L)$, where $L/\mathbb{Q}_p$ is a finite extension and $\mathfrak{X}$ is the weight space, i.e. $\mathfrak{X}(L) = \mathrm{Hom}_{\mathrm{cts}}(T(\mathbb{Z}_p), L^\times)$. Namely, if we define

$$A_{\kappa,r}(I_m, L) = \{f : I_m \longrightarrow L \mid f(n'tn) = \lambda(t)f(n),\ f(n) \text{ locally analytic of radius } p^{-r}\},$$

then $W_\kappa(L) \hookrightarrow A_{\kappa,r}(I_m, L)$ for arithmetic κ. Note that one can recover $W_\kappa(L)$ from $A_{\kappa,r}(I_m, L)$'s, namely

$$W_\kappa(L) = \ker(A_{\kappa,0}(I_m, L) \xrightarrow{\text{differentiate } k_1 - k_2 + 1 \text{ times}} A_{\kappa,0}(I_m, L)).$$

As $A_{\kappa,r}$'s have actions by I_m and as $\Gamma_0(p^m) \cap \Gamma_1(N) \subset \Gamma_1(p^m N)$, we have

$$M_k(\Gamma_1(N), L) \subset H^1(\Gamma_1(p^m N), A_{\lambda,r}(I_m, L)).$$

We still have to know how to interpolate $A_\lambda(I_m, L)$'s in terms of λ.

An approach of p-adic interpolation uses p-adic distributions instead,

$$D_{\lambda,r}(I_m, L) = \mathrm{Hom}_{\mathrm{cts}}(A_{\lambda,r}(I_m, L), L),$$

so that they have $W_\kappa^\vee \cong W_\kappa$ as quotients instead. This has an advantage that the classicality of p-adic modular forms is very easy to see, granted that the cohomology groups are nice spaces with an action of U_p-operator so that there is a so-called **slope decomposition** (nowadays one usually does everything on the level of complexes, not cohomologies). In particular, in this case, because $\Gamma_1(p^m N)$ has cohomological dimension 1, we have an exact sequence

$$H^1(\Gamma_1(p^m N), D_{\kappa,r})^{\leq h-\kappa} \xrightarrow{\theta_\kappa^*} H^1(\Gamma_1(p^m N), D_{\kappa,r})^{\leq h} \longrightarrow H^1(\Gamma_1(p^m N), W_\kappa)^{\leq h} \longrightarrow 0,$$

where $\leq h$ denotes the slope-less-than-h part of the cohomologies and θ_κ^* is the dual of the "θ-operator." Thus, if $h < \kappa$, the image of θ_κ^* is trivially zero (slope is always nonnegative), so the exactness of the sequence gives you classicality.

Remark 5.1. In this setting, which is "dual" to the setting of p-adic modular forms à la Katz, it is a bit trickier to see that a classical modular form is a p-adic modular form.

After a simple speculation we can develop a strategy of how to vary p-adic automorphic forms V_k with respect to weight. A problem is that p^k is, as a function of k, not an analytic function. We can however try to do the following.

- For a sufficiently regular weight, we have an ordering of Frobenius eigenvalues $a_1, \cdots, a_n$ which is ordered so that slopes are increasing. They are expected to have slopes being certain functions coming from the weight, calculable from Hodge-Tate weights.
- The "unit root" (slope 0 part) is expected to vary p-adically anayltically.
- More generally, $D_{\mathrm{cris}}(\wedge^i V_k)$ has Frobenius eigenvalues $a_{j_1} \cdots a_{j_i}$ for $j_1 < \cdots < j_i$, and it has a unique eigenvalue of smallest slope, namely $a_1 \cdots a_i$.
- Thus, $D_{\mathrm{cris}}(\wedge^i V_k(\text{appropriate twist}))$ must also have "unit root", and this should vary p-adically analytically.
- In turn, this gives p-adic analytic variations of a_1, $a_1 a_2 p^{\text{some function}}$, $\cdots$, $a_1 a_2 \cdots a_n p^{\text{some function}}$, or rather said differently, $a_1, a_2 p^{\text{some function}}, \cdots, a_n p^{\text{some function}}$.

This will be justified by Kisin's work on crystalline periods.

5.2. **Finite slope automorphic forms.** We make things precise in the case of $G = \mathrm{U}(\Phi_{a,b})$. Let p be a prime that splits in K, $p = v\overline{v}$. For a cuspidal automorphic representation π of $G(\mathbb{A})$, suppose that π_p, seen as a representation of $\mathrm{GL}_d(\mathbb{Q}_p)$, is unramified, so that π_p sits inside an unramified principle series $I(\chi)$, where χ is some unramified character (really the Jacquet functor of π_p) and $I(\chi)$ is the unitary induction.

For

$$t \in T^+(\mathbb{Q}_p) = \left\{ \begin{pmatrix} t_1 & & \\ & \cdots & \\ & & t_d \end{pmatrix} \mid \left| \frac{t_i}{t_{i+1}} \right|_p \leq 1 \right\},$$

we will associate the analogue of U_p-operator, u_t, as $I_0(p)tI_0(p)$, where $I_0(p)$ is the Iwahori subgroup

$$I_0(p) = \{g \in \mathrm{GL}_d(\mathbb{Z}_p) \mid g(\mathrm{mod}\, p) \in B(\mathbb{F}_p)\}.$$

Let $\mathscr{U}_p$ be the $\mathbb{Z}_p$-algebra generated by u_t's inside $C_c(I_0(p)\backslash \mathrm{GL}_d(\mathbb{Q}_p)/I_0(p))$. This is a commutative algebra as $u_{t_1}u_{t_2} = u_{t_1t_2}$, and it acts on $I(\chi)^{I_0(p)}$ that preserves $\pi_p^{I_0(p)}$. Note that u_t can be written also as $u_t = \coprod_{n \in N(\mathbb{Z}_p)/tN(\mathbb{Z}_p)t^{-1}} ntI_0(p)$ (because what happens on the lower-triangular unipotent and the diagonal has no effect, and $tN(\mathbb{Z}_p)t^{-1} \subset N(\mathbb{Z}_p)$ precisely because $t \in T^+(\mathbb{Q}_p)$).

A nice thing about Iwahori subgroup is that we have a Bruhat decomposition $\mathrm{GL}_d(\mathbb{Q}_p) = \coprod_{w \in W_{\mathrm{GL}_d}} B(\mathbb{Q}_p)wI_0(p)$ coming from the Bruhat decomposition of $\mathrm{GL}_d(\mathbb{F}_p)$.

Proposition 5.1. $\dim I(\chi)^{I_0(p)} = d!$. *More precisely, there is an $\mathscr{U}_p$-eigenfunction $\varphi_w \in I(\chi)^{I_0(p)}$ such that $u_z\varphi_w = \chi(t)\delta_B(t)^{-1/2}\varphi_w$.*

Proof. Note that $I(\chi)_N^{B(\mathbb{Z}_p)} = \oplus \chi^w \delta_B \delta^{1/2}$, so the u_t-eigenvalue on it is

$$\chi^w(t)\delta_B^{1/2}(t)\#\{\text{involved } n \in N(\mathbb{Z}_p)/tN(\mathbb{Z}_p)t^{-1}\},$$

and the number of involved n's is $\delta_B^{-1}(t)$. □

Thus, $\mathscr{U}_p$-eigenvalues of π_p are contained in the set of possible $\mathscr{U}_p$-eigenvalues of $I(\chi)$. This means that, for an $\mathscr{U}_p$-eigenvector $v \in \pi_p^{I_0(p)}$, then there exist $\alpha_1, \cdots, \alpha_d$ such that $u_tv = \prod_i \alpha_i^{\mathrm{ord}_p(t_i)}v$, where $\alpha_i = \delta^{-1/2}\chi^w \begin{pmatrix} 1 & & & & \\ & \cdots & & & \\ & & p & & \\ & & & \cdots & \\ & & & & 1 \end{pmatrix}$. For example, if $\pi_p = I(\chi)$ (when π_p is tempered), this is exactly the same as choosing an ordering of Satake parameters.

Definition 5.1. *A* ***finite slope*** *automorphic representation of $G(\mathbb{A})$ is an automorphic representation π of $G(\mathbb{A})$ such that*

1. *π_∞ is a discrete series (e.g. D_W^+ with $W = (c_{b+1} \geq \cdots \geq c_d; c_1 \geq \cdots \geq c_b)$, $c_b - c_{b+1} \geq d$),*
2. *$\pi_p^{I_0(p)} \neq 0$, together with $\theta : \mathscr{U}_p \longrightarrow \mathbb{C}$ a homomorphism such that $\theta(u_t) \neq 0$ for all $t \in T^+(\mathbb{Q}_p)$,*
3. *there exists $0 \neq v \in \pi_p^{I_0(p)}$ a $\mathscr{U}_p$-eigenvector with eigenvalues given by θ.*

Then, $\pi^p \otimes \theta$ can be seen as a representation of $G(\mathbb{A}_f^p) \otimes \mathscr{U}_p$ occuring in $\pi^{I_0(p)} = \pi^p \otimes \pi_p^{I_0(p)}$. This is the one that will be p-adically interpolated.

Example 5.1. If $\pi_p \cong I(\chi)$ unramified, then the choice of θ is exactly the same as the choice of χ^w, or the ordering of parameters.

A justification that this can be hoped to be p-adically interpolated is as follows. Note that $M_W(K^pI_0(p))$, the space of holomorphic modular forms of weight W, sits inside, by the Eichler-Shimura map, $H^{ab}(\mathrm{Sh}_G(K^pI_0(p)), L)$, where L is the local system coming from the representation of $G(\mathbb{C}) \cong \mathrm{GL}_d(\mathbb{C})$ with highest weight

$$\lambda = (c_1 - a \geq \cdots \geq c_b - a \geq c_{b+1} + b \geq \cdots \geq c_d + b),$$

(e.g. $(k, 0)$ on U(1, 1) corresponds to $\mathrm{Sym}^{k-2} \otimes \det$ as usual) and a $\mathscr{U}_p$-eigenvector $v^p \otimes v_p \in \pi^{K^pI_0(p)}$ corresponding to θ goes into a cohomology class which is also an $\mathscr{U}_p$-eigenvector with eigenvalues $\tilde{\theta}(t) = \theta(t)\left|\prod t_i^{\lambda_i}\right|_p^{-1}$. This now satisfies nice p-integrality as every Hecke action comes from correspondences, so this has a hope of being p-adically interpolated.

Let us be more precise. If the Hodge-Tate weights of a crystalline Galois representation V are given by $(k_1, \cdots, k_d)$, and if $\alpha_1, \cdots, \alpha_d$ are the eigenvalues of the crystalline Frobenius on $D_{\mathrm{cris}}(V)$, then the **slope** of the family of the deformation of the Galois representation is $s = (s_1, \cdots, s_d)$ where $s_i = \mathrm{ord}_p \beta_i$ and $\alpha_i = \beta_i p^{k_i}$. In terms of the weight of the automorphic form, if π is a cuspidal automorphic representation of $G(\mathbb{A})$, $G = \mathrm{U}(\Phi_{a,b})$, with $\pi_\infty = D_W^+$, $W = (c_{b+1} \geq \cdots \geq c_d; c_1 \geq \cdots \geq c_b)$, $c_b - c_{b+1} \geq d$, then the Hodge-Tate weights of $\rho_\pi|_{G_{K_v}}$ is $c_1 + d - 1 - a > \cdots > c_b + d - b - a > c_{b+1} + d - (b+1) + b > \cdots > c_d + b$. Also, $L(\rho_\pi, s) = L(\mathrm{BC}(\pi)^\vee, s + \frac{1-d}{2})$, and the local L-factor of $L(\mathrm{BC}(\pi)^\vee, s + \frac{1-d}{2})$ at p is $\prod(1 - \chi_i^{-1}(p)p^{\frac{d-1}{2}-s})$, where $\mathrm{BC}(\pi_p)_v \hookrightarrow \mathrm{Ind}(\chi)$ for an unramified character χ. Thus, in terms of the Langlands parameter, the slopes of Galois representation are

$$\mathrm{ord}_p\left(\chi_i^{-1}(p)p^{\frac{d-1}{2}-\left(c_i+(d-i)+\begin{cases}-a & i \leq b \\ b & i > a\end{cases}\right)}\right).$$

How is this, coming from the Galois side, compatible with our previous discussion of slopes in the automorphic side? Suppose that our choice of ordering is just compatible with χ. Then $\tilde{\theta}(t) = \chi(t)\delta_B(t)^{-1/2}|\lambda_W(t)|_p^{-1}$, where δ_B has the formula $\delta_B(t) = |\prod t_i^{d-2i+1}|_p$, and

$$\lambda_W = (c_1 - a, \cdots, c_b - a, c_{b+1} + b, \cdots, c_d + b),$$

is the highest weight of the algebraic representation of G corresponding to the local system over the unitary Shimura variety into which the weight W holomorphic automorphic forms are embedded via Eichler-Shimura,

$$M_W(K^pI_0(p)) \hookrightarrow H^{ab}(\mathrm{Sh}_G(K^pI_0(p)), L_{\lambda_W}).$$

This evaluated at $\begin{pmatrix} 1 & & & & \\ & \cdots & & & \\ & & p & & \\ & & & \cdots & \\ & & & & 1 \end{pmatrix}$, p at the i-th place, is

$$\chi_i(p)|p|_p^{-\left(\frac{d-2i+1}{2}\right)}|p|_p^{-\left(c_i+\begin{cases}-a & i\le b\\ b & i>a\end{cases}\right)},$$

which has exactly ord_p the negative of the slope we calculated above. Thus, that a p-adic family should have the constant slope is compatible with our previous expectation that $\tilde{\theta}(t)$ should be the one that can be varied p-adic analytically.

5.3. **Eigenvarieties.** The weight space $\mathfrak{X}$ is a rigid space over $\mathbb{Q}_p$ such that $\mathfrak{X}(L) = \mathrm{Hom}_{\mathrm{cts}}(T(\mathbb{Z}_p), L^\times)$.

Definition 5.2. *An* ***eigenvariety*** *for an open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$ is a rigid analytic variety $\mathscr{E}_{K^p}$ that sits inside a commutative diagram*

$$\begin{tikzcd} \mathscr{E}_{K^p} \arrow[r] \arrow[d, "s"'] & \mathrm{MaxSpec}(R^S \otimes \mathscr{U}_p) \times \mathfrak{X} \arrow[dl, "\mathrm{proj}_2"] \\ \mathfrak{X} & \end{tikzcd}$$

where

- *S is the set of primes such that*
  - *$\ell = p$, or*
  - *ℓ divides $\mathrm{disc}\, K$, or*
  - *$\dim \Phi_{a,b}$ is not an ℓ-unit, or*
  - *K_ℓ^p is not maximal,*
- *$R^S = \otimes'_{\ell\notin S} C_c(K_\ell \backslash G(\mathbb{Q}_\ell)/K_\ell, \mathbb{Z}_p)$,*

such that it satisfies the following properties.

- *s is flat and generically finite; the irreducible components of $\mathcal{E}_{K^p}$ have the same dimension, $\dim \mathfrak{X} = d$.*
- *If π is an automorphic representation with π_∞ holomorphic discrete series of weight W such that $\pi^{K^p I_0(p)} \neq 0$ with θ a p-stabilization of π_p, then there is $x \in \mathscr{E}_{K^p}(\overline{\mathbb{Q}}_p)$ such that $x \mapsto (\sigma_\pi, \tilde{\theta}, \lambda_W)$,*
- *If $x \in \mathscr{E}(\overline{\mathbb{Q}}_p)$ giving rise to $(\sigma_x, \theta_x, \lambda_W)$, and if θ_x is* ***non-critical****, which means $|s_i - s_{i+1}| < 2|c_i - c_{i+1} - 1|$, then $(\sigma_x, \theta_x, \lambda_w) = (\sigma_\pi, \tilde{\theta}, \lambda_W)$.*

Such object, at least in our setting, is constructed by Urban.

Example 5.2. Given an elliptic curve with Hecke polynomial at p, $x^2 - a_p x + p$, with roots α_1, α_2 where α_1 is a p-unit, there are two stabilizations, (α_1, α_2) and (α_2, α_1). Then the corresponding slopes are the ord_p's of $(\alpha_1, \alpha_2 p^{-1})$ and $(\alpha_2, \alpha_1 p^{-1})$, which are $(0,0)$ (ordinary stabilization) and $(1,-1)$ (critical stabilization; $4 = 2 \times |2-0-1|$).

Now start with π, $\pi_\infty = D^+_{W_0}$, and a normalized p-stabilization $\tilde{\theta}$ of $(\pi_p$, and the existence of eigenvariety means there is a point $x_0 \in \mathscr{E}(\overline{\mathbb{Q}}_p)$. Let $\lambda_0 = \lambda_{W_0} \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$. Then, the properties of eigenvariety imply that there is a small ball $\mathscr{U} \subset \mathfrak{X}$ of λ_0 and an affinoid neighborhood $\mathscr{A} \subset \mathscr{E}$ of x_0, admitting a finite flat map to $\mathscr{U}$. This map gives a homomorphism $R^S \otimes \mathscr{U}_p \longrightarrow \mathcal{O}(\mathscr{A})$ such that, for any $\phi : \mathcal{O}(\mathscr{A}) \longrightarrow \overline{\mathbb{Q}}_p$ sitting over an algebraic homomorphism $\lambda_W \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$, the composition $R^S \otimes \mathscr{U}_p \longrightarrow \mathcal{O}(\mathscr{A}) \xrightarrow{\phi} \overline{\mathbb{Q}}_p$ comes from $\sigma_{\pi_\phi} \otimes \tilde{\theta}_{\pi_\phi}$ for an automorphic form π_ϕ of weight W, if $\tilde{\theta}_{\pi_\phi}$ is non-critical.

Remark 5.2.
- Non-critical points are dense as the slope stays constant and a dense set of points has arbitrarily highly regular weights.
- The use of this kind of p-adic family is important in constructing Galois representations. For automorphic forms of low weight, we can try to put it in an eigenvariety and obtain a p-adically varying pseudocharacters which specialize to actual geometric Galois representations at sufficiently regular weights. From these we can realize a specialization at the desired low weight to be a pseudocharacter with desired properties so that we could construct a Galois representation with desired properties.
- The p-adic limit process does not say much about properties at p, e.g. how do you know that the resulting Galois representation is geometric/crystalline/semistable, what are the (crystalline) Frobenius eigenvalues? Several tricks may solve this by using specialized subfamilies, e.g. you fix one crystalline Frobenius eigenvalue and let others vary.

Now we want to use the eigenvariety to get a p-adic deformation of Eisenstein series. This is done in the following steps.

(1) Find a p-stabilized Eisenstein series on the eigenvariety. This is not naively possible becuase for low regular weight the same Hecke eigensystem can appear in multiple cohomological degrees which is bad as Urban's eigenvariety is constructed by taking the alternating sum of traces over the full complex that computes the cohomology. On the other hand one has a trick of multiplying by high powers of a lift of Hasse invariant so that one finds the desired point by taking p-adic limit.
(2) Just that we have a point gives a deformation of Hecke eigensystem. Namely say we have a point $(\sigma_{\Pi_0}, \tilde{\theta}_0)$ over $w_0 \in \mathfrak{X}$, then there is an affinoid $w_0 \in \mathscr{U} \subset \mathfrak{X}$ and $x_0 \in \mathscr{A} \subset \mathscr{E}$ such that $\mathscr{A} = s^{-1}(\mathscr{U})$ and there is a p-adic deformation, namely $\sigma_{\mathscr{A}} : R^S \longrightarrow \mathcal{O}(\mathscr{A})$ and $\theta_{\mathscr{A}} : \mathscr{U}_p \longrightarrow \mathcal{O}(\mathscr{A})$.
(3) For a suitable p-stabilization the deformation we get is cuspidal. Note that this is special for overconvergent deformation. Indeed, even in the GL_2-case, one can find a family of p-adic modular forms of E^{crit}_{2k} by extracting an eigensystem from the finite slope projector applied to $\theta^{2k-1} E^{\mathrm{ord}}_{2-2k+2\ell}$. Now we note that if we take the p-stabilization so that the central two slopes, the spots we shoved a character into, are $(-1, 1)$, then we see that this is a gap that cannot be realized as slopes of Eisenstein series of sufficiently regular weight as p-stabilized Eisenstein series pick Hodge-Tate weights as U_p-eigenvaleus.

(4) From this we want to get a Galois representation $G_K \longrightarrow \mathrm{GL}_{d+2}(\mathcal{O}(\mathscr{A}))$. But this is also not literally possible precisely because the eigenvariety might not be étale over the weight space. The crutch here is that we take a 1-dimensional sub-deformation and then we can resolve the curve. Let $A = L\langle p^{-r}T\rangle$ and B be the preimage of A, finite integral over A. Then for any arithmetic weight $\phi \in \mathrm{Hom}_L(B, \overline{\mathbb{Q}}_p)$, the pseudocharacter $G_K \longrightarrow \overline{\mathbb{Q}}_p$ coming from the corresponding Galois representation can be recovered by $R^S \longrightarrow B \xrightarrow{\phi} \overline{\mathbb{Q}}_p$ as we know Hecke operators and Frobenii go to the same element. Thus, by the Zariski density of ϕ in B, we can construct a pesudocharacter $G_K \longrightarrow B$. By the theory of pseudocharacters we can then recover a semisimple Galois representation $\rho_B : G_K \longrightarrow \mathrm{GL}_{d+2}(\mathrm{Frac}(B))$. Now as G_K is compact and A is a Dedekind domain, we can invert finitely many primes so that ρ_B can be realized to land in $\mathrm{GL}_{d+2}(B')$ for some enlargement B' of B. In fact one can just shrink A enough so that we can avoid any bad places.

(5) The Galois deformation $\rho : G_K \longrightarrow \mathrm{GL}_{d+2}(B)$ is generically irreducible, where $\rho_{\phi_0}^{\mathrm{ss}} = \rho_\pi \oplus \chi' \oplus \chi'\varepsilon$, where Π_0 is induced from (π, χ). We can even choose our line A so that the deformation is generically crystalline. Suppose not; how can $V_1 \oplus V_2$ can degenerate into $\rho_\pi \oplus \chi' \oplus \chi'\varepsilon$? We might have V_1 a d-dimensional irreducible chunk, then V_2 is either a sum of characters or a 2-dimensional irreducible representation.
   - It cannot be a sum of characters, because these should also agree with one of the Hodge-Tate weights.
   - It cannot be two-dimesnional also, because the resulting extension gives a nonzero element in $H^1_f(K, \overline{\mathbb{Q}}_p(1) = 0$ (this uses that K is an imaginary quadratic field; does not work for a general CM field). That the nonsplit extension is in H^1_f because we know the deformation is generically crystalline and we know crystalline periods stay there (Kisin) so that D_{cris} has the full 2-dimension.

   Thus V_1 is $(d+1)$-dimensional and V_2 is a character. But again then V_2 must see one of the Hodge-Tate weights which is a contradiction. Thus, ρ has to be generically irreducible.

Remark 5.3. The generic irreducibility of ρ is nontrivial because an endoscopic form has a reducible Galois representation. Moreover, even though it is expected, the irreducibility of the Galois representaion associated to a stable cusp form is not proved in general.

Indeed, by thinking in the reverse way from the Bloch-Kato conjecture, any p-adic deformation of the Eisenstein series with nonvanishing critical L-value should end up being an endoscopic family.

Now we sketch why this gives a nonzero element in $H^1_f(K, V^\vee(1))$. By the lattice construction certainly we can take a free lattice inside ρ_B so that V arises as the quotient of the lattice. Then taking the dual and twist, we get a representation of form

$$\begin{pmatrix} V^\vee(1) & *_1 & *_2 \\ & \chi^{-1}\varepsilon & *_3 \\ & & \chi \end{pmatrix},$$

and a similar argument of "crystalline period of $\chi^{-1}\varepsilon$ survives in the p-adic limit," as well as the purity of $V^\vee(1)$ (to see that the crystalline periods are different from that of $\chi^{-1}\varepsilon$), shows that the extension $*_1$ is in H^1_f.