THE VORONOI FORMULA FOR $GL(n, \mathbb{R})$

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Abstract. A Voronoi formula is an identity $A = B$ where A is a weighted sum over Fourier coefficients of an automorphic form and B is another weighted sum involving the Fourier coefficients of the dual automorphic form. The weights in A are additive characters multiplied by a test function while the weights in B are Kloosterman sums multiplied by a suitable transform of the test function. We derive an explicit Voronoi formula for even Maass forms in $\mathcal{L}^2\big(SL(n,\mathbb{Z})\backslash GL(n,\mathbb{R})/O(n,\mathbb{R})\cdot\mathbb{R}^\times\big)$ for all $n \ge 3$.

§1. Introduction.

Let h be a smooth compactly supported function on $\mathbb{R}^+$ and let

$$r(m) = \#\Big\{m_1, m_2 \in \mathbb{Z} \;\Big|\; m_1^2 + m_2^2 = m\Big\},$$

where # denotes the cardinality of a set. The classical Voronoi formula (first proved in [V]) states that

$$\sum_{m=1}^{\infty} r(m)h(m) = \pi \int_0^{\infty} h(x)\,dx \;+\; \sum_{m=1}^{\infty} r(m)H(m) \tag{1.1}$$

where

$$H(y) = \pi \int_0^{\infty} h(x) J_0\left(2\pi\sqrt{xy}\right)\,dx.$$

This formula immediately implies (see [I-K, Corollary 4.9]) that

$$\sum_{1\le m\le x} r(m) = \pi x + \mathcal{O}\left(x^{\frac{1}{3}}\right),$$

which greatly improves on Gauss' estimate of $x^{\frac{1}{2}}$ for the error term in the classical circle problem.

The Voronoi formula (1.1) may be viewed as an identity arising from the fact that Eisenstein series are modular forms, and from this point of view, may be generalized (see [I-K], [M-S1], [M-S2]) to arbitrary automorphic forms on $GL(2)$. For example [M-S2], if

$$\sum_{m\neq 0} a_m \sqrt{y} K_\nu(2\pi|m|y) e^{2\pi i m x}$$

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is an even Maass form of type ν for $SL(2,\mathbb{Z})$, then the Voronoi formula takes the form

$$\sum_{m\neq 0} a_m e^{\frac{-2\pi i m a}{c}} h(n) \;=\; |c| \sum_{m\neq 0} \frac{a_m}{|m|} e^{\frac{2\pi i m \bar{a}}{c}} H^*\left(\frac{n}{c^2}\right), \tag{1.2}$$

where $a, c \in \mathbb{Z}$ with $c \neq 0$, $(a,c)=1$, $a\bar{a} \equiv 1 \pmod{c}$,

$$H^*(y) = \frac{1}{2\pi^2 i} \int\limits_{\Re(s)=\sigma} \frac{\Gamma\left(\frac{1+s+\nu}{2}\right)\Gamma\left(\frac{1+s-\nu}{2}\right)}{\Gamma\left(\frac{-s+\nu}{2}\right)\Gamma\left(\frac{-s-\nu}{2}\right)} \tilde{h}(-s)\, ds,$$

and

$$\tilde{h}(s) = \int\limits_0^\infty h(x)\, x^s \, \frac{dx}{x}$$

is the Mellin transform of h.

The Voronoi formula (1.2) was first generalized to $GL(3)$ by Miller and Schmidt [M-S2]. A simpler proof of the Miller-Schmidt $GL(3)$ Voronoi formula was found shortly after (see [G-L]) by the authors of this paper. In [G-L], a Voronoi formula for Maass forms (twisted by additive characters of prime conductor) on $GL(n)$ was obtained for all $n \geq 3$. It is the object of this paper to remove the restriction on prime conductor and derive a very general Voronoi formula for Maass forms for $GL(n)$ (twisted by additive characters) with $n \geq 3$. We now state the main theorem of this paper.

We freely adopt the notation of [G]. Let $e(x) = e^{2\pi i x}$ denote the standard exponential function. For $n \geq 2$, let

$$\mathfrak{h}^n = GL(n,\mathbb{R})/(O(n,\mathbb{R})\cdot \mathbb{R}^\times) \tag{1.3}$$

be the generalized upper half plane. Each element $z \in \mathfrak{h}^n$ takes the form $z = x \cdot y$ where

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & & x_{1,n} \\ & 1 & x_{2,3} & \cdots & & x_{2,n} \\ & & \ddots & & & \vdots \\ & & & & 1 & x_{n-1,n} \\ & & & & & 1 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}, \tag{1.4}$$

with $x_{i,j} \in \mathbb{R}$ for $1 \leq i < j \leq n$ and $y_i > 0$ for $1 \leq i \leq n-1$. We also adopt the notation that $\operatorname{diag}(d_1, d_2, \ldots, d_n)$ denotes the diagonal $n \times n$ matrix $\begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}$. For $n \geq 2$, let

$$f(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^\infty \cdots \sum_{m_{n-2}=1}^\infty \sum_{m_{n-1}\neq 0} \frac{A(m_1,\ldots,m_{n-1})}{\prod\limits_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}} W_J\left(M \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, \nu, \psi_{1,\ldots,1,\frac{m_{n-1}}{|m_{n-1}|}}\right) \tag{1.5}$$

be a Maass form of type ν for $SL(n,\mathbb{Z})$ (see [G, 9.1.2]) where W_J denotes the Jacquet Whittaker function and $M = \operatorname{diag}\big(m_1 \cdots m_{n-2}|m_{n-1}|,\ m_1 \cdots m_{n-2},\ \ldots\ , m_1,\ 1\big)$.

Main Theorem. *Fix $n \geq 3$. Let f be an even Maass form as in (1.5) and let ϕ be a smooth compactly supported function on R^+. Then for integers h, q with $q \neq 0$, $(h,q)=1$, we have*

$$\sum_{m\neq 0} A(1,\ldots,1,m)e\left(\frac{m\bar{h}}{q}\right)\phi(|m|)$$

$$= q\sum_{d_1|q}\sum_{d_2|\frac{q}{d_1}}\cdots\sum_{d_{n-2}|\frac{q}{d_1\cdots d_{n-3}}}\sum_{m\neq 0}\frac{A(m,d_{n-2},\ \ldots\ ,d_2,d_1)}{d_1\cdots d_{n-2}\,|m|}KL(h,m;d,q)\ \Phi\left(\frac{|m|\prod_{i=1}^{n-2} d_i^{n-i}}{q^n}\right),$$

where $h\bar{h}\equiv 1 \pmod q$, $d=(d_1,d_2,\ldots,d_{n-2})$,

$$KL(h,m;d,q) = \sum_{\substack{t_1=1\\(t_1,\frac{q}{d_1})=1}}^{\frac{q}{d_1}} e\left(\frac{(-1)^n h t_1}{q/d_1}\right)\sum_{\substack{t_2=1\\(t_1,\frac{q}{d_1d_2})=1}}^{\frac{q}{d_1d_2}} e\left(\frac{\bar{t}_1\,t_2}{q/(d_1d_2)}\right)$$

$$\cdots\sum_{\substack{t_{n-2}=1\\(t_{n-2},\frac{q}{d_1\cdots d_{n-2}})=1}}^{\frac{q}{d_1\cdots d_{n-2}}} e\left(\frac{\bar{t}_{n-3}\,t_{n-2}}{q/(d_1\cdots d_{n-2})}\right)\cdot e\left(\frac{m\,\bar{t}_{n-2}}{q/(d_1\cdots d_{n-2})}\right)$$

is the hyper Kloosterman sum, and $\Phi(x)$ is a certain transform of ϕ given in (7.4).

The proof of the main theorem above is given in §7 and is based on [J-P-S] (see also [B], [G]). It contains a new non-adelic proof of the functional equation (see (7.2) with $h=0$, $q=1$) of the standard L-function for $GL(n)$. We also remark that the method in this paper will also give a proof of the Voronoi formula for Eisenstein series on $GL(n)$, i.e., for n-tuples of divisor functions. In this case, however, much simpler proofs are known, (see [Iv], [B-B]). A potential application of the Voronoi formula is to break the convexity bound of the $GL(n)$ L-functions. In the second author's paper [Li], the Voronoi formula on $GL(3)$ was used to break the convexity bound of the triple L-functions in the splitting case. Other applications to cancellations of additively twisted sums on $GL(n)$ were made by Miller [Mi].

§2. Notation.

For a field $\mathbb{F}$, let $U_n(\mathbb{F}) = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$ denote the unipotent upper triangular matrices in $GL(n,\mathbb{F})$. We introduce the following matrices in $GL(n,\mathbb{R})$, with $n \geq 3$.

$$w = \begin{pmatrix} (-1)^n & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & (-1)^{n-2} \\ \vdots & \vdots & \vdots & \iddots \iddots & \vdots \\ \vdots & \vdots & \iddots & (-1)^2 & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}, \quad w^{-1} = \begin{pmatrix} (-1)^n & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & (-1)^{n-2} & \cdots & 0 & 0 \end{pmatrix} \tag{2.1}$$

$$\hat{u}_n = \begin{pmatrix} 1 & 0 & u_{1,3} & \cdots & u_{1,n} \\ & 1 & u_{2,3} & \cdots & u_{2,n} \\ & & 1 & \cdots & \vdots \\ & & & \ddots & u_{n-1,n} \\ & & & & 1 \end{pmatrix}, \tag{2.2}$$

For integers h, q with $(h, q) = 1$ and $q \neq 0$, we define the matrix:

$$A_{h/q} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{h}{q} & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{2.3}$$

§3. The basic identity.

Fix $n \geq 2$. For $z \in \mathfrak{h}^n$, let $f(z)$ be a Maass form of type $\nu = (\nu_1, ..., \nu_{n-1})$ for $SL(n, \mathbb{Z})$. The dual Maass form $\tilde{f}$ is defined by

$$\tilde{f}(z) = f\left(w_0 \cdot {}^t z^{-1}\right),$$

is a Maass form of type $\tilde{\nu} = (\nu_{n-1}, ..., \nu_1)$ where

$$w_0 = \begin{pmatrix} & & & \pm 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$$

is the long element of the Weyl group associated to $SL(n)$. Since f is automorphic it is easily seen that

$$f(z) = \tilde{f}\left(w \cdot {}^t z^{-1}\right),$$

where w is given by (2.1). Fix rational integers h, q with $q \neq 0$ and $(h,q) = 1$. We now choose $z = A_{h/q} \cdot \hat{u}y$ with $A_{h/q}$ given by (2.3), $\hat{u} = \hat{u}_n$ given by (2.2), and y given by (1.4). The basic identity we employ for computing the Voronoi formula is

$$\int_0^1 \cdots \int_0^1 f\left(A_{h/q} \cdot \hat{u}y\right) e\big(-qu_{2,3} - u_{3,4} - u_{4,5} - \cdots - u_{n-1,n}\big)\, d\hat{u}^*$$

(3.1)

$$= \int_0^1 \cdots \int_0^1 \tilde{f}\left(w \cdot {}^t\left(A_{h/q} \cdot \hat{u}y\right)^{-1}\right) e\big(-qu_{2,3} - u_{3,4} - u_{4,5} - \cdots - u_{n-1,n}\big)\, d\hat{u}^*.$$

Here

$$d\hat{u}^* = \prod_{\substack{1 \le i < j \le n \\ (i,j) \neq (1,2)}} du_{ij},$$

is the standard Lebesgue measure satisfying $\int_0^1 du_{ij} = 1$ for each $1 \le i < j \le n$, and $(i,j) \neq (1,2)$.

The computation of the left hand side of (3.1) is much easier than the computation of the right hand side. The identity that arises from the two computations (after taking a suitable Mellin transform) is the Voronoi formula.

§4. Computation of the left hand side of (3.1).

Let

$$F(y) = \int_0^1 \cdots \int_0^1 f\left(A_{h/q} \cdot \hat{u}y\right) e\big(-qu_{2,3} - u_{3,4} - u_{4,5} - \cdots - u_{n-1,n}\big)\, d\hat{u}^*$$

denote the left hand side of (3.1).

Proposition 4.1. *Fix $n \geq 3$. Let f be an even Maass form of type ν for $SL(n,\mathbb{Z})$ as in (1.5). Then we have*

$$F(y) = \sum_{m \neq 0} \frac{A(1,\ldots,1,m)}{|m|^{\frac{n-1}{2}}} e\left(m\frac{\bar{h}}{q}\right) \cdot W_J\left(\begin{pmatrix} q^{-1}|m|y_1\cdots y_{n-1} & & & & & \\ & qy_1\cdots y_{n-2} & & & & \\ & & y_1\cdots y_{n-3} & & & \\ & & & \ddots & & \\ & & & & y_1 & \\ & & & & & 1 \end{pmatrix}, \nu, \psi_{1,\ldots,1,1}\right).$$

Proof: Let

$$\gamma = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{pmatrix} \in U_{n-1}(\mathbb{Z})\backslash SL(n-1,\mathbb{Z}).$$

Define

$$\gamma' \ = \ \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot A_{h/q} = \begin{pmatrix} a'_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & 0 \\ a'_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a'_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

where

$$\begin{aligned} a'_{1,1} &= a_{1,1} + a_{1,2}\frac{h}{q}, \\ a'_{2,1} &= a_{2,1} + a_{2,2}\frac{h}{q}, \\ &\vdots \\ a'_{n-1,1} &= a_{n-1,1} + a_{n-1,2}\frac{h}{q}. \end{aligned}$$

It follows from [G, Theorem 9.4.7] that

$$\begin{aligned} F(y) &= \sum_{\gamma'} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1}\neq 0} \frac{A(m_1, \ldots, m_{n-1})}{\prod\limits_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}} \\ &\qquad \cdot \int_0^1 \cdots \int_0^1 W_J\left(M\gamma'\hat{u}y, \nu, \psi_{1,\ldots,1,\frac{m_{n-1}}{|m_{n-1}|}}\right) e\big(-qu_{2,3} - u_{3,4} - u_{4,5} - \ \cdots \ - u_{n-1,n}\big)\, d\hat{u}^* \\ &= \sum_{\gamma'} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1}\neq 0} \frac{A(m_1, \ldots, m_{n-1})}{\prod\limits_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}} \\ &\qquad \cdot \int_0^1 \cdots \int_0^1 e\Big(m_1\big(a'_{n-1,1}u_{1,n} + a'_{n-1,2}u_{2,n} + \cdots + a'_{n-1,n-1}u_{n-1,n}\big)\Big) \\ &\qquad\qquad \cdot e\Big(m_2 u^{\gamma'}_{n-2,n-1} + m_3 u^{\gamma'}_{n-3,n-2} + \ \cdots \ + m_{n-1}u^{\gamma'}_{1,2}\Big) \\ &\qquad\qquad\quad \cdot W_J\left(My^{\gamma'}, \nu, \psi_{1,\ldots 1}\right) e\big(-qu_{2,3} - u_{3,4} - u_{4,5} - \ \cdots \ - u_{n-1,n}\big)\, d\hat{u}^*, \end{aligned} \tag{4.2}$$

where by the Iwasawa decomposition,

$$\gamma'\hat{u}y \ = \ \hat{u}^{\gamma'} \cdot y^{\gamma'} \quad \text{mod } \Big(O(n,\mathbb{R}) \cdot \mathbb{R}^{\times}\Big)$$

and

$$\hat{u}^{\gamma'} = \begin{pmatrix} 1 & u_{1,2}^{\gamma'} & u_{1,3}^{\gamma'} & \cdots & & u_{1,n}^{\gamma'} \\ & 1 & u_{2,3}^{\gamma'} & \cdots & & u_{2,n}^{\gamma'} \\ & & \ddots & & & \vdots \\ & & & & 1 & u_{n-1,n}^{\gamma'} \\ & & & & & 1 \end{pmatrix}, \qquad y^{\gamma'} = \begin{pmatrix} y_1^{\gamma'} y_2^{\gamma'} \cdots y_{n-1}^{\gamma'} & & & & \\ & y_1^{\gamma'} y_2^{\gamma'} \cdots y_{n-2}^{\gamma'} & & & \\ & & \ddots & & \\ & & & y_1^{\gamma'} & \\ & & & & 1 \end{pmatrix}.$$

Since $\int_0^1 e(\alpha\nu)d\nu = 0$ unless $\alpha = 0$, it immediately follows that the integral on the right hand side of (4.2) is zero unless

$$\begin{aligned} a'_{n-1,1} &= a_{n-1,1} + a_{n-1,2}\frac{h}{q} = 0 \\ a'_{n-1,2} &= a_{n-1,2} = 0 \\ a'_{n-1,3} &= a_{n-1,3} = 0 \\ &\vdots \\ a'_{n-1,n-2} &= a_{n-1,n-2} = 0 \\ a'_{n-1,n-1} &= a_{n-1,n-1} = 1, \quad m_1 = 1. \end{aligned}$$

It follows that the integral on the right hand side of (4.2) vanishes unless γ' takes the form

$$\gamma' = \begin{pmatrix} a'_{1,1} & a_{1,2} & \cdots & a_{1,n-2} & 0 & 0 \\ a'_{2,1} & a_{2,2} & \cdots & a_{2,n-2} & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ a'_{n-2,1} & a_{n-2,2} & \cdots & a_{n-2,n-2} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

We may now proceed by induction. Suppose γ' takes the form

$$\gamma' = \begin{pmatrix} a'_{1,1} & a_{1,2} & \cdots & a_{1,k} & 0 & \cdots & 0 \\ a'_{2,1} & a_{2,2} & \cdots & a_{2,k} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a'_{k,1} & a_{k,2} & \cdots & a_{k,k} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix},$$

for $3 \le k \le n-1$ and $m_{n-k-1} = 1$.

Then by the Iwasawa decomposition, we have

$$u_{k,k+1}^{\gamma'} = a'_{k,1}u_{1,k+1} + a_{k,2}u_{2,k+1} + \cdots + a_{k,k}u_{k,k+1}.$$

Again, the integral on the right hand side of (4.2) vanishes unless

$$\begin{aligned} a'_{k,1} &= a_{k.1} + a_{k,2}\frac{h}{q} = 0, \\ a_{k,2} &= 0, \\ &\vdots \\ a_{k,k} &= 1, \quad m_{n-k} = 1. \end{aligned}$$

It follows by induction, that the integral on the right hand side of (4.2) is zero unless

$$\gamma' = \begin{pmatrix} a'_{1,1} & a_{1,2} & 0 & \cdots & 0 \\ a'_{2,1} & a_{2,2} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \qquad m_1 = m_2 = \cdots = m_{n-3} = 1.$$

But

$$u_{2,3}^{\gamma'} = a'_{2,1}u_{1,3} + a_{2,2}u_{2,3}.$$

The $u_{1,3}$-integral on the right hand side of (4.2) will vanish unless $a'_{2,1} = a_{2,1} + a_{2,2}\frac{h}{q} = 0$. Similarly, the $u_{2,3}$-integral will vanish unless $a_{2,2} = q$ which implies that $a_{2,1} = -h$ and $m_{n-2} = 1$.

It follows that the integral on the right side of (4.2) vanishes unless

$$\gamma' = \begin{pmatrix} q^{-1} & \bar{h} & & & \\ 0 & q & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

and

$$y_1^{\gamma'} = y_1, \quad \ldots \quad , y_{n-3}^{\gamma'} = y_{n-3}, \qquad y_{n-2}^{\gamma'} = q y_{n-2}, \qquad y_{n-1}^{\gamma'} = \frac{1}{q^2} y_{n-1}.$$

Proposition 4.1 immediately follows.

□

§5. Computation of the right hand side of (3.1).

Let

$$F(y) = \int_0^1 \cdots \int_0^1 \tilde{f}\left(w \cdot {}^t\left(A_{h/q}\cdot \hat{u}y\right)^{-1}\right) e\big(-qu_{2,3} - u_{3,4} - u_{4,5} - \cdots - u_{n-1,n}\big)\, d\hat{u}^* \tag{5.1}$$

denote the right hand side of (3.1). The computation of (5.1) is based on two elementary lemmas arising in the classical theory of Fourier series.

Lemma 5.2. *Let $G:\mathbb{R}\to\mathbb{C}$ be a smooth function satisfying the periodicity relation $G(t+1)=G(t)$ for all $t\in\mathbb{R}$. Then for all $\alpha\in\mathbb{R}$, we have $\int_0^1 G(t)\,dt=\int_0^1 G(t+\alpha)\,dt$.*

Proof: Differentiate $I(\alpha):=\int_0^1 G(t+\alpha)\,dt$ with respect to α to get 0. Hence $I(\alpha)$ is a constant.
□

Lemma 5.3 *Let $G:\mathbb{R}\to\mathbb{C}$ be a smooth function satisfying the periodicity relation $G(t+1)=G(t)$ for all $t\in\mathbb{R}$. Then for all $\alpha\in\mathbb{R}$,*

$$G(\alpha)=\sum_{m\in\mathbb{Z}}\int_0^1 G(t+\alpha)e(-tm)\,dt.$$

Proof: By the Fourier expansion $G(\alpha)$ is equal to:

$$\sum_{m\in\mathbb{Z}}\left(\int_0^1 G(t)e(-tm)\,dt\right)\cdot e(m\alpha)=\sum_{m\in\mathbb{Z}}\int_0^1 G(t)e\big((-t+\alpha)m\big)\,dt=\sum_{m\in\mathbb{Z}}\int_0^1 G(t+\alpha)e(-tm)\,dt.$$

□

Proposition 5.4. *Fix $n\geq 3$. Let f be an even Maass form of type ν for $SL(n,\mathbb{Z})$ as in (1.5). Then we have*

$$F(y)=\sum_{d_1|q}\sum_{d_2|\frac{q}{d_1}}\cdots\sum_{d_{n-2}|\frac{q}{d_1\cdots d_{n-3}}}\frac{q^{\frac{(n-3)(n-2)}{2}}}{\prod\limits_{k=1}^{n-2}d_k^{\frac{(n-k)(n-3)+2}{2}}}\sum_{m\neq 0}\frac{A(m,d_{n-2},\,\dots\,,d_2,d_1)}{|m|^{\frac{n-1}{2}}}KL(h,m;d,q)$$

$$\cdot\int_{u_{13}=-\infty}^{\infty}\cdots\int_{u_{1n}=-\infty}^{\infty}W_J\left(\begin{pmatrix}\frac{d_1^{n-1}d_2^{n-2}\cdots d_{n-2}^2\,m}{q^{n-2}} & & & & \\ & q & & & \\ & & \ddots & & \\ & & & q & \\ & & & & 1\end{pmatrix}\begin{pmatrix}1 & & & & & \\ u_{1,n} & 1 & & & & \\ u_{1,n-1} & & 1 & & & \\ \vdots & & & \ddots & & \\ u_{1,3} & & & & 1 & \\ 0 & & & & & 1\end{pmatrix}\right.$$

$$\left.\cdot\begin{pmatrix}\frac{(-1)^n}{y_1\cdots y_{n-1}} & 0 & \cdots & & 0\\ 0 & & & & (-1)^{n-2}\\ & & & \frac{(-1)^{n-3}}{y_1} & \\ \vdots & & \iddots & & \\ & & \frac{-1}{y_1\cdots y_{n-3}} & & \\ 0 & \frac{1}{y_1\cdots y_{n-2}} & & & \end{pmatrix},\tilde{\nu},\psi_{1,\dots,1,1}\right)du_{1,3}du_{1,4}\cdots du_{1,n}.$$

where $d = (d_1, d_2, \ldots, d_{n-2})$ *and*

$$KL(h,m;d,q) = \sum_{\substack{t_1=1 \\ \left(t_1, \frac{q}{d_1}\right)=1}}^{\frac{q}{d_1}} e\left(\frac{(-1)^n h t_1}{q/d_1}\right) \sum_{\substack{t_2=1 \\ \left(t_1, \frac{q}{d_1 d_2}\right)=1}}^{\frac{q}{d_1 d_2}} e\left(\frac{\bar{t}_1\, t_2}{q/d_1 d_2}\right) \cdots$$

$$\cdots \sum_{\substack{t_{n-2}=1 \\ \left(t_{n-2}, \frac{q}{d_1\cdots d_{n-2}}\right)=1}}^{\frac{q}{d_1\cdots d_{n-2}}} e\left(\frac{\bar{t}_{n-3}\, t_{n-2}}{q/d_1\cdots d_{n-2}}\right) \cdot e\left(\frac{m\,\bar{t}_{n-2}}{q/d_1\cdots d_{n-2}}\right)$$

is the hyper Kloosterman sum.

Proof: Note that

$$^t\hat{u}_n = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ u_{1,3} & u_{2,3} & 1 & & \\ \vdots & \vdots & & \ddots & \\ u_{1n} & u_{2n} & \cdots & u_{n-1,n} & 1 \end{pmatrix} = \begin{pmatrix} ^t\hat{u}_{n-1} & 0 \\ \beta & 1 \end{pmatrix}$$

where

$$\beta = (u_{1,n},\ u_{2,n},\ \cdots\ u_{n-1,n}).$$

Since

$$\begin{pmatrix} ^t\hat{u}_{n-1}^{-1} & 0 \\ & \\ -\beta\, ^t\hat{u}_{n-1}^{-1} & 1 \end{pmatrix} \begin{pmatrix} ^t\hat{u}_{n-1} & 0 \\ & \\ \beta & 1 \end{pmatrix} = \begin{pmatrix} I_{n-1} & \\ & 1 \end{pmatrix},$$

where I_{n-1} denotes the $(n-1)\times(n-1)$ identity matrix, it follows that

$$^t\hat{u}_n^{-1} = \begin{pmatrix} ^t\hat{u}_{n-1}^{-1} & 0 \\ & \\ -\beta\, ^t\hat{u}_{n-1}^{-1} & 1 \end{pmatrix}.$$

We may write

$$^t\hat{u}_{n-1}^{-1} = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ -u_{1,3}+u_{1,3}^* & -u_{2,3} & & & \\ -u_{1,4}+u_{1,4}^* & -u_{2,4}+u_{2,4}^* & & & \\ \vdots & \vdots & & \ddots & \\ & & & 1 & \\ -u_{1,n-1}+u_{1,n-1}^* & -u_{2,n-1}+u_{2,n-1}^* & \cdots & -u_{n-2,n-1} & 1 \end{pmatrix}$$

for certain $u^*_{i,j}$ which are polynomial expressions in $u_{k,\ell}$ with $(k,\ell)\neq(i,j)$.

Similarly, $-\beta\, {}^t\hat{u}_{n-1}^{-1}$ is equal to

$$-\Big(u_{1,n},u_{2,n},\dots,u_{n-1,n}\Big)\cdot\begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ -u_{1,3}+u^*_{1,3} & -u_{2,3} & & & \\ -u_{1,4}+u^*_{1,4} & -u_{2,4}+u^*_{2,4} & & & \\ \vdots & \vdots & & \ddots & \\ & & & & 1 \\ -u_{1,n-1}+u^*_{1,n-1} & u_{2,n-1}+u^*_{2,n-1} & \cdots & -u_{n-2,n-1} & 1 \end{pmatrix}$$

$$=\Big(-u_{1,n}+u^*_{1,n},-u_{2,n}+u^*_{2,n},\quad\dots\quad,-u_{n-1,n}+u^*_{n-1,n}\Big).$$

It follows that

$$w\,{}^t\hat{u}_n^{-1}w^{-1}=\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -u_{1,n}+u^*_{1,n} & 1 & u_{n-1,n} & -u_{n-2,n}+u^*_{n-2,n} & \cdots & (-1)^{n-2}(u_{2,n}+u^*_{2,n}) \\ u_{1,n-1}+u^*_{1,n-1} & 0 & 1 & u_{n-2,n-1} & \cdots & \\ -u_{1,n-2}+u^*_{1,n-2} & 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & u_{2,3} \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{5.5}$$

Also, by a direct computation we have

$$w\cdot{}^tA_{h/q}^{-1}\cdot w^{-1}=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \frac{(-1)^{n+1}h}{q} \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{5.6}$$

Now, by (5.5) and (5.6) we obtain

$$\tilde{f}\left(w\cdot{}^t\left(A_{h/q}\cdot\hat{u}y\right)^{-1}\right)=\tilde{f}\left(\left(w\cdot{}^tA_{h/q}^{-1}\cdot w^{-1}\right)\cdot(w\cdot{}^t\hat{u}^{-1}\cdot w^{-1})\cdot w\,{}^ty^{-1}\right) \tag{5.7}$$

$$=\tilde{f}\left(\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & \frac{(-1)^{n+1}h}{q} \\ -u_{1,n}+u^*_{1,n} & 1 & u_{n-1,n} & -u_{n-2,n}+u^*_{n-2,n} & \cdots & (-1)^{n-2}(u_{2,n}+u^*_{2,n}) \\ u_{1,n-1}+u^*_{1,n-1} & 0 & 1 & u_{n-2,n-1} & \cdots & \\ -u_{1,n-2}+u^*_{1,n-2} & 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & u_{2,3} \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} w\,{}^ty^{-1}\right).$$

By repeated application of Lemma 5.2 we will show that $F(y)$ is, in fact, equal to

(5.8)
$$\int_0^1 \cdots \int_0^1 \tilde f\left(\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & \frac{(-1)^{n+1}h}{q} \\ & 1 & u_{n-1,n} & u_{n-2,n} & \cdots & u_{2,n} \\ & & 1 & u_{n-2,n-1} & \cdots & u_{2,n-1} \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & u_{2,3} \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ u_{1,n} & 1 & 0 & \cdots & 0 & 0 \\ u_{1,n-1} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{1,3} & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} w\, {}^t y^{-1}\right)$$
$$\cdot\, e\big(- qu_{2,3} - u_{3,4} - u_{4,5} - \ \cdots \ - u_{n-1,n}\big)\, d\hat u^*.$$

We now briefly indicate how to prove (5.8). For $1 \le i \le n-1$, and $1 \le j \le n$, $i \ne j$, let $I_{i,j}$ denote the $n \times n$ matrix which differs from the identity at only the position (i,j) where it takes the value 1. Now

$$\tilde f\left(I_{i,j}\cdot w \cdot {}^t\left(A_{h/q}\cdot \hat u y\right)^{-1}\right) = \tilde f\left(w \cdot {}^t\left(A_{h/q}\cdot \hat u y\right)^{-1}\right). \tag{5.9}$$

Choosing $i = n-k+2,\ j = n$ in (5.9), we see from (5.7) that $\tilde f\left(w \cdot {}^t\left(A_{h/q}\cdot \hat u y\right)^{-1}\right)$ is periodic in $u_{2,k}$ for $3 \le k \le n$. By Lemma 5.2, we may replace $u_{2,k} + u^*_{2,k}$ (for $3 \le k \le n$) by $u_{2,k}$ everywhere on the right side of (5.7). Similarly, by choosing $i = n-k+2,\ j = 1$ in (5.9) it follows from (5.7) that $\tilde f\left(w \cdot {}^t\left(A_{h/q}\cdot \hat u y\right)^{-1}\right)$ is periodic in $u_{1,k}$ for $3 \le k \le n$. By Lemma 5.2, we may thus replace $\pm u_{1,k} + u^*_{1,k}$ by $\pm u_{1,k}$ everywhere on the right side of (5.7). One may continue this process by choosing $i = n-k+2,\ j = n-1$, etc. and eventually show periodicity in all $u_{i,j}$ with $i \ne j$ so that all $u^*_{i,j}$ may be removed on the right side of (5.7) and replaced by any expression not involving $u_{i,j}$. Finally, $\pm u_{i,j}$ can be changed to $u_{i,j}$ by making the transformation $u_{i,j} \to \pm u_{i,j}$. Each such transformation does not change the value of $F(y)$ due to the periodicity.

We now proceed with the computation of (5.8). Since

$$\tilde f\left(I_{1,n}\cdot w \cdot {}^t\left(A_{h/q}\cdot \hat u y\right)^{-1}\right) = \tilde f\left(w \cdot {}^t\left(A_{h/q}\cdot \hat u y\right)^{-1}\right),$$

it easily follows from Lemma 5.3 and (5.8) that

(5.10)
$$F(y) \ = \ \sum_{m\in\mathbb{Z}} \int_0^1 \cdots \int_0^1 \int_{x_n=0}^1 e\left(\frac{(-1)^{n+1}hm}{q}\right)$$
$$\cdot\, \tilde f\left(\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & x_n \\ & 1 & u_{n-1,n} & u_{n-2,n} & \cdots & u_{2,n} \\ & & 1 & u_{n-3,n-1} & \cdots & u_{2,n-1} \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & u_{2,3} \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ u_{1,n} & 1 & 0 & \cdots & 0 & 0 \\ u_{1,n-1} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{1,3} & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} w\, {}^t y^{-1}\right)$$
$$\cdot\, e\big(- qu_{2,3} - u_{3,4} - u_{4,5} - \ \cdots \ - u_{n-1,n}\big) e(-mx_n)\, d\hat u^*\, dx_n.$$

Let $d_1=(m,q)$. Then there exists integers A,B with $1\leqslant B\leqslant q/d_1$ such that

$$A\frac{q}{d_1}-B\frac{m}{d_1}\;=\;1.$$

Define the matrix

$$M_{m,q,d_1}:=\begin{pmatrix} A & 0 & \cdots & B & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \frac{m}{d_1} & 0 & \cdots & \frac{q}{d_1} & 0\\ 0 & 0 & \cdots & 0 & 1\end{pmatrix}\;\in\;SL(n,\mathbb{Z}). \tag{5.11}$$

Later on we denote m/d_1 by m. It immediately follows from (5.10) and (5.11) that

(5.12)

$$F(y)\;=\sum_{d_1|q}\;\sum_{\substack{m\in\mathbb{Z}\\ \left(m,\frac{q}{d_1}\right)=1}}\;\int_0^1\cdots\int_0^1\int_{x_n=0}^1 e\left(\frac{(-1)^{n+1}hm}{q}\right)$$

$$\cdot\tilde f\left(M_{m,q,d_1}\begin{pmatrix}1 & 0 & 0 & 0 & \cdots & x_n\\ & 1 & u_{n-1,n} & u_{n-2,n} & \cdots & u_{2,n}\\ & & 1 & u_{n-3,n-1} & \cdots & u_{2,n-1}\\ & & & \ddots & \ddots & \vdots\\ & & & & 1 & u_{2,3}\\ & & & & & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & \cdots & 0 & 0\\ u_{1,n} & 1 & 0 & \cdots & 0 & 0\\ u_{1,n-1} & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ u_{1,3} & 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & 0 & \cdots & 0 & 1\end{pmatrix} w\,{}^t y^{-1}\right)$$

$$\cdot\, e\big(-qu_{2,3}-u_{3,4}-u_{4,5}-\;\cdots\;-u_{n-1,n}\big)e(-md_1x_n)\;d\hat u^*\,dx_n.$$

After some elementary transformations, (5.12) takes the form:

(5.13)

$$F(y)\;=\sum_{d_1|q}\;\sum_{\substack{m\in\mathbb{Z}\\ \left(m,\frac{q}{d_1}\right)=1}} e\left(\frac{(-1)^{n+1}hm}{q}\right)\cdot\left(\frac{d_1}{q}\right)^{n-3}\int_0^1\cdots\int_0^1\int_{x_n=0}^1\int_{u_{3,n}=0}^{\frac{q}{d_1}}\int_{u_{3,n-1}=0}^{\frac{q}{d_1}}\cdots\int_{u_{3,4}=0}^{\frac{q}{d_1}}$$

$$\cdot\tilde f\left(\begin{pmatrix}A & 0 & 0 & \cdots & B & Ax_n+Bu_{2,3}\\ 0 & 1 & u_{n-1,n} & \cdots & u_{3,n} & u_{2,n}\\ 0 & 0 & 1 & \ddots & \vdots & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ m & 0 & 0 & \cdots & \frac{q}{d_1} & mx_n+\frac{q}{d_1}u_{2,3}\\ 0 & 0 & 0 & \cdots & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & \cdots & 0 & 0\\ u_{1,n} & 1 & 0 & \cdots & 0 & 0\\ u_{1,n-1} & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ u_{1,3} & 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & 0 & \cdots & 0 & 1\end{pmatrix} w\,{}^t y^{-1}\right)$$

$$\cdot\, e\big(-qu_{2,3}-u_{3,4}-u_{4,5}-\;\cdots\;-u_{n-1,n}\big)e(-md_1x_n)\;d\hat u^*\,dx_n.$$

In the above multiple integral, we make the following transformations:

$$Ax_n + Bu_{2,3} \to x_n, \qquad mx_n + \frac{q}{d_1}u_{2,3} \to u_{2,3} \tag{5.14}$$

$$u_{3,n} \to \frac{q}{d_1}u_{3,n}, \qquad u_{3,n-1} \to \frac{q}{d_1}u_{3,n-1}, \qquad \cdots \qquad u_{3,4} \to \frac{q}{d_1}u_{3,4}.$$

When the transformations (5.14) are applied to (5.13), we obtain:

$$F(y) = \sum_{d_1|q} \sum_{\substack{m\in\mathbb{Z}\\ \left(m,\frac{q}{d_1}\right)=1}} \int_0^1 \cdots \int_0^1 \int_{x_n=0}^1 \int_{u_{3,n}=0}^1 \int_{u_{3,n-1}=0}^1 \cdots \int_{u_{3,4}=0}^1 e\left(\frac{(-1)^{n+1}hm}{q}\right) \tag{5.15}$$

$$\cdot \tilde{f}\left(\begin{pmatrix} A+Bu_{1,3} & 0 & 0 & \cdots & B & x_n \\ u_{1,n}+u_{1,n}^* & 1 & u_{n-1,n} & \cdots & \frac{q}{d_1}u_{3,n} & u_{2,n} \\ u_{1,n-1}+u_{1,n-1}^* & & & & & \\ & & \ddots & & & \\ u_{1,4}+u_{1,4}^* & & & 1 & \frac{q}{d_1}u_{3,4} & u_{2,4} \\ m+\frac{q}{d_1}u_{1,3} & & & & \frac{q}{d_1} & u_{2,3} \\ 0 & & & & 0 & 1 \end{pmatrix} w\,{}^t y^{-1}\right)$$

$$\cdot\, e\left(-d_1u_{2,3} - \frac{q}{d_1}u_{3,4} - u_{4,5} - \cdots - u_{n-1,n}\right)\, d\hat{u}^*\, dx_n.$$

Let

$$\mathfrak{m} = \begin{pmatrix} A+Bu_{1,3} & 0 & 0 & \cdots & B & x_n \\ u_{1,n}+u_{1,n}^* & 1 & u_{n-1,n} & \cdots & \frac{q}{d_1}u_{3,n} & u_{2,n} \\ u_{1,n-1}+u_{1,n-1}^* & & & & & \\ & & \ddots & & & \\ u_{1,4}+u_{1,4}^* & & & 1 & \frac{q}{d_1}u_{3,4} & u_{2,4} \\ m+\frac{q}{d_1}u_{1,3} & & & & \frac{q}{d_1} & u_{2,3} \\ 0 & & & & 0 & 1 \end{pmatrix}.$$

Define $I_{i,j}(\beta)$ (for $i \neq j$) to be the $n \times n$ matrix which differs from the identity at only the position (i,j) where it takes the value β. It follows that $\tilde{f}\left(I_{k,n}(1)\cdot \mathfrak{m}w\,{}^t y^{-1}\right) = \tilde{f}\left(\mathfrak{m}w\,{}^t y^{-1}\right)$ for any $2 \leq k \leq n-1$, so that $\tilde{f}\left(\mathfrak{m}w\,{}^t y^{-1}\right)$ is periodic in $u_{2,\ell}$ for all $3 \leq \ell \leq n$. Lemma 5.2, tells us that we may replace $u_{2,\ell}$ (for $3 \leq \ell \leq n$) on the right side of (5.15) by $u_{2,\ell}+\alpha$ for any real number α. In a similar manner $\tilde{f}\left(I_{n,n-1}(-B)\cdot I_{n,1}(q/d_1)\cdot \mathfrak{m}w\,{}^t y^{-1}\right) = \tilde{f}\left(\mathfrak{m}w\,{}^t y^{-1}\right)$ from which it follows that

(5.16)

$$F(y) = \sum_{d_1|q} \sum_{\substack{m\in\mathbb{Z}\\ \left(m,\frac{q}{d_1}\right)=1}} e\left(\frac{(-1)^{n+1}hm}{q}\right) \int_0^1 \cdots \int_0^1$$

$$\cdot\, \tilde{f}\left(\begin{pmatrix} A+Bu_{1,3} & 0 & 0 & \cdots & B & x_n \\ u_{1,n}+u^*_{1,n} & 1 & u_{n-1,n} & \cdots & \frac{q}{d_1}u_{3,n} & u_{2,n} \\ u_{1,n-1}+u^*_{1,n-1} & & & & & \\ & & \ddots & & & \\ u_{1,4}+u^*_{1,4} & & & 1 & \frac{q}{d_1}u_{3,4} & u_{2,4} \\ m+\frac{q}{d_1}u_{1,3} & & & & \frac{q}{d_1} & u_{2,3} \\ 1 & & & & 0 & \frac{q}{d_1}x_n - Bu_{2,3}+1 \end{pmatrix} w\, {}^t y^{-1}\right)$$

$$\cdot\, e\left(-d_1u_{2,3} - \frac{q}{d_1}u_{3,4} - u_{4,5} - \cdots - u_{n-1,n}\right)\, d\hat{u}^*\, dx_n.$$

Since $\tilde{f}$ is left invariant by $I_{k,n}(1)$ for $2 \le k \le n-2$, it follows that the right side of (5.16) is periodic in $u_{1,\ell}$ for $4 \le \ell \le n$. We can then use Lemma 5.3 to remove the $u^*_{1,\ell}$ terms ($3 \le \ell \le n$) on the right side of (5.16) and replace them with arbitrary real numbers. We also write

$$m = k\frac{q}{d_1} + t_1, \qquad \text{with } \left(t_1, \frac{q}{d_1}\right) = 1.$$

Then (5.16) can be reexpressed in the form:

(5.17)

$$F(y) = \sum_{d_1|q} \sum_{\substack{t_1=1\\ \left(t_1,\frac{q}{d_1}\right)=1}}^{\frac{q}{d_1}} e\left(\frac{(-1)^{n+1}ht_1}{q/d_1}\right) \sum_{k\in\mathbb{Z}} \int_0^1 \cdots \int_0^1$$

$$\cdot\, \tilde{f}\left(\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & x_n \\ & 1 & u_{n-1,n} & \cdots & u_{3,n} & u_{2,n} \\ & & & \ddots & & \vdots \\ & & & & 1 & u_{1,3} \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} A+Bu_{1,3} & 0 & \cdots & B & 0 \\ u_{1,n} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m+\frac{q}{d_1}u_{1,3} & 0 & \cdots & \frac{q}{d_1} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} w\, {}^t y^{-1}\right)$$

$$\cdot\, e\left(-d_1u_{2,3} - \frac{q}{d_1}u_{3,4} - u_{4,5} - \cdots - u_{n-1,n}\right)\, d\hat{u}^*\, dx_n.$$

In (5.17) we make the change of variables:

$$m + \frac{q}{d_1} u_{1,3} = \frac{q}{d_1} u'_{1,3}$$

which implies that

$$A + B u_{1,3} = A + B\left(u'_{1,3} - \frac{d_1}{q} m\right) = \frac{d_1}{q} + B u'_{1,3}.$$

Then (5.17) can be rewritten as

(5.18)

$$F(y) = \sum_{d_1 | q} \sum_{\substack{t_1=1 \\ \left(t_1, \frac{q}{d_1}\right)=1}}^{\frac{q}{d_1}} e\left(\frac{(-1)^{n+1} h t_1}{q/d_1}\right) \int_0^1 \cdots \int_0^1 \int_{u_{1,3}=-\infty}^{\infty}$$

$$\cdot \tilde{f}\left(\begin{pmatrix} 1 & 0 & \cdots & 0 & x_n \\ & 1 & \cdots & u_{3,n} & u_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & u_{1,3} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} \frac{d_1}{q} + B u_{1,3} & 0 & \cdots & B & 0 \\ u_{1,n} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{q}{d_1} u_{1,3} & 0 & \cdots & \frac{q}{d_1} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} w\, {}^t y^{-1}\right)$$

$$\cdot e\left(-d_1 u_{2,3} - \frac{q}{d_1} u_{3,4} - u_{4,5} - \cdots - u_{n-1,n}\right) d\hat{u}^*\, dx_n.$$

Recall that $m = k\frac{q}{d_1} + t_1$ and $A\frac{q}{d_1} - Bm = 1$. Let us define $\bar{t}_1$ by $t_1 \cdot \bar{t}_1 \equiv 1 \left(\operatorname{mod} \frac{q}{d_1}\right)$, so that $\bar{t}_1 = -B$. It follows that (5.18) can be rewritten as

(5.19)

$$F(y) = \sum_{d_1 | q} \sum_{\substack{t_1=1 \\ \left(t_1, \frac{q}{d_1}\right)=1}}^{\frac{q}{d_1}} e\left(\frac{(-1)^{n+1} h t_1}{q/d_1}\right) \int_0^1 \cdots \int_0^1 \int_{u_{1,3}=-\infty}^{\infty}$$

$$\cdot \tilde{f}\left(\begin{pmatrix} 1 & 0 & \cdots & 0 & x_n \\ & 1 & \cdots & u_{3,n} & u_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & u_{2,3} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} \frac{d_1}{q} & 0 & \cdots & -\bar{t}_1 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{q}{d_1} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ u_{1,n} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{1,3} & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} w\, {}^t y^{-1}\right)$$

$$\cdot e\left(-d_1 u_{2,3} - \frac{q}{d_1} u_{3,4} - u_{4,5} - \cdots - u_{n-1,n}\right) d\hat{u}^*\, dx_n.$$

Next change $-t_1 \to t_1$.

Proof by induction: By induction, suppose that

(5.20)

$$F(y) = \sum_{d_1|q} \sum_{d_2|\frac{q}{d_1}} \cdots \sum_{d_k|\frac{q}{d_1d_2\cdots d_{k-1}}} \frac{q^{\frac{(k-1)k}{2}}}{d_1^{\frac{(k-1)k}{2}} d_2^{\frac{(k-2)(k-1)}{2}} \cdots d_{k-1}} \sum_{\substack{t_1=1\\ \left(t_1,\frac{q}{d_1}\right)=1}}^{\frac{q}{d_1}} e\left(\frac{(-1)^n h t_1}{q/d_1}\right)$$

$$\cdot \sum_{\substack{t_2=1\\ \left(t_2,\frac{q}{d_1d_2}\right)=1}}^{\frac{q}{d_1d_2}} e\left(\frac{\bar t_1 t_2}{q/d_1d_2}\right) \sum_{\substack{t_3=1\\ \left(t_3,\frac{q}{d_1d_2d_3}\right)=1}}^{\frac{q}{d_1d_2d_3}} e\left(\frac{\bar t_2 t_3}{q/d_1d_2d_3}\right) \cdots \sum_{\substack{t_k=1\\ \left(t_k,\frac{q}{d_1d_2\cdots d_k}\right)=1}}^{\frac{q}{d_1d_2\cdots d_k}} e\left(\frac{\bar t_{k-1} t_k}{q/d_1d_2\cdots d_k}\right)$$

$$\cdot \int_{u_{1,3}=-\infty}^{\infty} \int_{u_{1,4}=-\infty}^{\infty} \cdots \int_{u_{1,k+2}=-\infty}^{\infty} \int_0^1 \cdots \int_0^1$$

$$\cdot \tilde f\left(\begin{pmatrix} 1 & 0 & \cdots & 0 & x_{n-k+1} & \cdots & x_{n-1} & x_n \\ 0 & 1 & u_{n-1,n} & \cdots & u_{k+1,n} & \cdots & u_{3,n} & u_{2,n} \\ 0 & 0 & 1 & & & & & \\ & & & \ddots & & & & \vdots \\ & & & & & & 1 & u_{2,3} \\ & & & & & & & 1 \end{pmatrix} \mathfrak{D}_k \begin{pmatrix} 1 & & & & \\ u_{1,n} & 1 & & & \\ \vdots & & & & \\ u_{1,k+2} & & \ddots & & \\ u_{1,k+1} & & & & \\ \vdots & & & & \\ u_{1,3} & & & 1 & \\ 0 & & & & 1 \end{pmatrix} w\, {}^t y^{-1}\right)$$

$$\cdot e\left(-d_1 u_{2,3} - d_2 u_{3,4} - \cdots - d_k u_{k+1,k+2} - \frac{q}{d_1 d_2 \cdots d_k} u_{k+2,k+3} - u_{k+3,k+4} - \cdots - u_{n-1,n}\right)$$
$$\cdot du^* dx_n\, dx_{n-1} \cdots dx_{n-k+1}.$$

where

$$\mathfrak{D}_k = \begin{pmatrix} \frac{d_1^k d_2^{k-1}\cdots d_k}{q^k} & & \overbrace{\bar t_k}^{\text{position } n-k} & & \cdots & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & \frac{q}{d_1d_2\cdots d_k} & & & & \\ & & & & \ddots & & & \\ & & & & & \frac{q}{d_1d_2} & & \\ & & & & & & \frac{q}{d_1} & \\ & & & & & & & 1 \end{pmatrix}.$$

We may also assume, by induction, that the expression on the right side of (5.20) is periodic in $x_{n-k+1}, \ldots, x_{n-1}, x_n$. Now, we must show that (5.20) also holds when k is replaced by $k+1$.

We temporarily define a new function $G(x_{n-k})$ to be equal to

$$\tilde{f}\left(\begin{pmatrix} 1 & 0 & \cdots & 0 & x_{n-k} & x_{n-k+1} & \cdots & x_n \\ 0 & 1 & u_{n-1,n} & \cdots & u_{k+1,n} & u_{k,n} & \cdots & u_{2,n} \\ 0 & 0 & 1 & & & & & \\ & & & \ddots & & & & \vdots \\ & & & & & & & \\ & & & & & & 1 & u_{2,3} \\ & & & & & & & 1 \end{pmatrix} \mathfrak{D}_k \begin{pmatrix} 1 & & & \\ u_{1,n} & 1 & & \\ \vdots & & & \\ u_{1,k+2} & & \ddots & \\ u_{1,k+1} & & & \\ \vdots & & & \\ u_{1,3} & & & 1 \end{pmatrix} w\, {}^t y^{-1}\right).$$

Since $\tilde{f}$ is left invariant by the matrix $I_{1,n-k}(1)$ it is clear that $G(x_{n-k}) = G(x_{n-k}+1)$. Then by Lemma 5.3 (with $\alpha = 0$) we obtain $G(0) = \sum\limits_{m\in\mathbb{Z}} \int\limits_0^1 G(t)e(-mt)\,dt$. It follows that the right side of (5.20) can be rewritten in the form

(5.21)

$$F(y) = \sum_{d_1|q} \sum_{d_2|\frac{q}{d_1}} \cdots \sum_{d_k|\frac{q}{d_1d_2\cdots d_{k-1}}} \frac{q^{\frac{(k-1)k}{2}}}{d_1^{\frac{(k-1)k}{2}} d_2^{\frac{(k-2)(k-1)}{2}} \cdots d_{k-1}} \sum_{\substack{t_1=1 \\ \left(t_1,\frac{q}{d_1}\right)=1}}^{\frac{q}{d_1}} e\left(\frac{(-1)^n h t_1}{q/d_1}\right)$$

$$\cdot \sum_{\substack{t_2=1 \\ \left(t_2,\frac{q}{d_1d_2}\right)=1}}^{\frac{q}{d_1d_2}} e\left(\frac{\bar{t}_1 t_2}{q/d_1d_2}\right) \cdot \sum_{\substack{t_3=1 \\ \left(t_3,\frac{q}{d_1d_2d_3}\right)=1}}^{\frac{q}{d_1d_2d_3}} e\left(\frac{\bar{t}_2 t_3}{q/d_1d_2d_3}\right) \cdots \sum_{\substack{t_k=1 \\ \left(t_k,\frac{q}{d_1d_2\cdots d_k}\right)=1}}^{\frac{q}{d_1d_2\cdots d_k}} e\left(\frac{\bar{t}_{k-1} t_k}{q/d_1d_2\cdots d_k}\right)$$

$$\cdot \int\limits_{u_{1,3}=-\infty}^{\infty} \int\limits_{u_{1,4}=-\infty}^{\infty} \cdots \int\limits_{u_{1,k+2}=-\infty}^{\infty} \int\limits_0^1 \cdots \int\limits_0^1 \sum_{m\in\mathbb{Z}} \int\limits_{x_{n-k}=0}^{1}$$

$$\cdot\, \tilde{f}\left(\begin{pmatrix} 1 & 0 & \cdots & 0 & x_{n-k} & x_{n-k+1} & \cdots & x_n \\ 0 & 1 & u_{n-1,n} & \cdots & u_{k+2,n} & u_{k+1,n} & \cdots & u_{2,n} \\ 0 & 0 & 1 & & & & & \\ & & & \ddots & \vdots & \vdots & & \vdots \\ & & & 1 & u_{k+2,k+3} & u_{k+1,k+3} & \cdots & u_{2,k+3} \\ & & & & & & & \vdots \\ & & & & & & 1 & u_{2,3} \\ & & & & & & & 1 \end{pmatrix} \mathfrak{D}_k \begin{pmatrix} 1 & & & & \\ u_{1,n} & 1 & & & \\ \vdots & & & & \\ u_{1,k+2} & & \ddots & & \\ u_{1,k+1} & & & & \\ \vdots & & & & \\ u_{1,3} & & & 1 & \\ 0 & & & & 1 \end{pmatrix} w\, {}^t y^{-1}\right)$$

$$\cdot\, e\left(-d_1u_{2,3} - d_2u_{3,4} - \cdots - d_ku_{k+1,k+2} - \frac{q}{d_1d_2\cdots d_k}u_{k+2,k+3} - u_{k+3,k+4} - \cdots - u_{n-1,n}\right)$$
$$\cdot\, e(-mx_{n-k})\, du^* dx_n\, dx_{n-1} \cdots dx_{n-k+1}\, dx_{n-k}.$$

Let $d_{k+1} = \left(m,\ \frac{q}{d_1\cdots d_k}\right)$, and choose integers C, D, with $1 \leqslant D \leqslant \frac{q}{d_1\ldots d_{k+1}}$, satisfying

$$C\frac{q}{d_1\cdots d_{k+1}} - D\frac{m}{d_{k+1}} \;=\; 1.$$

Later on we denote m/d_{k+1} by m. Now, replace the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & x_{n-k} & x_{n-k+1} & \cdots & x_n \\ 0 & 1 & u_{n-1,n} & \cdots & u_{k+2,n} & u_{k+1,n} & \cdots & u_{2,n} \\ 0 & 0 & 1 & & & & & \\ & & & \ddots & \vdots & \vdots & & \vdots \\ & & & 1 & u_{k+2,k+3} & u_{k+1,k+3} & \cdots & u_{2,k+3} \\ & & & & & & & \vdots \\ & & & & & & 1 & u_{2,3} \\ & & & & & & & 1 \end{pmatrix}$$

which occurs on the right side of (5.20) by the product of matrices

$$\begin{pmatrix} C & & & & D & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ m & & & & \frac{q}{d_1\cdots d_{k+1}} & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 & x_{n-k} & x_{n-k+1} & \cdots & x_n \\ 0 & 1 & u_{n-1,n} & \cdots & u_{k+2,n} & u_{k+1,n} & \cdots & u_{2,n} \\ 0 & 0 & 1 & & & & & \\ & & & \ddots & \vdots & \vdots & & \vdots \\ & & & 1 & u_{k+2,k+3} & u_{k+1,k+3} & \cdots & u_{2,k+3} \\ & & & & & & & \vdots \\ & & & & & & 1 & u_{2,3} \\ & & & & & & & 1 \end{pmatrix},$$

where D occurs at position $(1, n-k-1)$ and m occurs at position $(n-k-1, 1)$. This replacement does not change the value of the right side of (5.21) because $\tilde{f}$ is automorphic.

Next, we make the change of variables

$$Cx_{n-k} + Du_{k+2,k+3} \longrightarrow x_{n-k}, \qquad mx_{n-k} + \frac{q}{d_1 d_2\cdots d_{k+1}}u_{k+2,k+3} \longrightarrow u_{k+2,k+3}.$$

Also, because the right side of (5.21) is periodic in $u_{k+3,n}, u_{k+3,n-1}, \ldots, u_{k+3,k+4}$ we may rewrite

$$\int\limits_{u_{k+3,n}=0}^{1} \cdots \int\limits_{u_{k+3,k+4}=0}^{1} \;=\; \left(\frac{d_1\cdots d_{k+1}}{q}\right)^{n-k-3} \int\limits_{u_{k+3,n}=0}^{\frac{q}{d_1\cdots d_{k+1}}} \cdots \int\limits_{u_{k+3,k+4}=0}^{\frac{q}{d_1\cdots d_{k+1}}}.$$

It follows that (5.21) can be rewritten as

$$
\begin{aligned}
F(y) \;=\;& \sum_{d_1 \mid q} \sum_{d_2 \mid \frac{q}{d_1}} \cdots \sum_{d_k \mid \frac{q}{d_1 d_2 \cdots d_{k-1}}} \frac{q^{\frac{k(k-1)}{2}}}{d_1^{\frac{(k-1)k}{2}} d_2^{\frac{(k-2)(k-1)}{2}} \cdots d_{k-1}} \\
&\cdot \sum_{\substack{t_1=1 \\ \left(t_1, \frac{q}{d_1}\right)=1}}^{\frac{q}{d_1}} e\left(\frac{(-1)^n h t_1}{q/d_1}\right) \cdots \sum_{\substack{t_k=1 \\ \left(t_k, \frac{q}{d_1 d_2 \cdots d_k}\right)=1}}^{\frac{q}{d_1 d_2 \cdots d_k}} e\left(\frac{\bar{t}_{k-1} t_k}{q/d_1 d_2 \cdots d_k}\right) \\
&\cdot \left(\frac{d_1 \cdots d_{k+1}}{q}\right)^{n-k-3} \int\limits_{u_{1,3}=-\infty}^{\infty} \cdots \int\limits_{u_{1,k+2}=-\infty}^{\infty} \int\limits_0^1 \cdots \int\limits_0^1 \sum_{\substack{m \in \mathbb{Z} \\ \left(m, \frac{q}{d_1 \cdots d_{k+1}}\right)=1}} \int\limits_{u_{k+3,n}=0}^{\frac{q}{d_1 \cdots d_{k+1}}} \cdots \int\limits_{u_{k+3,k+4}=0}^{\frac{q}{d_1 \cdots d_{k+1}}}
\end{aligned}
$$

(5.22)

$$
\begin{aligned}
&\cdot \tilde{f}\left(\left(\begin{matrix}
C & 0 & \cdots & & & D & x_{n-k} & \cdots & x_n \\
 & 1 & u_{n-1,n} & \cdots & & u_{k+3,n} & u_{k+2,n} & \cdots & u_{2,n} \\
 & & \ddots & & & & & & \\
 & & & 1 & & & & & \vdots \\
m & & & & 1 & \frac{q}{d_1 \cdots d_{k+1}} & u_{k+2,k+3} & \cdots & u_{2,k+3} \\
 & & & & & 1 & & & \\
 & & & & & & \ddots & & \vdots \\
 & & & & & & & 1 & u_{2,3} \\
 & & & & & & & & 1
\end{matrix}\right)\right. \\
&\cdot \begin{pmatrix}
\frac{d_1^k d_2^{k-1} \cdots d_k}{q^k} & 0 & \cdots & \overbrace{\bar{t}_k}^{\text{position } n-k} & & \cdots & 0 \\
u_{1,n} & 1 & \cdots & 0 & & \cdots & 0 \\
\vdots & & \ddots & \vdots & & & \\
u_{1,k+3} & 0 & & 1 \;\; 0 & & \cdots & 0 \\
0 & & & \frac{q}{d_1 \cdots d_k} & & \cdots & 0 \\
\vdots & & & & \ddots & & \vdots \\
0 & & & & & \frac{q}{d_1} & 0 \\
0 & & & & & 0 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & & & & \\
0 & 1 & & & \\
\vdots & & & & \\
0 & & & & \\
u_{1,k+2} & & \ddots & & \\
u_{1,k+1} & & & & \\
\vdots & & & 1 & \\
u_{1,3} & & & & 1 \\
0 & & & & & 1
\end{pmatrix} w\, {}^t y^{-1} \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\\1\\1\\1\end{matrix}}\right) \\
&\cdot e\left(-d_1 u_{2,3} - d_2 u_{3,4} - \cdots - d_k u_{k+1,k+2} - \frac{q}{d_1 d_2 \cdots d_k} u_{k+2,k+3} - u_{k+3,k+4} - \cdots - u_{n-1,n}\right) \\
&\qquad\qquad\qquad \cdot e(-m d_{k+1} x_{n-k})\, du^* dx_n\, dx_{n-1} \cdots dx_{n-k}.
\end{aligned}
$$

In (5.22) we make the change of variables

$$
u_{k+3,n} \longrightarrow \frac{q}{d_1 \cdots d_{k+1}} u_{k+3,n}, \qquad \cdots \qquad , u_{k+3,k+4} \longrightarrow \frac{q}{d_1 \cdots d_{k+1}} u_{k+3,k+4}.
$$

Consequently

$$F(y) = \sum_{d_1|q} \cdots \sum_{d_{k+1}|\frac{q}{d_1 d_2\cdots d_k}} \frac{q^{\frac{k(k-1)}{2}}}{d_1^{\frac{(k-1)k}{2}} d_2^{\frac{(k-2)(k-1)}{2}} \cdots d_{k-1}}$$

$$\cdot \sum_{\substack{t_1=1 \\ \left(t_1,\frac{q}{d_1}\right)=1}}^{\frac{q}{d_1}} e\left(\frac{(-1)^n h t_1}{q/d_1}\right) \cdots \sum_{\substack{t_k=1 \\ \left(t_k,\frac{q}{d_1 d_2\cdots d_k}\right)=1}}^{\frac{q}{d_1 d_2 \cdots d_k}} e\left(\frac{\bar{t}_{k-1} t_k}{q/d_1 d_2\cdots d_k}\right)$$

(5.23)

$$\cdot \int_{u_{1,3}=-\infty}^{\infty} \cdots \int_{u_{1,k+2}=-\infty}^{\infty} \int_0^1 \cdots \int_0^1 \sum_{\substack{m\in\mathbb{Z} \\ \left(m,\frac{q}{d_1\cdots d_{k+1}}\right)=1}} \int_{u_{k+3,n}=0}^{1} \cdots \int_{u_{k+3,k+4}=0}^{1}$$

$$\cdot \tilde{f}\left(\left(\begin{pmatrix} C\frac{d_1^k\cdots d_k}{q^k} + Du_{1,k+3} & 0 & \cdots & D & C\bar{t}_k + \frac{q}{d_1\cdots d_k}x_{n-k} & \frac{q}{d_1\cdots d_{k-1}}x_{n-k+1} & \cdots & x_n \\ u_{1,n} + u_{1,n}^* & 1 & u_{n-1,n} & \frac{q u_{k+3,n}}{d_1\cdots d_{k+1}} & \frac{q}{d_1\cdots d_d}u_{k+2,n} & u_{k+1,n} & \cdots & u_{2,n} \\ & & \ddots & & & & & \\ & & & & & & & \vdots \\ \frac{m d_1^k\cdots d_k}{q^k} + \frac{q\,u_{1,k+3}}{d_1\cdots d_{k+1}} & 0 & \cdots & \frac{q}{d_1\cdots d_{k+1}} & m\bar{t}_k + \frac{q\,u_{k+2,k+3}}{d_1\cdots d_k} & & \cdots & u_{2,k+3} \\ & & & & \frac{q}{d_1\cdots d_k} & & & \vdots \\ & & & & & \ddots & & \\ & & & & & & 1 & u_{2,3} \\ & & & & & & & 1 \end{pmatrix}\right.\right.$$

$$\left.\left.\cdot \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ \vdots & & & & & \\ 0 & & & & & \\ u_{1,k+2} & & 1 & & & \\ u_{1,k+1} & & & \frac{q}{d_1\cdots d_k} & & \\ \vdots & & & & \ddots & \\ u_{1,3} & & & & & \frac{q}{d_1} \\ 0 & & & & & & 1 \end{pmatrix} w\,{}^t y^{-1}\right)\right) \cdot e\Big(- d_1 u_{2,3} - d_2 u_{3,4} - \cdots - d_k u_{k+1,k+2}$$

$$- d_{k+1} u_{k+2,k+3} - \frac{q}{d_1 d_2 \cdots d_{k+1}} u_{k+3,k+4} - \cdots - u_{n-1,n}\Big) \cdot du^* dx_n\, dx_{n-1} \cdots dx_{n-k}.$$

As we did previously, one may show that the right side of (5.23) is periodic with respect to $u_{1,n}, \ldots u_{1,k+4}$ and $u_{2,n}, \ldots, u_{2,3}$. We can, therefore, drop the starred terms in (5.23). After making the transformations

$$C\bar{t}_k + \frac{q}{d_1\cdots d_k}x_{n-k} \longrightarrow \frac{q}{d_1\cdots d_k}x_{n-k}, \qquad m\bar{t}_k + \frac{q}{d_1\cdots d_k}u_{k+2,k+3} \longrightarrow \frac{q}{d_1\cdots d_k}u_{k+2,k+3},$$

it follows that (5.23) may be rewritten in the form

$$
\begin{aligned}
F(y) \;=\; & \sum_{d_1|q} \cdots \sum_{d_{k+1}|\frac{q}{d_1 d_2\cdots d_k}} \frac{q^{\frac{k(k-1)}{2}}}{d_1^{\frac{(k-1)k}{2}} d_2^{\frac{(k-2)(k-1)}{2}} \cdots d_{k-1}} \\
& \cdot \sum_{\substack{t_1=1\\ \left(t_1,\frac{q}{d_1}\right)=1}}^{\frac{q}{d_1}} e\left(\frac{(-1)^n h t_1}{q/d_1}\right) \cdots \sum_{\substack{t_k=1\\ \left(t_k,\frac{q}{d_1 d_2\cdots d_k}\right)=1}}^{\frac{q}{d_1 d_2\cdots d_k}} e\left(\frac{\bar{t}_{k-1} t_k}{q/d_1 d_2\cdots d_k}\right) \\
& \cdot \int\limits_{u_{1,3}=-\infty}^{\infty} \cdots \int\limits_{u_{1,k+2}=-\infty}^{\infty} \int\limits_0^1 \cdots \int\limits_0^1 \sum_{\substack{m\in\mathbb{Z}\\ \left(m,\frac{q}{d_1\cdots d_{k+1}}\right)=1}} \int\limits_{u_{k+3,n}=0}^{1} \cdots \int\limits_{u_{k+3,k+4}=0}^{1} \\
& \cdot \tilde{f}\left(\begin{pmatrix}
1 & 0 & \cdots & & 0 & x_{n-k} & x_{n-k+1} & \cdots & x_n \\
 & 1 & u_{n-1,n} & & \cdots & u_{k+2,n} & u_{k+1,n} & \cdots & u_{2,n} \\
 & & 1 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & 1 & u_{k+2,k+3} & u_{k+1,k+3} & \cdots & u_{2,k+3} \\
 & & & & & 1 & & & \\
 & & & & & & \ddots & & \\
 & & & & & & & 1 & u_{2,3} \\
 & & & & & & & & 1
\end{pmatrix}\right. \\
& \cdot \left.\begin{pmatrix}
C\frac{d_1^k\cdots d_k}{q^k} + D u_{1,k+3} & & \cdots & D & 0 & \cdots & 0 \\
u_{1,n} & 1 & \cdots & 0 & 0 & \cdots & \\
u_{1,n-1} & & & & & & \\
\vdots & & \ddots & & & & \\
 & & & 1 & 0 & 0 & \\
m\frac{d_1^k\cdots d_k}{q^k} + \frac{q}{d_1\cdots d_{k+1}} u_{1,k+3} & 0 & \cdots & \frac{q}{d_1\cdots d_{k+1}} & 0 & \cdots & 0 \\
 & & & & \frac{q}{d_1\cdots d_k} & & \\
 & & & & & \ddots & \\
 & & & & & & 1
\end{pmatrix}
\begin{pmatrix}
1 & & & & \\
0 & 1 & & & \\
\vdots & & & & \\
0 & & & & \\
u_{1,k+2} & 1 & & & \\
u_{1,k+1} & & \frac{q}{d_1\cdots d_k} & & \\
\vdots & & & \ddots & \\
u_{1,3} & & & \frac{q}{d_1} & \\
0 & & & & 1
\end{pmatrix} w\, {}^t y^{-1}\right) \\
& \cdot e\left(-d_1 u_{2,3} - d_2 u_{3,4} - \cdots - d_{k+1} u_{k+2,k+3} - \frac{q}{d_1 d_2\cdots d_{k+1}} u_{k+3,k+4} - \cdots - u_{n-1,n}\right) \\
& \cdot e\left(\frac{-m\bar{t}_k}{q/d_1\cdots d_{k+1}}\right) du^* dx_n\, dx_{n-1}\cdots dx_{n-k}.
\end{aligned} \tag{5.24}
$$

Let

$$
m\frac{d_1^k\cdots d_k}{q^k} + \frac{q}{d_1\cdots d_{k+1}} u_{1,k+3} \longrightarrow \frac{q}{d_1\cdots d_{k+1}} u_{1,k+3}.
$$

Then

$$C\frac{d_1^k\cdots d_k}{q^k} \;+ Du_{1,k+3} \quad\longrightarrow\quad \frac{d_1^{k+1}\cdots d_k^2\, d_{k+1}}{q^{k+1}} \;+\; Du_{1,k+3}.$$

It follows that (5.24) can be rewritten in the form

$$\begin{aligned}
F(y) \;=\; &\sum_{d_1|q}\;\cdots \sum_{d_{k+1}|\frac{q}{d_1d_2\cdots d_k}} \frac{q^{\frac{k(k-1)}{2}}}{d_1^{\frac{(k-1)k}{2}}\, d_2^{\frac{(k-2)(k-1)}{2}}\cdots d_{k-1}} \\
&\cdot \sum_{\substack{t_1=1\\ \left(t_1,\frac{q}{d_1}\right)=1}}^{\frac{q}{d_1}} e\left(\frac{(-1)^n h t_1}{q/d_1}\right)\;\cdots \sum_{\substack{t_k=1\\ \left(t_k,\frac{q}{d_1d_2\cdots d_k}\right)=1}}^{\frac{q}{d_1d_2\cdots d_k}} e\left(\frac{\bar{t}_{k-1}t_k}{q/d_1d_2\cdots d_k}\right)
\end{aligned}$$

(5.25)

$$\cdot \int\limits_{u_{1,3}=-\infty}^{\infty}\cdots\int\limits_{u_{1,k+2}=-\infty}^{\infty}\int_0^1\cdots\int_0^1 \sum_{\substack{m\in\mathbb{Z}\\ \left(m,\frac{q}{d_1\cdots d_{k+1}}\right)=1}}\;\int\limits_{u_{1,k+3}\,=\,m\frac{d_1^{k+1}d_2^k\cdots d_k^2 d_{k+1}}{q^{k+1}}}^{1+m\frac{d_1^{k+1}d_2^k\cdots d_k^2 d_{k+1}}{q^{k+1}}}$$

$$\cdot\,\tilde{f}\left(\begin{pmatrix}
1 & 0 & \cdots & & 0 & x_{n-k} & x_{n-k+1} & \cdots & x_n\\
 & 1 & u_{n-1,n} & & \cdots & u_{k+2,n} & u_{k+1,n} & \cdots & u_{2,n}\\
 & & 1 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & 1 & u_{k+2,k+3} & u_{k+1,k+3} & \cdots & u_{2,k+3}\\
 & & & & & 1 & & & \\
 & & & & & & \ddots & & \\
 & & & & & & & 1 & u_{2,3}\\
 & & & & & & & & 1
\end{pmatrix}\right.$$

$$\left.\cdot\begin{pmatrix}
\frac{d_1^{k+1}\cdots d_{k+1}}{q^{k+1}} & & \cdots & D & \cdots & & \\
 & 1 & & & & & \\
 & & \ddots & \vdots & & & \\
 & & & 1 & & & \\
 & & & \frac{q}{d_1\cdots d_{k+1}} & & & \\
 & & & & \ddots & & \\
 & & & & & \frac{q}{d_1} & \\
 & & & & & & 1
\end{pmatrix}
\begin{pmatrix}
1 & & & & \\
u_{1,n} & 1 & & & \\
u_{1,n-1} & & 1 & & \\
\vdots & & & \ddots & \\
u_{1,k+3} & & & & 1 \\
0 & & & & \\
\vdots & & & & \ddots \\
0 & & & & & 1
\end{pmatrix}
\begin{pmatrix}
1 & & & & & \\
0 & 1 & & & & \\
\vdots & & & & & \\
0 & & & & & \\
u_{1,k+2} & & 1 & & & \\
u_{1,k+1} & & & 1 & & \\
\vdots & & & & \ddots & \\
u_{1,3} & & & & & 1 \\
0 & & & & & & 1
\end{pmatrix} w\,{}^t y^{-1}\right)$$

$$\cdot\, e\left(-d_1u_{2,3}-d_2u_{3,4}-\cdots-d_{k+1}u_{k+2,k+3}-\frac{q}{d_1d_2\cdots d_{k+1}}u_{k+3,k+4}-\cdots-u_{n-1,n}\right)$$

$$\cdot\, e\left(\frac{-m\bar{t}_k}{q/d_1\cdots d_{k+1}}\right)\, du^*dx_n\,dx_{n-1}\cdots dx_{n-k}.$$

In (5.25) let $m = \ell \cdot \frac{q^{k+1}}{d_1^{k+1} d_2^k \cdots d_k^2 d_{k+1}} + t_{k+1}, \quad \left(t_{k+1}, \frac{q}{d_1 d_2 \cdots d_{k+1}}\right) = 1$. Then

(5.26)

$$F(y) = \sum_{d_1|q} \cdots \sum_{d_{k+1}|\frac{q}{d_1 d_2 \cdots d_k}} \frac{q^{\frac{(k-1)k}{2}}}{d_1^{\frac{(k-1)k}{2}} d_2^{\frac{(k-2)(k-1)}{2}} \cdots d_{k-1}} \sum_{\substack{t_1=1 \\ \left(t_1, \frac{q}{d_1}\right)=1}}^{\frac{q}{d_1}} e\left(\frac{(-1)^n h t_1}{q/d_1}\right)$$

$$\cdot \frac{q^k}{d_1^k d_2^{k-1} \cdots d_k} \sum_{\substack{t_2=1 \\ \left(t_2, \frac{q}{d_1 d_2}\right)=1}}^{\frac{q}{d_1 d_2}} e\left(\frac{\bar{t}_1 t_2}{q/d_1 d_2}\right) \cdots \sum_{\substack{t_{k+1}=1 \\ \left(t_{k+1}, \frac{q}{d_1 d_2 \cdots d_{k+1}}\right)=1}}^{\frac{q}{d_1 d_2 \cdots d_{k+1}}} e\left(\frac{-\bar{t}_k t_{k+1}}{q/d_1 d_2 \cdots d_{k+1}}\right)$$

$$\cdot \sum_{\ell \in \mathbb{Z}} \int_0^1 \cdots \int_0^1 \int_{u_{1,3}=-\infty}^{\infty} \cdots \int_{u_{1,k+2}=-\infty}^{\infty} \int_{u_{1,k+3} = \ell + t_{k+1} \frac{d_1^{k+1} d_2^k \cdots d_k^2 d_{k+1}}{q^{k+1}}}^{1+\ell+t_{k+1} \frac{d_1^{k+1} d_2^k \cdots d_k^2 d_{k+1}}{q^{k+1}}}$$

$$\cdot \tilde{f}\left(\left(\begin{matrix} 1 & 0 & \cdots & & 0 & x_{n-k} & x_{n-k+1} & \cdots & x_n \\ & 1 & u_{n-1,n} & & \cdots & u_{k+2,n} & u_{k+1,n} & \cdots & u_{2,n} \\ & & 1 & & & & & & \\ & & & \ddots & & & & & \\ & & & & 1 & u_{k+2,k+3} & u_{k+1,k+3} & \cdots & u_{2,k+3} \\ & & & & & 1 & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & u_{2,3} \\ & & & & & & & & 1 \end{matrix}\right)\right.$$

$$\left. \cdot \begin{pmatrix} \frac{d_1^{k+1} \cdots d_{k+1}}{q^{k+1}} & & \cdots & D & \cdots & \\ & 1 & & & & \\ & & \ddots & \vdots & & \\ & & & 1 & & \\ & & & & \frac{q}{d_1 \cdots d_{k+1}} & & \\ & & & & & \ddots & \\ & & & & & & \frac{q}{d_1} \\ & & & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ u_{1,n} & 1 & & & \\ \vdots & & & & \\ u_{1,k+2} & & \ddots & & \\ u_{1,k+1} & & & & \\ \vdots & & & & \\ u_{1,3} & & & 1 & \\ 0 & & & & 1 \end{pmatrix} w\, {}^t y^{-1} \right)$$

$$\cdot e\left(-d_1 u_{2,3} - d_2 u_{3,4} - \cdots - d_{k+1} u_{k+2,k+3} - \frac{q}{d_1 d_2 \cdots d_{k+1}} u_{k+3,k+4} - \cdots - u_{n-1,n}\right)$$

$$\cdot du^* dx_n \, dx_{n-1} \cdots dx_{n-k}.$$

Recall that $m = \ell \frac{q^{k+1}}{d_1^{k+1} d_2^k \cdots d_k^2 d_{k+1}} + t_{k+1}$ and $C\frac{q}{d_1\cdots d_{k+1}} - Dm = 1$. We may define $\bar{t}_{k+1}$ by

$$t_{k+1}\bar{t}_{k+1} \;\equiv\; 1 \left(\mathrm{mod}\ \frac{q}{d_1\cdots d_{k+1}}\right),$$

so that $\bar{t}_{k+1} = -D$. Next, change $t_{k+1} \to -t_{k+1}$ and sum over ℓ in (5.26). We obtain

(5.27)

$$F(y) \;=\; \sum_{d_1|q}\cdots \sum_{d_{k+1}|\frac{q}{d_1 d_2\cdots d_k}} \frac{q^{\frac{k(k+1)}{2}}}{d_1^{\frac{k(k+1)}{2}} d_2^{\frac{(k-1)k}{2}} \cdots d_{k-1}^3 d_k} \sum_{\substack{t_1=1\\ \left(t_1,\frac{q}{d_1}\right)=1}}^{\frac{q}{d_1}} e\left(\frac{(-1)^n h t_1}{q/d_1}\right)$$

$$\cdot \sum_{\substack{t_2=1\\ \left(t_2,\frac{q}{d_1 d_2}\right)=1}}^{\frac{q}{d_1 d_2}} e\left(\frac{\bar{t}_1 t_2}{q/d_1 d_2}\right) \cdots \sum_{\substack{t_{k+1}=1\\ \left(t_{k+1},\frac{q}{d_1 d_2\cdots d_{k+1}}\right)=1}}^{\frac{q}{d_1 d_2\cdots d_{k+1}}} e\left(\frac{\bar{t}_k t_{k+1}}{q/d_1 d_2\cdots d_{k+1}}\right) \int_0^1\cdots\int_0^1 \int_{u_{1,3}=-\infty}^{\infty}\cdots\int_{u_{1,k+3}=-\infty}^{\infty}$$

$$\cdot\, \tilde{f}\left(\begin{pmatrix} 1 & 0 & \cdots & & 0 & x_{n-k} & x_{n-k+1} & \cdots & x_n \\ & 1 & u_{n-1,n} & & \cdots & u_{k+2,n} & u_{k+1,n} & \cdots & u_{2,n} \\ & & 1 & & & & & & \\ & & & \ddots & & & & & \\ & & & & 1 & u_{k+2,k+3} & u_{k+1,k+3} & \cdots & u_{2,k+3} \\ & & & & & 1 & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & u_{2,3} \\ & & & & & & & & 1 \end{pmatrix}\right.$$

$$\cdot \begin{pmatrix} \frac{d_1^{k+1}\cdots d_{k+1}}{q^{k+1}} & & \cdots & \bar{t}_{k+1} & \cdots & & \\ & 1 & & & & & \\ & & \ddots & \vdots & & & \\ & & & 1 & & & \\ & & & & \frac{q}{d_1\cdots d_{k+1}} & & \\ & & & & & \ddots & \\ & & & & & & \frac{q}{d_1} \\ & & & & & & & 1 \end{pmatrix} \left.\begin{pmatrix} 1 & & & & \\ u_{1,n} & 1 & & & \\ \vdots & & & & \\ u_{1,k+2} & & \ddots & & \\ u_{1,k+1} & & & & \\ \vdots & & & & \\ u_{1,3} & & & 1 & \\ 0 & & & & 1 \end{pmatrix} w\, {}^t y^{-1}\right)$$

$$\cdot\, e\left(-d_1 u_{2,3} - d_2 u_{3,4} - \cdots - d_{k+1}u_{k+2,k+3} - \frac{q}{d_1 d_2\cdots d_{k+1}} u_{k+3,k+4} - \cdots - u_{n-1,n}\right)$$

$$\cdot\, du^* dx_n\, dx_{n-1}\cdots dx_{n-k}.$$

This completes the proof by induction of (5.20). It follows that for $k = n-3$, the identity (5.27) takes the form

(5.28)

$$
\begin{aligned}
F(y) \;=\; & \sum_{d_1|q}\cdots\sum_{d_{n-2}|\frac{q}{d_1d_2\cdots d_{n-3}}} \frac{q^{\frac{(n-3)(n-2)}{2}}}{d_1^{\frac{(n-3)(n-2)}{2}}\, d_2^{\frac{(n-4)(n-3)}{2}}\cdots d_{n-4}^3 d_{n-3}} \sum_{\substack{t_1=1\\ \left(t_1,\frac{q}{d_1}\right)=1}}^{\frac{q}{d_1}} e\left(\frac{(-1)^n h t_1}{q/d_1}\right)\\
& \cdot \sum_{\substack{t_2=1\\ \left(t_2,\frac{q}{d_1d_2}\right)=1}}^{\frac{q}{d_1d_2}} e\left(\frac{\bar t_1 t_2}{q/d_1d_2}\right)\cdots \sum_{\substack{t_{n-2}=1\\ \left(t_{n-2},\frac{q}{d_1d_2\cdots d_{n-2}}\right)=1}}^{\frac{q}{d_1d_2\cdots d_{n-2}}} e\left(\frac{\bar t_{n-3} t_{n-2}}{q/d_1d_2\cdots d_{n-2}}\right)\int_0^1\cdots\int_0^1 \int_{u_{1,3}=-\infty}^{\infty}\cdots\int_{u_{1,n}=-\infty}^{\infty}\\
& \cdot \tilde f\left(\begin{pmatrix} 1 & 0 & x_3 & x_4 & & \cdots & x_n\\ & 1 & u_{n-1,n} & u_{n-2,n} & \cdots & & u_{2,n}\\ & & 1 & & & & \vdots\\ & & & \ddots & & & \\ & & & & & & \\ & & & & & 1 & u_{2,3}\\ & & & & & & 1\end{pmatrix}\right.\\
& \cdot \left.\begin{pmatrix} \frac{d_1^{n-2}\cdots d_{n-2}}{q^{n-2}} & & \cdots & \bar t_{n-2} & \cdots & & \\ & 1 & & & & & \\ & & \ddots & \vdots & & & \\ & & & 1 & & & \\ & & & \frac{q}{d_1\cdots d_{n-2}} & & & \\ & & & & \ddots & & \\ & & & & & \frac{q}{d_1} & \\ & & & & & & 1\end{pmatrix}\begin{pmatrix} 1 & & & & \\ u_{1,n} & 1 & & & \\ u_{1,n-1} & & & & \\ \vdots & & \ddots & & \\ u_{1,3} & & & 1 & \\ 0 & & & & 1\end{pmatrix} w\,{}^t y^{-1}\right)\\
& \cdot e\big(-d_1u_{2,3}-d_2u_{3,4}-\cdots-d_{n-2}u_{n-1,n}\big)\, du^*dx_n\, dx_{n-1}\cdots dx_3.
\end{aligned}
$$

The computation of (5.28) we requires a lemma. Note that the $du^*dx_n\cdots dx_3$ integrals in (5.28) give an integral of the form [G, Theorem 9.4.9]. We follow [G] to compute these integrals.

Lemma 5.29. *Let $\hat{u}_n$ be given by (2.2). Then*

$$\int_0^1 \cdots \int_0^1 \tilde{f}(\hat{u}z) e^{-2\pi i\left(d_1 u_{n-1,n} + \cdots + d_{n-2}u_{2,3}\right)} \, d^*\hat{u}$$
$$= \sum_{m\neq 0} \frac{A(m, d_{n-2}, \ldots, d_2, d_1)}{|m|^{\frac{n-1}{2}} \cdot \prod_{k=1}^{n-2} d_k^{\frac{k(n-k)}{2}}} \cdot W_J\left(\begin{pmatrix} d_1 d_2 \cdots d_{n-2}\, m & & & & \\ & d_1 d_2 \cdots d_{n-2} & & & \\ & & \ddots & & \\ & & & d_1 d_2 & & \\ & & & & d_1 & \\ & & & & & 1 \end{pmatrix} z;\ \tilde{\nu},\ \psi_{1,\ldots,1}\right).$$

Proof: We may apply the method of [G, Theorem 9.4.9] which picks off an infinite sum of Fourier coefficients of $\tilde{f}$. Note that because we have the dual form $\tilde{f}$ we get the Fourier coefficient $A(m, d_{n-2}, \ldots, d_2, d_1)$ instead of $A(d_1, d_2, \ldots, d_{n-2}, m)$. It follows that

$$\int_0^1 \cdots \int_0^1 \tilde{f}(\hat{u}z) e^{-2\pi i\left(d_1 u_{n-1,n} + \cdots + d_{n-2}u_{2,3}\right)} \, d^*\hat{u}$$
$$= \sum_{m\neq 0} \frac{A(m, d_{n-2}, \ldots, d_2, d_1)}{|m|^{\frac{n-1}{2}} \cdot \prod_{k=1}^{n-2} d_k^{\frac{k(n-k)}{2}}} \cdot W_J\left(\begin{pmatrix} d_1 d_2 \cdots d_{n-2}\, |m| & & & & \\ & d_1 d_2 \cdots d_{n-2} & & & \\ & & \ddots & & \\ & & & d_1 d_2 & & \\ & & & & d_1 & \\ & & & & & 1 \end{pmatrix} z;\ \tilde{\nu},\ \psi_{1,\ldots,1,\frac{m}{|m|}}\right).$$

In order to complete the proof of Lemma 5.29, we require the identity

$$\begin{aligned} W_J&\left(\mathrm{diag}\Big(|m|, 1, \ldots, 1\Big) z; \tilde{\nu}, \psi_{1,\ldots,1,\frac{m}{|m|}}\right) \\ &= W_J\left(\mathrm{diag}\Big(-1, 1, \ldots, 1\Big)\mathrm{diag}\Big(|m|, 1, \ldots, 1\Big) z; \tilde{\nu}, \psi_{1,\ldots,1,\frac{-m}{|m|}}\right) \\ &= W_J\left(\mathrm{diag}\Big(m, 1, \ldots, 1\Big) z; \tilde{\nu}, \psi_{1,\ldots,1,1}\right). \end{aligned}$$

It is clear that the above identity holds when m is either positive or negative.
□

It immediately follows from Lemma 5.29 that

(5.30)

$$F(y) = \sum_{d_1|q}\cdots\sum_{d_{n-2}|\frac{q}{d_1d_2\cdots d_{n-3}}} \frac{q^{\frac{(n-3)(n-2)}{2}}}{d_1^{\frac{(n-3)(n-2)}{2}} d_2^{\frac{(n-4)(n-3)}{2}}\cdots d_{n-4}^3 d_{n-3}} \sum_{\substack{t_1=1\\ \left(t_1,\frac{q}{d_1}\right)=1}}^{\frac{q}{d_1}} e\left(\frac{(-1)^n h t_1}{q/d_1}\right)$$

$$\cdot \sum_{\substack{t_2=1\\ \left(t_2,\frac{q}{d_1d_2}\right)=1}}^{\frac{q}{d_1d_2}} e\left(\frac{\bar{t}_1 t_2}{q/d_1d_2}\right)\cdots \sum_{\substack{t_{n-2}=1\\ \left(t_{n-2},\frac{q}{d_1d_2\cdots d_{n-2}}\right)=1}}^{\frac{q}{d_1d_2\cdots d_{n-2}}} e\left(\frac{\bar{t}_{n-3}t_{n-2}}{q/d_1d_2\cdots d_{n-2}}\right)$$

$$\cdot \sum_{m\neq 0} \frac{A(m,d_{n-2},\ \ldots\ ,d_2,d_1)}{|m|^{\frac{n-1}{2}}\cdot\prod\limits_{k=1}^{n-2} d_k^{\frac{k(n-k)}{2}}}\, e\left(\frac{m\bar{t}_{n-2}}{q/d_1\cdots d_{n-2}}\right) \int\limits_{u_{1,3}=-\infty}^{\infty}\cdots\int\limits_{u_{1,n}=-\infty}^{\infty}$$

$$\cdot W_J\left(\begin{pmatrix} d_1d_2\cdots d_{n-2}\,m & & & & & \\ & d_1d_2\cdots d_{n-2} & & & & \\ & & \ddots & & & \\ & & & d_1d_2 & & \\ & & & & d_1 & \\ & & & & & 1\end{pmatrix}\begin{pmatrix} \frac{d_1^{n-2}\cdots d_{n-2}}{q^{n-2}} & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & \frac{q}{d_1\cdots d_{n-2}} & & \\ & & & & & \ddots & \\ & & & & & & \frac{q}{d_1} & \\ & & & & & & & 1\end{pmatrix}\begin{pmatrix} 1 & & & & \\ u_{1,n} & 1 & & & \\ u_{1,n-1} & & & & \\ \vdots & & \ddots & & \\ u_{1,3} & & & 1 & \\ 0 & & & & 1\end{pmatrix} w\,{}^t y^{-1},\ \tilde{\nu},\psi_{1,\ldots,1}\right)$$

$$\cdot\, e\big(-d_1u_{2,3}-d_2u_{3,4}-\cdots-d_{n-2}u_{n-1,n}\big)\, du_{1,3}\ \cdots\ du_{1,n}.$$

Finally, a simple computation shows that

$$w\,{}^t y^{-1} = \begin{pmatrix} (-1)^n & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & (-1)^{n-2}\\ \vdots & \vdots & \vdots & \iddots\ \iddots & \vdots\\ \vdots & \vdots & \iddots & (-1)^2 & 0\\ 0 & 0 & -1 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\end{pmatrix}\begin{pmatrix} \frac{1}{y_1y_2\cdots y_{n-1}} & & & & \\ & \frac{1}{y_1y_2\cdots y_{n-2}} & & & \\ & & \ddots & & \\ & & & \frac{1}{y_1} & \\ & & & & 1\end{pmatrix}$$

$$= \begin{pmatrix} \frac{(-1)^n}{y_1\cdots y_{n-1}} & 0 & & \cdots & & 0\\ 0 & & & & & (-1)^{n-2}\\ & & & & \frac{(-1)^{n-3}}{y_1} & \\ \vdots & & & \iddots & & \\ & & \frac{-1}{y_1\cdots y_{n-3}} & & & \\ 0 & \frac{1}{y_1\cdots y_{n-2}} & & & & \end{pmatrix}.$$

This completes the proof of Proposition 5.4.

§6. Taking Mellin transforms.

Taken together, Propositions 4.1 and 5.4 give the identity

(6.1)

$$F(y) = \sum_{m\neq 0} \frac{A(1,\dots,1,m)}{|m|^{\frac{n-1}{2}}} e\left(m\frac{\bar{h}}{q}\right) \cdot W_J\left(\begin{pmatrix} q^{-1}|m|y_1\cdots y_{n-1} & & & & & \\ & qy_1\cdots y_{n-2} & & & & \\ & & y_1\cdots y_{n-3} & & & \\ & & & \ddots & & \\ & & & & y_1 & \\ & & & & & 1 \end{pmatrix}, \ \nu, \ \psi_{1,\dots,1} \right)$$

$$= \sum_{d_1|q} \sum_{d_2|\frac{q}{d_1}} \cdots \sum_{d_{n-2}|\frac{q}{d_1\cdots d_{n-3}}} \frac{q^{\frac{(n-3)(n-2)}{2}}}{\prod_{k=1}^{n-2} d_k^{\frac{(n-k)(n-3)+2}{2}}} \sum_{m\neq 0} \frac{A(m,d_{n-2},\dots,d_2,d_1)}{|m|^{\frac{n-1}{2}}} KL(h,m;d,q)$$

$$\cdot \int_{u_{13}=-\infty}^{\infty} \cdots \int_{u_{1n}=-\infty}^{\infty} W_J\left(\begin{pmatrix} \frac{d_1^{n-1}d_2^{n-2}\cdots d_{n-2}^2\, m}{q^{n-2}} & & & & \\ & q & & & \\ & & \ddots & & \\ & & & q & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & & \\ u_{1,n} & 1 & & & & \\ u_{1,n-1} & & 1 & & & \\ \vdots & & & \ddots & & \\ u_{1,3} & & & & 1 & \\ 0 & & & & & 1 \end{pmatrix} \right.$$

$$\left. \cdot \begin{pmatrix} \frac{(-1)^n}{y_1\cdots y_{n-1}} & 0 & \cdots & & 0 \\ 0 & & & & (-1)^{n-2} \\ & & & \frac{(-1)^{n-3}}{y_1} & \\ \vdots & & \iddots & & \\ & & \frac{-1}{y_1\cdots y_{n-3}} & & \\ 0 & \frac{1}{y_1\cdots y_{n-2}} & & & \end{pmatrix}, \ \tilde{\nu}, \ \psi_{1,\dots,1} \right) du_{1,3}du_{1,4}\cdots du_{1,n}.$$

To obtain a more useful result, we must replace the Whittaker function W_J in (6.1) with an arbitrary test function. The first step in doing this is to take a suitable Mellin transform in a complex variable s of both sides of (6.1). Accordingly, we shall consider

$$\int_{y_1=0}^{\infty} \cdots \int_{y_{n-1}=0}^{\infty} F(y) \left(y_1^{n-1}\cdots y_{n-1}\right)^{s+\frac{n-1}{2}} \prod_{k=1}^{n-1} y_k^{-k(n-k)} \frac{dy_k}{y_k}. \tag{6.2}$$

It follows from (6.1) and (6.2) that

(6.3)

$$F(y) = \sum_{m\neq 0} \frac{A(1,\ldots,1,m)}{|m|^{\frac{n-1}{2}}} e\left(m\frac{\bar{h}}{q}\right)\cdot \mathcal{I}^*(s)$$

$$= \sum_{d_1|q}\sum_{d_2|\frac{q}{d_1}}\cdots\sum_{d_{n-2}|\frac{q}{d_1\cdots d_{n-3}}} \frac{q^{\frac{(n-3)(n-2)}{2}}}{\prod\limits_{k=1}^{n-2} d_k^{\frac{(n-k)(n-3)+2}{2}}} \sum_{m\neq 0}\frac{A(m,d_{n-2},\ \ldots\ ,d_2,d_1)}{|m|^{\frac{n-1}{2}}} KL(h,m;d,q)\cdot\mathcal{I}(s),$$

where

(6.4)

$$\mathcal{I}^*(s) = \int\limits_{y_1=0}^{\infty}\cdots\int\limits_{y_{n-1}=0}^{\infty} W_J\left(\begin{pmatrix} q^{-1}|m|y_1\cdots y_{n-1} & & & & & \\ & qy_1\cdots y_{n-2} & & & & \\ & & y_1\cdots y_{n-3} & & & \\ & & & \ddots & & \\ & & & & y_1 & \\ & & & & & 1\end{pmatrix},\ \nu,\ \psi_{1,\ldots,1}\right)$$

$$\cdot\left(y_1^{n-1}\cdots y_{n-1}\right)^{s-\frac{n-1}{2}}\prod_{k=1}^{n-1} y_k^{-k(n-k)}\ \frac{dy_k}{y_k},$$

and

(6.5)

$$\mathcal{I}(s) = \int\limits_{y_1=0}^{\infty}\cdots\int\limits_{y_{n-1}=0}^{\infty}\int\limits_{u_{13}=-\infty}^{\infty}\cdots\int\limits_{u_{1n}=-\infty}^{\infty} W_J\left(\begin{pmatrix}\frac{d_1^{n-1}d_2^{n-2}\cdots d_{n-2}^2\,m}{q^{n-2}} & & & & \\ & q & & & \\ & & \ddots & & \\ & & & q & \\ & & & & 1\end{pmatrix}\right.$$

$$\cdot\begin{pmatrix} 1 & & & & & \\ u_{1,n} & 1 & & & & \\ u_{1,n-1} & & 1 & & & \\ \vdots & & & \ddots & & \\ u_{1,3} & & & & 1 & \\ 0 & & & & & 1\end{pmatrix}\left.\begin{pmatrix}\frac{(-1)^n}{y_1\cdots y_{n-1}} & 0 & & \cdots & & 0\\ 0 & & & & & (-1)^{n-2}\\ & & & & \frac{(-1)^{n-3}}{y_1} & \\ \vdots & & & \iddots & & \\ & & \frac{-1}{y_1\cdots y_{n-3}} & & & \\ 0 & \frac{1}{y_1\cdots y_{n-2}} & & & & \end{pmatrix},\ \tilde{\nu},\ \psi_{1,\ldots,1}\right)$$

$$\cdot\left(y_1^{n-1}\cdots y_{n-1}\right)^{s+\frac{n-1}{2}}\prod_{k=1}^{n-1} y_k^{-k(n-k)}\ \frac{dy_k}{y_k}\ du_{1,3}du_{1,4}\cdots du_{1,n}.$$

The rest of this section will be devoted to explicitly evaluating (6.4) and (6.5). The results are given in the following two lemmas.

Lemma 6.6. *Let $\mathcal{I}^*(s)$ be given by (6.4). Then we have*

$$\mathcal{I}^*(s) \;=\; |m|^{-s+\frac{n-1}{2}} q^{-2} G^*(s)$$

where $G^*(s)$ is equal to

$$\int\limits_{y_1=0}^{\infty}\cdots\int\limits_{y_{n-1}=0}^{\infty} W_J\left(\begin{pmatrix} y_1y_2\cdots y_{n-1} & & & & \\ & y_1y_2\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1\end{pmatrix},\, \nu,\, \psi_{1,\dots,1}\right)\left(y_1^{n-1}\cdots y_{n-1}\right)^{s+\frac{n-1}{2}}\prod_{k=1}^{n-1} y_k^{-k(n-k)}\,\frac{dy_k}{y_k}.$$

Proof: Direct computation.
□

Lemma 6.7. *Let $\mathcal{I}(s)$ be given by (6.5). Then we have*

$$\mathcal{I}(s) \;=\; q^{-ns-\frac{n^2}{2}+\frac{5n}{2}-4}\left(d_1^{n-1}d_2^{n-2}\cdots d_{n-2}^2|m|\right)^{s+\frac{n-3}{2}}\hat{G}(s),$$

where

$$\hat{G}(s) \;=\; \int\limits_{y_1=0}^{\infty}\cdots\int\limits_{y_{n-1}=0}^{\infty}\int\limits_{u_{13}=-\infty}^{\infty}\cdots\int\limits_{u_{1n}=-\infty}^{\infty}$$

$$\cdot\, W_J\left(\begin{pmatrix} \frac{(-1)^n}{y_1\cdots y_{n-1}} & 0 & & \cdots & & 0 \\ \frac{(-1)^n u_{1,n}}{y_1\cdots y_{n-1}} & & & & & (-1)^{n-2} \\ \frac{(-1)^n u_{1,n-1}}{y_1\cdots y_{n-1}} & & & & \frac{(-1)^{n-3}}{y_1} & \\ \vdots & & & \iddots & & \\ \frac{(-1)^n u_{1,3}}{y_1\cdots y_{n-1}} & 0 & \frac{-1}{y_1\cdots y_{n-3}} & & & \\ 0 & \frac{1}{y_1\cdots y_{n-2}} & & & & \end{pmatrix},\, \tilde{\nu},\, \psi_{1,\dots,1}\right)$$

$$\cdot\left(y_1^{n-1}\cdots y_{n-1}\right)^{s+\frac{n-1}{2}}\prod_{k=1}^{n-1} y_k^{-k(n-k)}\,\frac{dy_k}{y_k}\,du_{1,3}du_{1,4}\cdots du_{1,n}.$$

Proof: It follows immediately from (6.5) that

$$\mathcal{I}(s) = \int\limits_{y_1=0}^{\infty} \cdots \int\limits_{y_{n-1}=0}^{\infty} \int\limits_{u_{13}=-\infty}^{\infty} \cdots \int\limits_{u_{1n}=-\infty}^{\infty}$$

$$\cdot W_J\left(\begin{pmatrix} \frac{(-1)^n d_1^{n-1} d_2^{n-2} \cdots d_{n-2}^2\, m}{q^{n-2} y_1 \cdots y_{n-1}} & 0 & \cdots & & 0 \\ \frac{(-1)^n q\, u_{1,n}}{y_1 \cdots y_{n-1}} & & & & (-1)^{n-2} q \\ \frac{(-1)^n q\, u_{1,n-1}}{y_1 \cdots y_{n-1}} & & & \frac{(-1)^{n-3} q}{y_1} & \\ \vdots & & \iddots & & \\ \frac{(-1)^n q\, u_{1,3}}{y_1 \cdots y_{n-1}} & 0 & \frac{-q}{y_1 \cdots y_{n-3}} & & \\ 0 & \frac{1}{y_1 \cdots y_{n-2}} & & & \end{pmatrix}, \tilde{\nu}, \psi_{1,\dots,1}\right)$$

$$\cdot\, du_{1,3} du_{1,4} \cdots du_{1,n} \left(y_1^{n-1} \cdots y_{n-1}\right)^{s - \frac{n-1}{2}} \prod_{k=1}^{n-1} y_k^{-k(n-k)} \, \frac{dy_k}{y_k}$$

$$= \int\limits_{y_1=0}^{\infty} \cdots \int\limits_{y_{n-1}=0}^{\infty} \int\limits_{u_{13}=-\infty}^{\infty} \cdots \int\limits_{u_{1n}=-\infty}^{\infty}$$

$$\cdot W_J\left(\begin{pmatrix} \frac{(-1)^n d_1^{n-1} d_2^{n-2} \cdots d_{n-2}^2\, m}{q^{n-1} y_1 \cdots y_{n-1}} & 0 & \cdots & & 0 \\ \frac{(-1)^n\, u_{1,n}}{y_1 \cdots y_{n-1}} & & & & (-1)^{n-2} \\ \frac{(-1)^n\, u_{1,n-1}}{y_1 \cdots y_{n-1}} & & & \frac{(-1)^{n-3}}{y_1} & \\ \vdots & & \iddots & & \\ \frac{(-1)^n\, u_{1,3}}{y_1 \cdots y_{n-1}} & 0 & \frac{-1}{y_1 \cdots y_{n-3}} & & \\ 0 & \frac{1}{q\, y_1 \cdots y_{n-2}} & & & \end{pmatrix}, \tilde{\nu}, \psi_{1,\dots,1}\right)$$

$$\cdot\, du_{1,3} du_{1,4} \cdots du_{1,n} \left(y_1^{n-1} \cdots y_{n-1}\right)^{s - \frac{n-1}{2}} \prod_{k=1}^{n-1} y_k^{-k(n-k)} \, \frac{dy_k}{y_k}.$$

Lemma 6.7 follows after making the change of variables

$$q y_{n-2} \to y_{n-2}, \qquad \frac{d_1^{n-1} d_2^{n-2} \cdots d_{n-2}^2\, |m|}{q^{n-2} y_{n-1}} \to \frac{1}{y_{n-1}}$$

$$u_{1,3} \to \frac{d_1^{n-1} d_2^{n-2} \cdots d_{n-2}^2\, m}{q^{n-1}} u_{1,3}, \qquad u_{1,4} \to \frac{d_1^{n-1} d_2^{n-2} \cdots d_{n-2}^2\, m}{q^{n-1}} u_{1,4},$$
$$\cdots \quad u_{1,n} \to \frac{d_1^{n-1} d_2^{n-2} \cdots d_{n-2}^2\, m}{q^{n-1}} u_{1,n}.$$

□

§7. Proof of the Voronoi formula.

Let

$$f(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1}\neq 0} \frac{A(m_1,\ldots,m_{n-1})}{\prod\limits_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}} W_J\left(M \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, \nu, \psi_{1,\ldots,1,\frac{m_{n-1}}{|m_{n-1}|}}\right)$$

be an even Maass form, of type ν, for $SL(n,\mathbb{Z})$ with $n \geq 2$ where $W_J(z)$ denotes the Jacquet Whittaker function and $M = \mathrm{diag}\big(m_1 \cdots m_{n-2}|m_{n-1}|,\ m_1 \cdots m_{n-2}, \quad \ldots \quad, m_1,\ 1\big)$. By the method of templates [G, §10.8], we may define

$$G(s) = \pi^{\frac{-ns}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{s-\lambda_i(\nu)}{2}\right), \qquad \tilde{G}(s) = \pi^{\frac{-ns}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{s-\tilde{\lambda}_i(\nu)}{2}\right)$$

for certain linear functions $\lambda_i(\nu),\ \tilde{\lambda}_i(\nu)$ $(1 \leq i \leq n)$, where

$$\frac{G^*(s)}{\hat{G}(s)} \;=\; \frac{G(s)}{\tilde{G}(1-s)}.$$

with $G^*(s)$ given in Lemma 6.6, while $\hat{G}(s)$ is given in Lemma 6.7. It now follows from (6.3) and Lemmas 6.6, 6.7 that

(7.1)
$$\sum_{m\neq 0} \frac{A(1,\ldots,1,m)}{|m|^s} e\left(m\frac{\bar{h}}{q}\right) \cdot G(s)$$
$$= \sum_{d_1|q} \sum_{d_2|\frac{q}{d_1}} \cdots \sum_{d_{n-2}|\frac{q}{d_1\cdots d_{n-3}}} \frac{q^{n^2-ns-n+1}}{\prod\limits_{i=1}^{n-2} d_i^{n^2-(n-1)(i+1)-(n-i)s}} \sum_{m\neq 0}$$
$$\cdot \frac{A(m,d_{n-2},\ \ldots\ ,d_2,d_1)}{|m|^{1-s}} KL(h,m;d,q) \cdot \tilde{G}(1-s).$$

After obtaining analytic continuation of the two series, we have the following functional equation

$$
\begin{aligned}
& q^{-ns+1} \sum_{d_1 | q} \sum_{d_2 | \frac{q}{d_1}} \cdots \sum_{d_{n-2} | \frac{q}{d_1 \cdots d_{n-3}}} \prod_{i=1}^{n-2} d_i^{-1+s(n-i)} \\
& \qquad \cdot \sum_{m \neq 0} \frac{A(m, d_{n-2}, \ \ldots \ , d_2, d_1)}{|m|^{1-s}} KL(h, m, d, q) \tilde{G}(1-s) \\
& \qquad\qquad = \sum_{m \neq 0} \frac{A(1, \ldots , 1, m)}{|m|^s} e\left(m \frac{\bar{h}}{q}\right) \cdot G(s).
\end{aligned} \tag{7.2}
$$

Let ϕ be a smooth compactly supported function on $\mathbb{R}^+$. For $s \in \mathbb{C}$ with $\Re(s)$ sufficiently large, let

$$\tilde{\phi}(s) = \int_0^\infty \phi(x)\, x^s \, \frac{dx}{x}$$

denote the Mellin transform of ϕ.

By taking the inverse Mellin transform it follows that for $\sigma > 0$, sufficiently large

$$\frac{1}{2\pi i} \int\limits_{\Re(s)=\sigma} \tilde{\phi}(s) \sum_{m \neq 0} \frac{A(1, \ldots , 1, m)}{|m|^s} e\left(\frac{m\bar{h}}{q}\right) ds \ = \ \sum_{m \neq 0} A(1, \ldots , 1, m) e\left(m \frac{\bar{h}}{q}\right) \phi\left(|m|\right). \tag{7.3}$$

After shifting the line of integration above to $-\sigma$ and applying (7.2), we obtain

$$
\begin{aligned}
& \frac{1}{2\pi i} \int\limits_{\Re(s)=-\sigma} \tilde{\phi}(s) q^{-ns+1} \sum_{d_1 | q} \sum_{d_2 | \frac{q}{d_1}} \cdots \sum_{d_{n-2} | \frac{q}{d_1 \cdots d_{n-3}}} \prod_{i=1}^{n-2} d_i^{-1+s(n-i)} \\
& \qquad \cdot \sum_{m \neq 0} \frac{A(m, d_{n-2}, \ \ldots \ , d_2, d_1)}{|m|^{1-s}} KL(h, m; d, q) \frac{\tilde{G}(1-s)}{G(s)} \\
& = q \sum_{d_1 | q} \sum_{d_2 | \frac{q}{d_1}} \cdots \sum_{d_{n-2} | \frac{q}{d_1 \cdots d_{n-3}}} \sum_{m \neq 0} \frac{A(m, d_{n-2}, \ \ldots \ , d_2, d_1)}{d_1 \cdots d_{n-2}\, |m|} KL(h, m; d, q) \Phi\left(\frac{|m| \prod\limits_{i=1}^{n-2} d_i^{n-i}}{q^n} \right),
\end{aligned} \tag{7.4}
$$

where

$$\Phi(x) = \frac{1}{2\pi i} \int\limits_{\Re(s)=-\sigma} \tilde{\phi}(s)\, x^s \, \frac{\tilde{G}(1-s)}{G(s)} \, ds.$$

Combining (7.2), (7.3), and (7.4) gives the Voronoi formula

$$\sum_{m\neq 0} A(1,\ldots,1,m)e\left(\frac{m\bar{h}}{q}\right)\phi(|m|)$$

$$= q\sum_{d_1|q}\sum_{d_2|\frac{q}{d_1}}\cdots\sum_{d_{n-2}|\frac{q}{d_1\cdots d_{n-3}}}\sum_{m\neq 0}\frac{A(m,d_{n-2},\ \ldots\ ,d_2,d_1)}{d_1\cdots d_{n-2}\,|m|}KL(h,m;d,q)\Phi\left(\frac{|m|\prod\limits_{i=1}^{n-2} d_i^{n-i}}{q^n}\right).$$

§8. References.

[B-B] Beineke, Jennifer; Bump, Daniel; *A summation formula for divisor functions associated to lattices,* Forum Math, (2006), 869–906.

[B] Bump, Daniel; *Automorphic Forms on* $GL(3,\mathbb{R})$*,* Lecture Notes in Math. **1083** (1984), Springer-Verlag.

[G] Goldfeld, Dorian; *Automorphic forms and L-functions for the group* $GL(n,\mathbb{R})$, Cambridge Studies in Advanced Mathematics, Vol. 99, 2006, Cambridge University Press.

[G-L] Goldfeld, Dorian; Li, Xiaoqing; *Voronoi formulas on* $GL(n)$, International Mathematics Research Notices, vol. 2006, Article ID 86295, 25 pages, (2006).

[Iv] Ivić, Aleksandar; *On the ternary additive divisor problem and the sixth moment of the zeta-function,* Sieve methods, exponential sums, and their applications in number theory (Cardiff, 1995), 205–243, London Math. Soc. Lecture Note Ser., 237, Cambridge Univ. Press, Cambridge, 1997.

[I-K] Iwaniec, Henryk; Kowalski, Emmanuel; *Analytic number theory,* American Mathematical Society Colloquium Publications, 53. American Mathematical Society, Providence, RI, 2004.

[J-P-S] Jacquet, Hervé; Piatetski-Shapiro, Ilya; Shalika, Joseph; *Automorphic forms on* $GL(3)$*, I, II,* Annals of Math. **109** (1979), 169–258.

[Li] Li, Xiaoqing; *The central value of the Rankin-Selberg L-functions,* to appear in GAFA.

[Mi] Miller, Stephen D.; *Cancellation in additively twisted sums on* $GL(n)$*,* Amer. J. Math. 128, no. 3 (2006), 699–729.

[M-S1] Miller, Stephen D.; Schmid, Wilfried; *Summation formulas, from Poisson and Voronoi to the present,* Noncommutative harmonic analysis, 419–440, Progr. Math., 220, Birkhuser Boston, Boston, MA, 2004.

[M-S2] Miller, Stephen D.; Schmid, Wilfried; *Automorphic distributions, L-functions, and Voronoi summation for* $GL(3)$*,* Ann. of Math. (2) **164** (2006), no. 2, 423–488.

[**V**] Voronoï, Georges; *Sur une fonction transcendante et ses applications á la sommation de quelques séries,* (French) Ann. Sci. École Norm. Sup. (3) **21** (1904), 207–267, 459–533.

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