

Topics in Number theory

Dorian Goldfeld

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Lecture 1

2010-1-19

1 Introduction

Topic: Automorphic representations for $GL(n, \mathbb{A})$, focus when mainly $n > 2$.

Recommended prior knowledge

- modular forms
- adeles
- representation theory

References

- Automorphic forms and L -function for $GL(n, \mathbb{R})$, Goldfeld
- Representation theory for $GL(n)$ over non-archimedean local fields, D.Prasad, A.Raghuram
- Zeta functions for simple algebras, Godement, Jacquet

Topics:

1. Classical automorphic forms for $GL(n, \mathbb{R})$
2. Automorphic representations for $GL(n, \mathbb{A})$ for \mathbb{A} , an adèle ring over \mathbb{Q}
3. One-to-one correspondence between automorphic representations and classical automorphic forms
4. Adelic automorphic forms, Fourier-Whittaker expansions, and L -functions
5. Admissible representations of $GL(n, \mathbb{Q}_p)$ and $GL(n, \mathbb{R})$
6. Parabolic induction, Jacquet Modules
7. Supercuspidal representations
8. Bernstein-Zelavinski classification
9. Matrix coefficients
10. Classification of representations by growth of matrix coefficients
11. Local-Global L -functions as integrals of matrix coefficients (Godement-Jacquet)

For 1 \sim 4, we have global theory, and after 5 we have local theory.

1.1 Automorphic forms

Reference: paper by Borel

Let $G =$ group and $X =$ topological space, and G acts on X , for $g \in G$, $x \in X$,

$$g.x \in X$$

where \cdot is an action. Assume that there exists a function, $\psi : G \times X \rightarrow \mathbb{C}$. We look at the space of automorphic functions with multiplier ψ :

$$\{f : X \rightarrow \mathbb{C}, f(g.x) = \psi(g, x)f(x)\}.$$

$$\begin{aligned} f(g_1 g_2 . x) &= \psi(g_1 g_2, x) f(x) \\ &= f(g_1 . (g_2 . x)) = \psi(g_1, g_2 . x) f(g_2 . x) = \psi(g_1, g_2 . x) \psi(g_2, x) f(x) \\ \Rightarrow \text{Cocycle relation: } \psi(g_1 g_2, x) &= \psi(g_1, g_2 . x) \psi(g_2, x), f(x) \neq 0 \end{aligned}$$

(Cohomology: cocycle relation)

Example: $G = GL(2, \mathbb{R})$ and $X = \mathbb{H} = \{z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}, y > 0\}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . z = \frac{az + b}{cz + d}, \text{ linear fractional action}$$

one-cocycle: $j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) := cz + d$

We have famous classical modular forms,

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z)$$

for $k = 2, 4, \dots$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$.

1.2 Automorphic representations

Gelfand-Gruer, P.Shapiro, 1970 Jacquet-Langlands

Representation = vector space V with a group G acting on V .

Roughly, If $V =$ space of automorphic function then we have an automorphic representations

$$V = \{f : X \rightarrow \mathbb{C} \mid f(g.x) = \psi(g, x)f(x)\}$$

with the right action,

$$h.f(g) = f(gh), g, h \in G$$

1.3 Adelic automorphic forms

We'll be looking at the group

$$GL(n, \mathbb{A}) = \text{group of } n \times n \text{ matrices, } \det \neq 0 \\ \text{with coefficients in } \mathbb{A}$$

where

$$\mathbb{A} = \left\{ \{a_\infty, a_2, \dots, a_p, \dots\}, a_\infty \in \mathbb{R}, a_p \in \mathbb{Q}_p, \right. \\ \left. a_p \in \mathbb{Z}_p \text{ for almost all } p \right\}$$

- $g \in GL(n, \mathbb{Q})$ can be embedded diagonally in $GL(n, \mathbb{A})$

$$\{g, g, \dots, \} \in GL(n, \mathbb{A})$$

- $GL(n, \mathbb{Q}) = \{\{g, g, \dots, \}, g \in GL(n, \mathbb{Q})\}$.
- $GL(n, \mathbb{Q})$ acts on $GL(n, \mathbb{A})$. For $\gamma = \{\alpha, \alpha, \dots, \}$, $\alpha \in GL(n, \mathbb{Q})$ and $g \in GL(n, \mathbb{A})$ then we have the action:

$$\gamma.g = \gamma g, \text{ matrix multiplication}$$

Adelic Automorphic functions

$$f : GL(n, \mathbb{A}) \rightarrow \mathbb{C}$$

$$f(\gamma.g) = f(g), \forall \gamma \in GL(n, \mathbb{Q}), g \in GL(n, \mathbb{A}).$$

Forms vs. functions: Forms are eigenfunctions for certain Differential operators for Lie. At place ∞ (real place), we need a theory for (\mathfrak{g}, K) -modules.

Automorphic forms: automorphic functions with

- growth condition
- eigenfunction of certain differential operators

1.4 Local theory

We consider the vector spaces of functions

$$V_p = \{f : GL(n, \mathbb{Q}_p) \rightarrow \mathbb{C}\}$$

with the right action,

$$h.f(g) = f(gh), g, h \in GL(n, \mathbb{Q}_p).$$

Then we have local representations. (This is a group representation.) We want to classify the representations. At ∞ it is different.

$$V_\infty = \{f : GL(n, \mathbb{R}) \rightarrow \mathbb{C}\}$$

with an action by differential operator,

$$D.f = Df$$

Here D acts as a differential operator. (This is not a group representation. In a group, every element has inverse, but for differential operators, we don't have inverses. But it is an algebraic representation.)

1.5 Parabolic inductions

You can look at this form

$$P := \begin{pmatrix} m_1 & & & \\ & m_2 & * & \\ & & \ddots & \\ & & & m_r \end{pmatrix}, m_i \in GL(n_i), n_1 + \dots + n_r = n$$

Parabolic subgroups.

$$\text{Ind}_P^{GL(n, \mathbb{R})} \text{ powerful way of classifying local representations}$$

1.6 Matrix coefficients

Globally from a classical point of view

- Non-holomorphic modular forms:

$$V := \left\{ f : \mathbb{H} \rightarrow \mathbb{C}, \left| \begin{array}{l} f\left(\frac{az+b}{cz+d}\right) = f(z), \forall z \in \mathbb{H}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \\ \Delta f = \lambda f \end{array} \right. \right\}$$

where $\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$.

- Petersson Inner Product: $f_1, f_2 \in V$

$$\langle f_1, f_2 \rangle := \iint_{SL(2, \mathbb{Z}) \backslash \mathbb{H}} f_1(z) \overline{f_2(z)} \frac{dx dy}{y^2}$$

Hilbert space, Unitary representation.

Let

$$\pi : G \rightarrow GL(V)$$

where

$$\pi(h).f(g) := f(gh)$$

for all $g, h \in G$. Then the inner product satisfies the following:

$$\langle \pi(h).f_1, \pi(h).f_2 \rangle = \langle f_1, f_2 \rangle.$$

Once you have such a thing, then the matrix coefficient:

$$g \mapsto \langle \pi(g).f_1, f_2 \rangle.$$

Harish-Chandra discovered that local and global automorphic representations can be classified by growth of matrix coefficients. If the matrix coefficients is L^2 , $f_1, f_2 \in V$, fixed,

$$\int_G |\langle \pi(g).f_1, f_2 \rangle|^2 dg < \infty$$

then the representation is called square integrable by some people but other people say the representation is discrete series. For any $\epsilon > 0$,

$$\int_G |\langle \pi(g).f_1, f_2 \rangle|^{2+\epsilon} dg < \infty$$

then the representation is tempered.

1.7 L-functions

Let $V =$ vector space of automorphic functions (local or global) and $G =$ group. Usually, $G = GL(n, \mathbb{A})$, $GL(n, \mathbb{Q}_p)$ or $GL(n, \mathbb{R})$. Fix $v_1, v_2 \in V$ and look at matrix coefficients.

$$g \mapsto \langle \pi(g).v_1, v_2 \rangle$$

and integrate over G :

$$L(s; H, \pi, v_1, v_2) := \int_G \langle \pi(g).v_1, v_2 \rangle H(g) |\det g|^s dg$$

where H is a function with a rapid decay. This function is an L -function.

Globally the Petersson inner product is Hermitian and we have unitary representations. It is not hard to show that all the representations. The global L -function is factored into local L -functions.

Last remark If you are working with modular forms, then

$$f(z) = f\left(-\frac{1}{z}\right), \quad f(iy) = f\left(\frac{i}{y}\right)$$

Traditionally this is used to obtain the function equation. But if we work in the above way, then we don't need the relations. The proof is different.

Lecture 2

2010-1-21

2 Classical automorphic forms for $GL(n, \mathbb{R})$

2.1 Review of classical automorphic forms for $GL(2, \mathbb{R})$

Hecke.

$$\mathbb{H} = \{x + iy, x \in \mathbb{R}, y > 0\}$$

Let $f : \mathbb{H} \rightarrow \mathbb{C}$ be a holomorphic function that satisfies the following condition:

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then $f(z + 1) = f(z)$

$$\Rightarrow f(z) = \sum_{n \in \mathbb{Z}} a(n) e^{2\pi i n z} = \sum_{n \in \mathbb{Z}} a(n) e^{2\pi i n x} e^{-2\pi n y}$$

In this case if n is negative, then $e^{-2\pi n y}$ blows up! So we need Moderate Growth Assumption

Moderate Growth Assumption There exists $C, B > 0$ such that

$$|f(z)| \leq C|z|^B, \quad \text{as } |z| \rightarrow \infty$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a(n) e^{2\pi i n z}$$

Definition 2.1 (Holomorphic modular form of weight k for $SL(2, \mathbb{Z})$). A function $f : \mathbb{H} \rightarrow \mathbb{C}$ is a holomorphic modular form of weight k if f satisfies the following conditions:

- (1) $f : \mathbb{H} \rightarrow \mathbb{C}$ holomorphic
- (2) $|f(z)| \leq C|z|^B$ as $|z| \rightarrow \infty$ (moderate growth)
- (3) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$

Later Hecke generalize the definition: Fix an integer $N \geq 1$. Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

Fix a Dirichlet character

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times.$$

Definition 2.2 (Holomorphic modular form of weight k , level N , character χ modular N). A function $f : \mathbb{H} \rightarrow \mathbb{C}$ is a holomorphic modular form of weight k , level N , character χ modular N if f satisfies the following conditions:

- (1) $f : \mathbb{H} \rightarrow \mathbb{C}$ holomorphic
- (2) f has moderate growth
- (3) $f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$, $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$

The space of these functions is the vector space and it is a Hilbert space with the Petersson inner product.

There is no reason that these functions should be holomorphic. In 1940, Maass introduced non-holomorphic forms.

Maass 1940 Introduced non-holomorphic automorphic forms.

Definition 2.3 (Maass form of weight k , level N and character χ modular N). A function $f : \mathbb{H} \rightarrow \mathbb{C}$ is a Maass form of weight k if f satisfies the following conditions:

- (1) $f : \mathbb{H} \rightarrow \mathbb{C}$ smooth and satisfies $\Delta_k f = \lambda f$ where

$$\Delta_k = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \frac{\partial}{\partial x} \text{ (elliptic partial differential operator)}$$

and $\lambda \in \mathbb{R}$

- (2) Moderate growth: There exists $B, C > 0$ such that

$$|f(z)| \leq C|z|^B, \text{ as } |z| \rightarrow \infty.$$

- (3) $f\left(\frac{az+b}{cz+d}\right) = \chi(d) \left(\frac{cz+d}{|cz+d|}\right)^k f(z)$, $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$

Aside: Δ_k satisfies the following condition: if f satisfies (3) then $\Delta_k f$ also satisfies (3).

The third conditions for Maass forms and holomorphic modular forms are equivalent. Let $z = x + iy$ and $\text{Im}(z) = y$. Then

$$\text{Im}\left(\frac{az+b}{cz+d}\right) = \frac{y}{|cz+d|^2}, \text{ if } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$$

Let f be a holomorphic modular form and

$$F(z) := f(z)\text{Im}(z)^{\frac{k}{2}}$$

then $F\left(\frac{az+b}{cz+d}\right) = \chi(d) \left(\frac{cz+d}{|cz+d|}\right)^k F(z)$.

Let's look at the case when the weight $k = 0$.

$$\Delta(z) := \Delta_0(z) = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

Lemma 2.4.

$$\Delta\left(\frac{az+b}{cz+d}\right) = \Delta(z)$$

i.e., Δ does not change when $z \mapsto \frac{az+b}{cz+d}$ for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$

Proof.

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$$

Then

$$\Delta(z) = -4\text{Im}(z)^2 \frac{\partial}{\partial z} \circ \frac{\partial}{\partial \bar{z}}$$

$$z \mapsto \frac{az + b}{cz + d}, \quad \partial z \mapsto (cz + d)^2 \partial z, \quad \partial \bar{z} \mapsto (\overline{cz + d})^2 \partial \bar{z} \quad \text{and} \quad \text{Im}(z) \mapsto \frac{\text{Im}(z)}{|cz + d|^2}$$

□

We want to generalize Maass forms to $GL(n, \mathbb{R})$ for all $n \geq 2$.

2.2 Iwasawa decomposition

Every $g \in GL(n, \mathbb{R})$ can be uniquely written in the form

$$g = \tilde{g} \cdot k \cdot d$$

where $k \in O(n, \mathbb{R}) = \{h \in GL(n, \mathbb{R}) \mid h \cdot {}^t h = I_n\}$, d is a diagonal matrix, $d = \begin{pmatrix} \delta & & \\ & \ddots & \\ & & \delta \end{pmatrix}$ for $\delta \neq 0$ and $\tilde{g} \in \mathbb{H}^n$.

Definition 2.5.

$$\mathbb{H}^n = \left\{ x \cdot y \mid x = \begin{pmatrix} 1 & x_{ij} & \\ & \ddots & \\ & & 1 \end{pmatrix}, y = \begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & y_1 & \\ & & & 1 \end{pmatrix} \text{ and } x_{ij} \in \mathbb{R}, y_i > 0 \right\}$$

Proof. for Iwasawa decomposition.

Let $g \in GL(n, \mathbb{R})$. Consider $g \cdot {}^t g =$ positive definite non-singular matrix. Claim that there

exists an upper triangular matrix $u = \begin{pmatrix} 1 & * & \\ & \ddots & \\ & & 1 \end{pmatrix}$, and lower triangular matrix $l = \begin{pmatrix} 1 & & \\ * & \ddots & \\ & & 1 \end{pmatrix}$

and diagonal matrix $d = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{pmatrix}$ for $\delta_i \neq 0$ such that

$$(*) \quad u \cdot g \cdot {}^t g = l \cdot d.$$

For $n = 2$: $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, ${}^t g = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, $g {}^t g = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix}$, $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g {}^t g = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$.

$$(*) \Rightarrow g {}^t g = u^{-1} l d = {}^t (u^{-1} l d) = d {}^t l u^{-1}$$

$$\Rightarrow (**) \quad l d {}^t u = u d {}^t l$$

Then the LH is lower triangular and RH is upper triangular so they are diagonal. So

$$(**) \quad l d {}^t u = u d {}^t l = d$$

$$\Rightarrow (***) \quad l d = d {}^t u^{-1}$$

$$(***) \text{ into } (*) \Rightarrow u g {}^t g = d {}^t u^{-1} \Rightarrow u g {}^t g u = d = a^{-1t} a^{-1}$$

$$\Rightarrow a u g \cdot {}^t (a u g) = I_n$$

$$\Rightarrow a u g = k \in O(n, \mathbb{R}) \text{ Iwasawa decomposition}$$

□

2.3 Maass forms of weight 0 for $SL(n, \mathbb{Z})$

Let $n \geq 2$.

(1) $f : \mathbb{H}^n \rightarrow \mathbb{C}$ smooth and satisfy the set of differential equations $\Delta_i = \lambda_i f$ for $i = 1, 2, \dots, n-1$. Here $\Delta_i(z) = \Delta_i(\gamma z)$ for all $z \in \mathbb{H}^n$ $\gamma \in SL(n, \mathbb{R})$.

(2) Moderate growth condition: there exists $C > 0$, $B_1, B_2, \dots, B_{n-1} > 0$ such that

$$|f(z)| \leq C y_1^{B_1} \cdots y_{n-1}^{B_{n-1}}$$

as $y_1, \dots, y_{n-1} \rightarrow \infty$.

(3) $f(\gamma z) = f(z)$ for all $\gamma \in SL(n, \mathbb{Z})$ and

$$f(zkd) = f(z), \quad \forall z \in \mathbb{H}^n, k \in K_n = O(n, \mathbb{R}), d \in Z_n = \text{center}(GL(n, \mathbb{R})).$$

Let $K = O(n, \mathbb{R}) =$ orthogonal group.

$$f : GL(n, \mathbb{R})/O(n, \mathbb{R}) \cdot \mathbb{R}^\times \rightarrow \mathbb{C}$$

These Maass forms of weight 0 are K -invariant under right multiplication by $k \in K$ correspond to unramified representation.

Lecture 3

2010-01-26

Review

Iwasawa decomposition For any $g \in GL(n, \mathbb{R})$ we have unique

$$x = \begin{pmatrix} 1 & x_{ij} \\ & \ddots \\ 0 & & 1 \end{pmatrix}, y = \begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix}, k \in O(n, \mathbb{R})$$

and $d = \begin{pmatrix} \delta & & \\ & \ddots & \\ & & \delta \end{pmatrix}$ such that

$$g = xykd.$$

Using Iwasawa decomposition, we can define generalized upper half plane:

$$\mathbb{H}^n = \{xy\}$$

For example, for $n = 2$,

$$\mathbb{H}^2 = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, x \in \mathbb{R}, y > 0 \right\}.$$

Automorphic functions on \mathbb{H}^n of weight 0 for $GL(n, \mathbb{Z})$

$$\phi : \mathbb{H}^n \rightarrow \mathbb{C}, \quad \phi(\gamma z) = \phi(z), \quad \forall \gamma \in GL(n, \mathbb{Z}), z \in \mathbb{H}^n.$$

For example we have Ramanujan weight 12 function

$$\Delta \left(\frac{az + b}{cz + d} \right) = (cz + d)^{12} \Delta(z)$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$.

2.4 Generalize level and weights

Definition 2.6 (Jacquet, Shalika, Piatetski-Shapiro). Fix an integer $N \geq 1$. Fix $n \geq 2$. We define

$$\Gamma_0(N) := \left\{ \begin{pmatrix} A & B \\ C & d \end{pmatrix} \in GL(n, \mathbb{Z}) \mid \begin{array}{l} A \in GL(n-1, \mathbb{Z}), \\ C = (c_1, c_2, \dots, c_{n-1}) \equiv (0, 0, \dots, 0) \pmod{N}, d \in \mathbb{Z} \end{array} \right\} \quad (2.1)$$

Let $\gamma \in GL(n, \mathbb{R})^+$ and $z \in \mathbb{H}^n$. By the Iwasawa decomposition,

$$\gamma z = \widetilde{\gamma} g \cdot \kappa(\gamma, z) \cdot d$$

where $\widetilde{\gamma} g \in \mathbb{H}^n$ and $\kappa(\gamma, z) \in O(n, \mathbb{R})^+ =: K$ and $d = \begin{pmatrix} \delta & & \\ & \ddots & \\ & & d \end{pmatrix}$ for $\delta > 0$. i.e.,

$$\kappa : GL(n, \mathbb{R})^+ \times \mathbb{H}^n \rightarrow K.$$

For example, $n = 2$, $z = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{cx+d}{|cz+d|} & -\frac{cy}{|cz+d|} \\ \frac{cy}{|cz+d|} & \frac{cx+d}{|cz+d|} \end{pmatrix} \begin{pmatrix} |cz+d| & 0 \\ 0 & |cz+d| \end{pmatrix}$$

where $\begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} \in \mathbb{H}^2$ and $\begin{pmatrix} \frac{cx+d}{|cz+d|} & -\frac{cy}{|cz+d|} \\ \frac{cy}{|cz+d|} & \frac{cx+d}{|cz+d|} \end{pmatrix} \in SO(2, \mathbb{R})$.

Lemma 2.7. $\kappa(\gamma, z)$ is a one-cocycle, i.e.,

$$\kappa(\gamma\gamma', z) = \kappa(\gamma, \widetilde{\gamma}'z).$$

Proof.

$$\gamma\gamma'z = \widetilde{\gamma}'z \cdot \kappa(\gamma\gamma', z) \cdot d$$

and

$$\begin{aligned} \gamma(\gamma'z) &= \gamma \left(\widetilde{\gamma}'z \kappa(\gamma', z) d' \right) \\ &= \widetilde{\gamma}'z \cdot \kappa(\gamma, \widetilde{\gamma}'z) \cdot \kappa(\gamma', z) d'' \\ &= \widetilde{\gamma}'z \kappa(\gamma, \widetilde{\gamma}'z) \kappa(\gamma', z) \cdot d \end{aligned}$$

by the uniqueness of the Iwasawa decomposition, we are done. \square

Fix $n \geq 2$ and $r \geq 1$. Let

$$\rho : SO(n, \mathbb{R}) \rightarrow GL(r, \mathbb{C})$$

be an irreducible representation. Then ρ will be the analogue of the weight k .

Definition 2.8.

$$J_\rho(\gamma, z) = \rho \left(\kappa(\gamma, z)^{-1} \right) \quad (2.2)$$

for $z \in \mathbb{H}^n$, $\gamma \in GL(n, \mathbb{R})^+$. Furthermore,

$$J_\rho(\gamma\gamma', z) = J_\rho(\gamma, \widetilde{\gamma}'z) \cdot J_\rho(\gamma', z).$$

Example for $n = 2$:

$$SO(2, \mathbb{R}) = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \mid 0 \leq \theta \leq 2\pi \right\}$$

is an abelian group. The only representation of $SO(2, \mathbb{R})$ are characters, i.e., Define

$$\rho \left(\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right) := (\cos \theta + i \sin \theta)^k$$

for some integer $k \in \mathbb{Z}$. This is a one-dimensional representation of $SO(2, \mathbb{R})$.

$$J_\rho(\gamma, z) = \rho(\kappa(\gamma, z)^{-1}) = \rho \left(\begin{pmatrix} \frac{cx+d}{|cz+d|} & \frac{cy}{|cz+d|} \\ -\frac{cy}{|cz+d|} & \frac{cx+d}{|cz+d|} \end{pmatrix} \right) = \left(\frac{cz+d}{|cz+d|} \right)^k$$

2.5 Classical vector valued automorphic functions

$\Phi : \mathbb{H}^n \rightarrow \mathbb{C}^r$ and $\phi_i : \mathbb{H}^n \rightarrow \mathbb{C}$ for $i = 1, 2, \dots, r$ such that

$$\Phi(z) = \begin{pmatrix} \phi_1(z) \\ \vdots \\ \phi_r(z) \end{pmatrix}$$

Definition 2.9 (Slash operator).

$$(\Phi |_\rho)(z) := J_\rho(\gamma, z)^{-1} \Phi(\gamma, z)$$

Definition 2.10 (Classical vector valued automorphic functions of arbitrary weight $\rho : SO(n, \mathbb{R}) \rightarrow GL(r, \mathbb{C})$, level $N \geq 1$ and character $\chi \bmod N$). Fix integers $n \geq 2, r \geq 1$. Fix irreducible representation

$$\rho : SO(n, \mathbb{R}) \rightarrow GL(r, \mathbb{C}).$$

Let $\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix} : \mathbb{H}^n \rightarrow \mathbb{C}^r$. Fix $N \geq 1$ and a Dirichlet character $\chi \bmod N$. Then Φ is a classical automorphic function of weight ρ , level N and character χ if

$$(\Phi |_\gamma)(z) = \chi(d) \Phi(z)$$

for any $z \in \mathbb{H}^n$ and $\gamma = \begin{pmatrix} A & B \\ C & d \end{pmatrix} \in \Gamma_0(N)$.

Recall that an automorphic form is an automorphic function satisfying 2 extra conditions:

- (1) Moderate growth
- (2) Satisfy some differential equations

Definition 2.11 (Moderate growth). A smooth function $f : \mathbb{H}^n \rightarrow \mathbb{C}$ is of moderate growth if for each fixed $\sigma \in GL(n, \mathbb{Q})$ there exists $C > 0, B > 0$ such that

$$|f(\sigma z)| \leq C |y_1 y_2 \cdots y_{n-1}|^B$$

for $z = xy, y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix}$.

We'll work for $GL(n, \mathbb{R})^+$. Why? Explain by example below:

Lift from upper half-plane $\mathbb{H}^2 = \{x + iy, x \in \mathbb{R}, y > 0\}$ to $GL(2, \mathbb{R})^+$ Let $f : \mathbb{H}^2 \rightarrow \mathbb{C}$,
 $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$(f|_k g)(z) := \left(\frac{cz + d}{|cz + d|} \right)^{-k} f\left(\frac{az + b}{cz + d} \right).$$

If $g \in GL(2, \mathbb{R})^+$, by Iwasawa

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}$$

for $0 \leq \theta \leq 2\pi$, $\delta > 0$.

Definition 2.12.

$$\tilde{f}(g) = (f|_k g)(i) = e^{ik\theta} f(x + iy)$$

Then

$$\tilde{f} : GL(2, \mathbb{R})^+ \rightarrow \mathbb{C}.$$

If we use $GL(2, \mathbb{R})$ then \tilde{f} will be the combination of weight k and weight $-k$.

Let Φ be automorphic for $GL(n, \mathbb{R})$

Definition 2.13 (Lift to $GL(n, \mathbb{R})^+$).

$$\tilde{\Phi}(g) := (\Phi|_{\rho} g)(I_n), \quad I_n = n \times n \text{ identity matrix.}$$

Differential operators Let $\alpha \in \mathfrak{gl}(n, \mathbb{R})$ and $f : GL(n, \mathbb{R})^+ \rightarrow \mathbb{C}$.

$$D_{\alpha} f(g) := \lim_{t \rightarrow 0} \frac{1}{t} [f(g \cdot \exp(t\alpha)) - f(g)]$$

(Lie derivative) We are looking for a differential operator which takes automorphic functions to automorphic functions

Casimir

$E_{ij} = n \times n$ matrix which is 1 at position (i, j) and 0 every where else

Let $D_{ij} := D_{E_{ij}}$.

$$\sum_{i_1=1}^m \sum_{i_2=1}^m \cdots \sum_{i_m=1}^m D_{i_1, i_2} \circ \cdots \circ D_{i_m, i_1}$$

for $1 \leq m \leq n$. We have n different Casimir operators. The Casimir operator generate $\mathcal{Z}(\mathcal{U}(\mathfrak{g})) = \text{center of the universal enveloping algebra, where } \mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$.

Definition 2.14 (Maass form for $GL(n, \mathbb{R})$). Fix $n \geq 2$, $r \geq 1$, $\rho : SO(n, \mathbb{R}) \rightarrow GL(r, \mathbb{C})$, $N \geq 1$ and $\chi \pmod{N}$. Let

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix} : \mathbb{H}^n \rightarrow \mathbb{C}^r$$

Then Φ is a vector valued Masss form of weight ρ , level N and character χ . if

$$(1) (\Phi|_{\rho} \gamma)(z) = \chi(d)\Phi(z), \quad \forall \gamma = \begin{pmatrix} A & B \\ C & d \end{pmatrix} \in \Gamma_0(N), \quad z \in \mathbb{H}^n$$

(2) Φ has moderate growth.

(3) For any Casimir differential operator $D \in \mathcal{Z}(\mathcal{U}(\mathfrak{g}))$, $\exists \lambda_D \in \mathbb{C}$ such that

$$D\Phi = \lambda_D \Phi$$

(4) L^2 condition.

Lecture 4

2010-01-28

2.6 Casimir differential operators on $GL(n, \mathbb{R})$

For $\alpha \in \mathfrak{gl}(n, \mathbb{R})$ and $f : GL(n, \mathbb{R}) \rightarrow \mathbb{C}$ we define Lie Derivative

$$D_\alpha f(g) := \lim_{t \rightarrow 0} \frac{1}{t} [f(g \cdot \exp(t\alpha)) - f(g)].$$

For $m = 1, 2, \dots, n$ we have the Casimir operators

$$D_m := \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_m=1}^n D_{i_1, i_2} \circ D_{i_2, i_3} \circ \cdots \circ D_{i_m, i_1} \quad (2.3)$$

where $E_{i,j} = n \times n$ matrix with 1 at position (i, j) , zeros everywhere else and $D_{i,j} = D_{E_{i,j}}$.

Example for $n = 3$

$$z = x \cdot y = \begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} y_1 y_2 & x_2 y_1 & x_3 \\ 0 & y_1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{H}^3$$

$$E_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

then

$$\exp(tE_{2,3}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} + \text{higher powers}$$

Let's compute

$$\begin{aligned} D_{2,3} f(z) &= \lim_{t \rightarrow 0} \frac{1}{t} \left[f \left(z \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \right) - f(z) \right] \\ &= \frac{\partial}{\partial t} f \left(\begin{pmatrix} y_1 y_2 & x_2 y_1 & x_2 y_1 t + x_3 \\ 0 & y_1 & y_1 t + x_1 \\ 0 & 0 & 1 \end{pmatrix} \right) \Big|_{t=0} \\ &= x_2 y_1 \frac{\partial}{\partial x_3} f(z) + y_1 \frac{\partial}{\partial x_1} f(z) \end{aligned}$$

Then $D_{2,3} = x_2 y_1 \frac{\partial}{\partial x_3} + y_1 \frac{\partial}{\partial x_1}$.

Prove "Invariant" Let $V =$ vector space of all smooth functions $f : GL(n, \mathbb{R}) \rightarrow \mathbb{C}$ and

$$\pi : GL(n, \mathbb{R}) \rightarrow \text{End}(V)$$

given by right translation, i.e., for any $f \in V$, and for any $h \in GL(n, \mathbb{R})$ define the action

$$\pi(h).f(g) := f(gh).$$

Now we want to prove the following theorem:

Theorem 2.15. Fix $n \geq 2$ and for $1 \leq m \leq n$ let D_m be the Casimir differential operator defined in (2.3). Then

$$(\pi(h) \circ D_m) \cdot f(g) = (D_m \circ \pi(h)) \cdot f(g)$$

for any $g, h \in GL(n, \mathbb{R})$ and any $f \in V$.

Proof. Define

$$M = \begin{pmatrix} D_{1,1} & \cdots & D_{1,n} \\ \vdots & \cdots & \vdots \\ D_{n,1} & \cdots & D_{n,n} \end{pmatrix}$$

Claim: $\text{Trace}(M^m) = D_m$.

Need to prove that $\pi(h) \circ D_m \circ \pi(h)^{-1} = D_m$

Claim: $\pi(h) \circ D_\alpha \circ \pi(h)^{-1} = D_{h\alpha h^{-1}}$

Because

$$\begin{aligned} h \exp(\alpha) h^{-1} &= \exp(h\alpha h^{-1}) = \sum_{k=0}^{\infty} \frac{(h\alpha h^{-1})^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{h\alpha^k h^{-1}}{k!} \end{aligned}$$

π and D are linear operators and

$$\text{Trace}(\pi(h)M^m\pi(h)^{-1}) = \text{Trace}(M^m)$$

□

Easy to show

$$D_m \circ D_{i,j} = D_{i,j} \circ D_m, \forall i, j, m$$

$\Rightarrow D_m$ commutes with all differential operators.

Definition 2.16 (Maass forms for $GL(n, \mathbb{R})$ of weight ρ , level N and character χ). Smooth function $f : \mathbb{H}^n \rightarrow \mathbb{C}^r$, $\mathbb{H}^n = GL(n, \mathbb{R})/O(n, \mathbb{R}) \cdot \mathbb{R}^\times$.

$\rho : O(n, \mathbb{R}) \rightarrow GL(r, \mathbb{C})$, irreducible representation

$$\Gamma_0(N) = \left\{ \begin{pmatrix} A & B \\ C & d \end{pmatrix} \mid \begin{array}{l} A \in GL(n-1, \mathbb{Z}), \\ C = (c_1, \dots, c_{n-1}) \equiv (0, \dots, 0) \pmod{N}, \\ d \in \mathbb{Z} \end{array} \right\}$$

(1) $(f|_{\rho}\gamma)(z) = \chi(d)f(z)$, $\forall \gamma \in \Gamma_0(N)$

(2) Moderate growth

(3) $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that $D_m f = \lambda_m f$ where $D_m = m$ th Casimir operator.

(4) $\int_{\Gamma_0(N)\backslash\mathbb{H}^n} |f_i(z)|^2 d^*z < \infty$ where $f(z) = \begin{pmatrix} f_1(z) \\ \vdots \\ f_r(z) \end{pmatrix}$.

2.7 Cusp forms

Definition 2.17. Fix $n \geq 2$, $1 \leq m < n$. Let $\mathbb{H}^n = GL(n, \mathbb{R})/O(n, \mathbb{R}) \cdot \mathbb{R}^\times$. Define $X_m^n \subset \mathbb{H}^n$ by

$$X_m^n := \left\{ \begin{pmatrix} I_m & * \\ 0 & I_{n-m} \end{pmatrix} \right\}$$

where $I_k = k \times k$ identity matrix.

$$Y_m^n := \left\{ \begin{pmatrix} z'd & 0 \\ 0 & z'' \end{pmatrix} \left| z' \in \mathbb{H}^m, z'' \in \mathbb{H}^{m-n}, d = \begin{pmatrix} \delta & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \delta \end{pmatrix}, \delta > 0 \right\}.$$

Definition 2.18 (Constant terms of a classical automorphic form). Fix $n \geq 2$, $r > 1$, $N \geq 1$, $\chi_{\text{mod } N}$ and $\rho : O(n, \mathbb{R}) \rightarrow GL(r, \mathbb{C})$ (irreducible). Let

$$\Phi := \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix}$$

be a vector valued Maass form of weight ρ , level N and character χ . For $1 \leq m < n$ and $\gamma \in GL(n, \mathbb{Z})$ let $D(m, N, \gamma)$ be a fundamental domain for $X_m^n(\mathbb{Z}) \cap (\gamma^{-1}\Gamma_0(N)\gamma)$ acting on $X_m^n(\mathbb{R})$. i.e.,

$$D(m, N, \gamma) := (X_m^n(\mathbb{Z}) \cap \gamma^{-1}\Gamma_0(N)\gamma) \backslash X_m^n(\mathbb{R})$$

Then we define

$$\Phi_{m,\gamma}(y) := \int_{D(m,N,\gamma)} (\Phi |_{\rho}\gamma)(xy) dx = \begin{array}{l} \text{constant terms along} \\ X_m^n \text{ for } \gamma^{-1}\Gamma_0(N)\gamma \end{array}$$

Definition 2.19 (Maass cusp form for $GL(n, \mathbb{R})$). A vector valued Mass form $\Phi : \mathbb{H}^n \rightarrow \mathbb{C}^r$ is said to be a cusp form of

$$\Phi_{m,\gamma}(y) = 0$$

for all $1 \leq m < n$ and all $\gamma \in GL(n, \mathbb{Z})$.

Example for $n = 2$ In this case $m = 1$. Then

$$X_1^2 = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

$$X_1^2(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}, l \in \mathbb{Z} \right\}$$

Then

$$X_1^2(\mathbb{Z}) \cap \gamma^{-1}\Gamma_0(N)\gamma = \text{cyclic group generated by } \begin{pmatrix} 1 & m_{\mathfrak{a}} \\ 0 & 1 \end{pmatrix}$$

where $\mathfrak{a} \in \mathbb{Q} \cup \{\infty\}$ is a cusp, $\mathfrak{a} = \gamma\infty$. So

$$\int_{\begin{pmatrix} 1 & m_{\mathfrak{a}} \\ 0 & 1 \end{pmatrix} \backslash \begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix}} = \int_0^{m_{\mathfrak{a}}}$$

3 Adelic automorphic forms for $GL(n)$

3.1 Adelic rings

Adelic ring

$$\mathbb{A}_{\mathbb{Q}} = \{(a_{\infty}, a_2, \dots, a_p, \dots)\}$$

where $a_{\infty} \in \mathbb{R}$, $a_p \in \mathbb{Q}_p$ and $a_p \in \mathbb{Z}_p$ for all but finitely many p .

$$K := O(n, \mathbb{R}) \prod_p GL(n, \mathbb{Z}_p) \subset GL(n, \mathbb{A}_{\mathbb{Q}})$$

is the maximal compact subgroup.

$GL(n, \mathbb{Q})$ can be diagonally embedded in $GL(n, \mathbb{A})$ if $\alpha \in GL(n, \mathbb{Q})$ then $(\alpha, \alpha, \dots) \in GL(n, \mathbb{A})$.

3.2 Adelic automorphic functions

- $f : GL(n, \mathbb{A}) \rightarrow \mathbb{C}$
- $f(\gamma g) = f(g)$, $\forall \gamma \in GL(n, \mathbb{Q})$, $\forall g \in GL(n, \mathbb{A})$
- Smooth conditions: $f((g_{\infty}, g_2, \dots, g_p, \dots))$ should be differentiable at ∞ , and at p , a locally constant function.

Lecture 5

2010-2-2

3.3 Adelic automorphic forms for $GL(n, \mathbb{A}_{\mathbb{Q}})$

$$GL(n, \mathbb{A}_{\mathbb{Q}}) = \text{restricted product } GL(n, \mathbb{R}) \cdot \prod_p GL(n, \mathbb{Q}_p) \quad (3.1)$$

where restricted is with respect to $GL(n, \mathbb{Z}_p)$, i.e., if $g = \{g_{\infty}, g_2, \dots, g_p, \dots\} \in GL(n, \mathbb{A}_{\mathbb{Q}})$ then $g_p \in GL(n, \mathbb{Z}_p)$ for almost all finitely many p . In order to define automorphic forms, we need several conditions.

An Adelic Automorphic Form

$$f : GL(n, \mathbb{A}_{\mathbb{Q}}) \rightarrow \mathbb{C}$$

with central character $\omega : \mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}}^{\times} \rightarrow \mathbb{C}^{\times}$ (i.e., $\omega(\alpha z) = \omega(z)$ for any $z \in \mathbb{A}_{\mathbb{Q}}^{\times}$ and $\alpha \in \mathbb{Q}^{\times}$), satisfies several conditions.

- (1) f is smooth (i.e., f is smooth for ∞ place, and locally constant for finite places).
- (2) $f(\gamma g) = f(g)$, $\forall \gamma \in GL(n, \mathbb{Q})$, $g \in GL(n, \mathbb{A}_{\mathbb{Q}})$ where $\gamma = (\alpha, \alpha, \dots)$ for $\alpha \in GL(n, \mathbb{Q})$.
- (3) $f(zg) = \omega(z)f(g)$, $\forall z \in \mathbb{A}_{\mathbb{Q}}^{\times}$, $g \in GL(n, \mathbb{A}_{\mathbb{Q}})$
- (4) f is right K -finite, for $K = O(n, \mathbb{R}) \cdot \prod_p GL(n, \mathbb{Z}_p)$.
- (5) f is $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ -finite where $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, $\mathcal{U}(\mathfrak{g}) =$ universal enveloping algebra generated by D_{α} , $\alpha \in \mathfrak{g}$ and $\mathcal{Z}(\mathcal{U}(\mathfrak{g})) =$ center of $\mathcal{U}(\mathfrak{g})$.

(6) f is of moderate growth.

We're going to explain the conditions.

For $g \in GL(n, \mathbb{A}_{\mathbb{Q}})$, let $f(g) = f(g_{\infty}, \dots, g_p, \dots)$. Fix all g_v except at one place p , then smooth means that the function f restricted to the place p , $GL(n, \mathbb{Q}_p) \rightarrow \mathbb{C}$ is locally constant. For $v = \infty$, fix all $v < \infty$ then $f(g_{\infty}) : GL(n, \mathbb{R}) \rightarrow \mathbb{C}$ is smooth if this function is infinitely differentiable in all values $g_{i,j} \in \mathbb{R}$ for $g_{\infty} = (g_{i,j})_{1 \leq i,j \leq n}$.

Definition 3.1 (Right K -finite).

$$K = O(n, \mathbb{R}) \cdot \prod_p GL(n, \mathbb{Z}_p) = \begin{array}{l} \text{maximal compact} \\ \text{subgroup of } GL(n, \mathbb{A}_{\mathbb{Q}}) \end{array}$$

We say a function $f : GL(n, \mathbb{Q}) \rightarrow \mathbb{C}$ is right K -finite, if the vector space

$$\text{Span} \{f(gk), k \in K\}$$

is a finite dimensional vector space.

Definition 3.2 ($\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ -finite). Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ is the polynomial algebra generated by the Casimir differential operator D_1, \dots, D_n act on g_{∞} .

We say f is $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ -finite if the vector space

$$\text{Span} \{Df(g), \forall D \in \mathcal{Z}(\mathcal{U}(\mathfrak{g}))\}$$

is finite dimensional.

Norm on $GL(n, \mathbb{Q}_v)$ Let $a \in \mathbb{A}_{\mathbb{Q}}$,

$$\|a\| = \prod_{v \leq \infty} |a_v|_v \quad (3.2)$$

Let

$$g = \begin{pmatrix} g_{1,1} & \cdots & g_{1,n} \\ \vdots & \ddots & \vdots \\ g_{n,1} & \cdots & g_{n,n} \end{pmatrix} \in GL(n, \mathbb{Q}_v)$$

for $v \leq \infty$. For $v = \infty$, $GL(n, \mathbb{Q}_{\infty}) = GL(n, \mathbb{R})$. Define

$$\|g\|_v := \max(|g_{i,j}|_v, |\det g|_v)_{1 \leq i,j \leq n} \quad (3.3)$$

Let $g = \{g_{\infty}, g_2, \dots, g_p, \dots\} \in GL(n, \mathbb{A}_{\mathbb{Q}})$ then

$$\|g\| = \prod_{v \leq \infty} \|g_v\|_v. \quad (3.4)$$

3.4 Adelic lift of a classical automorphic form on $GL(n, \mathbb{R})$

Theorem 3.3. Fix integers $n \geq 2$, $r, N \geq 1$. Fix irreducible representation $\rho : O(n, \mathbb{R}) \rightarrow$

$GL(r, \mathbb{C})$. Fix a character $\chi : (N\mathbb{Z} \backslash \mathbb{Z})^{\times} \rightarrow \mathbb{C}^{\times}$. Let $\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix}$ be a classical vector valued

automorphic form (i.e., $(\Phi|_{\rho}\gamma)(z) = \chi(d) \cdot \Phi(z)$ for all $z \in \mathbb{H}^n \cong GL(n, \mathbb{R})/\mathbb{R}^{\times} \cdot O(n, \mathbb{R})$ and $\gamma = \begin{pmatrix} A & B \\ C & d \end{pmatrix} \in \Gamma_0(N)$). Then there exists r adelic automorphic forms which can be explicitly constructed from ϕ_1, \dots, ϕ_r which can be explicitly constructed from ϕ_1, \dots, ϕ_r .

Each $\phi_i \rightarrow \phi_{i,\text{adelic}}$.

The adelic requires strong approximation Two embeddings:

- $i_{\text{diag}}(\gamma) := (\gamma, \dots, \gamma, \dots) : GL(n, \mathbb{Q}) \rightarrow GL(n, \mathbb{A}_{\mathbb{Q}})$
- $i_{\infty}(g_{\infty}) := (g_{\infty}, I_n, I_n, \dots) : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{A}_{\mathbb{Q}})$

Lemma 3.4.

$$i_{\text{diag}}(SL(n, \mathbb{Q})) \cdot i_{\infty}(SL(n, \mathbb{R}))$$

is dense in $SL(n, \mathbb{A}_{\mathbb{Q}})$.

Sketch: Let $U_{i,j}$ be a matrix in $SL(n, \mathbb{A}_{\mathbb{Q}})$, 1 on the diagonal and $u_{i,j} \in \mathbb{A}_{\mathbb{Q}}^{\times}$ for (i, j) place, and otherwise 0. Then $SL(n, \mathbb{A}_{\mathbb{Q}})$ is generated by $U_{i,j}$. Use Tate Thesis for $n = 1$. Then we are done. \square

$$\mathbb{A}_{\text{finite}} = ((1)^\times, a_2, \dots, a_p, \dots), \quad a_p \in \mathbb{Q}_p \text{ and } a_p \in \mathbb{Z}_p \text{ for almost all } p$$

Definition 3.5. Let $N \geq 1$, $N = \prod_{i=1}^l p_i^{e_i}$. We define the compact subgroup $K_0(N) \subseteq GL(n, \mathbb{A}_{\mathbb{Q}})$ as follows:

$$K_0(N) \subseteq \prod_p GL(n, \mathbb{Z}_p) \subset GL(n, \mathbb{A}_{\text{finite}})$$

(for $k \in \prod_p GL(n, \mathbb{Z}_p)$, $k = (k_2, \dots, k_p, \dots)$) We define

$$K_0(N) = \left\{ k \in \prod_p GL(n, \mathbb{Z}_p) \left| \begin{array}{l} k_{p_i} = \begin{pmatrix} A_i & B_i \\ C_i & d_i \end{pmatrix}, \quad A_i \in GL(n-1, \mathbb{Z}_{p_i}), \\ C_i \in (p_i^{e_i} \mathbb{Z}_{p_i})^{n-1}, \quad d_i \in \mathbb{Z}_{p_i}, \text{ for } i = 1, \dots, l, \text{ otherwise } k_p \in GL(n, \mathbb{Z}_p) \end{array} \right. \right\}$$

(it is a compact open subset.)

Theorem 3.6 (Stronger approximate). Fix $N \geq 1$ and $n \geq 2$. Let $g \in GL(n, \mathbb{A}_{\mathbb{Q}})$. Then there exist $\gamma \in GL(n, \mathbb{Q})$ and $g_{\infty} \in GL(n, \mathbb{R})$, $k_N \in K_0(N)$ such that

$$g = i_{\text{diag}}(\gamma) i_{\infty}(g_{\infty}) k_N$$

(this decomposition may not be unique!)

Sketch of the proof. We first consider $g \in SL(n, \mathbb{A}_{\mathbb{Q}})$. Let U_{∞} be an open neighborhood of I_n in $SL(n, \mathbb{R})$. consider the set

$$g \cdot (i_{\infty}(U_{\infty}) \cdot K_0(N)) = \text{open set}$$

By strong approximate, this set contains $i_{\text{diag}}(\gamma) \cdot i_{\infty}(g'_{\infty})$ for some $\gamma \in SL(n, \mathbb{Q})$ and $g'_{\infty} \in SL(n, \mathbb{R})$. Then we are done. \square

Definition 3.7 (Idelic lifting of a classical Dirichlet character). Fix $N \geq 1$, $\chi : (N\mathbb{Z} \setminus \mathbb{Z})^{\times} \rightarrow \mathbb{C}^{\times}$, $N = \prod_{i=1}^l p_i^{e_i}$. Let $\chi_{p_i} : (p_i\mathbb{Z} \setminus \mathbb{Z})^{\times} \rightarrow \mathbb{C}^{\times}$ such that

$$\chi(d) = \prod_{i=1}^l \chi_i(d)^{e_i}$$

Let $k \in K_0(N)$ and $k = (k_2, \dots, k_p, \dots)$ and $k_p = \begin{pmatrix} A_p & B_p \\ C_p & d_p \end{pmatrix}$ for $p = p_i$. Then

$$\tilde{\chi}_{\text{idelic}} : K_0(N) \rightarrow \mathbb{C}^{\times}$$

is defined by

$$\tilde{\chi}_{\text{idelic}}(k) := \prod_{i=1}^l \chi_i(d_{p_i})^{-e_i}$$

Let $\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix}$ be a classical automorphic form of weight $\rho : O(n, \mathbb{R}) \rightarrow GL(n, \mathbb{C})$ with level N , character χ .

Definition 3.8.

$$\Phi_{\text{adelic}}(i_{\text{diag}}(\gamma) \cdot i_{\infty}(g_{\infty}) \cdot k_{\text{finite}}) := \tilde{\chi}_{\text{idelic}}(k_{\text{finite}}) \cdot (\Phi|_{\rho} g_{\infty})(I_n)$$

Lecture 6

slight mistakes last time. The "lifting" also works for $GL(1)$.

3.5 Adelic automorphic forms for $GL(1, \mathbb{A}_{\mathbb{Q}})$

$$GL(1, \mathbb{A}_{\mathbb{Q}}) = \mathbb{A}_{\mathbb{Q}}^{\times}$$

Definition 3.9 (Adelic automorphic forms for $GL(1, \mathbb{A}_{\mathbb{Q}})$). A smooth function

$$\phi : \mathbb{Q}^{\times} \backslash GL(1, \mathbb{A}_{\mathbb{Q}}) \rightarrow \mathbb{C}$$

is an adelic automorphic form with central character $\omega : \mathbb{Q}^{\times} \mathbb{A}_{\mathbb{Q}}^{\times} \rightarrow \mathbb{C}^{\times}$ if

- (1) $\phi(\gamma g) = \phi(g)$, for all $\gamma \in \mathbb{Q}^{\times}$ and $g \in GL(1, \mathbb{A}_{\mathbb{Q}})$
- (2) $\phi(zg) = \omega(z)\phi(g)$, for all $z \in \text{center of } GL(1, \mathbb{A}_{\mathbb{Q}}) = \mathbb{A}_{\mathbb{Q}}^{\times}$
- (3) ϕ is of moderate growth.

Examples Consider a Dirichlet character modular p^f for a prime p and $f \geq 1$,

$$\chi : (p^f \mathbb{Z} \backslash \mathbb{Z})^{\times} \rightarrow \mathbb{C}^{\times}$$

such that $\chi(ab) = \chi(a)\chi(b)$ and $|\chi(a)| = 1$ for all $a \in (p^f \mathbb{Z} \backslash \mathbb{Z})^{\times}$

Definition 3.10 (Adelic lift of Dirichlet series). Let χ be a Dirichlet character mod p^f for a prime p and $f \geq 1$. We defined an adelic lift of χ :

$$\chi_{\text{idelic}}(g) := \chi_{\infty}(g_{\infty})\chi_2(g_2) \cdots \chi_p(g_p) \cdots$$

where

$$\chi_{\infty}(g_{\infty}) = \begin{cases} 1, & \chi(-1) = 1 \\ 1, & \chi(-1) = -1, g_{\infty} > 0 \\ -1 & \chi(-1) = -1, g_{\infty} < 0 \end{cases}$$

and for $v < \infty$,

$$\chi_v(g_v) := \begin{cases} \chi(v)^m, & g_v \in v^m \mathbb{Z}_v^{\times}, & (v \neq p) \\ \chi(j)^{-1}, & g_v \in p^k (j + p^f \mathbb{Z}_p), j, k \in \mathbb{Z}, (j, p) = 1, & (v = p) \end{cases}$$

Then χ_{idelic} satisfies

$$\chi_{\text{idelic}}(ab) = \chi_{\text{idelic}}(a) \cdot \chi_{\text{idelic}}(b), \forall a, b \in \mathbb{A}_{\mathbb{Q}}^{\times}$$

$$\chi_{\text{idelic}}(ab) = \chi_{\text{idelic}}(b), \forall b \in \mathbb{A}_{\mathbb{Q}}^{\times}, \gamma \in \mathbb{Q}^{\times}$$

$$\chi_{\text{idelic}}(\alpha) = 1, \text{ if } \alpha \in \mathbb{Q}^{\times}$$

On $GL(1, \mathbb{A}_{\mathbb{Q}})$ every automorphic form is a character.

Characters for \mathbb{Q}_p^\times Characters $\psi : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$, $\psi(ab) = \psi(a)\psi(b)$.

Case 1 unramified

$$\psi(u) = 1, \forall u \in \mathbb{Z}_p^\times$$

Then $\psi(u)$ determines all possible character.

Case 2 ramified

$$\psi(u) \neq 1, \text{ for some } u \in \mathbb{Z}_p^\times$$

In this case there exists an open subgroup $U \subset \mathbb{Z}_p^\times$ such that for any $u \in U$, $\psi(u) = 1$. So $\psi(p)$ and $\psi(u)$ for $u \in \mathbb{Z}_p^\times/U$ determine ψ .

3.6 Adelic lifts to $GL(n, \mathbb{A}_\mathbb{Q})$

For real place, we have generalized upper half plane

$$\mathbb{H}^n = GL(n, \mathbb{R})/O(n, \mathbb{R}) \cdot \mathbb{R}^\times.$$

Fix $r \geq 1$, irreducible representation

$$\rho : O(n, \mathbb{R}) \rightarrow GL(r, \mathbb{C})$$

Fix $N \geq 1$,

$$\Gamma_0(N) = \left\{ \begin{pmatrix} A & B \\ C & d \end{pmatrix} \middle| \begin{array}{l} A \in GL(n-1, \mathbb{Z}), \\ C = (c_1, \dots, c_{n-1}) \equiv (0, \dots, 0) \pmod{N}, d \in \mathbb{Z} \end{array} \right\}$$

Then for any $\gamma \in \Gamma_0(N)$ and $z \in \mathbb{H}^n$, there exists $\tilde{\gamma}z \in \mathbb{H}^n$, $\kappa(\gamma, z) \in O(n, \mathbb{R})$ and $d = \begin{pmatrix} \delta & & \\ & \ddots & \\ & & \delta \end{pmatrix}$, $\delta \neq 0$ such that

$$\gamma z = \tilde{\gamma}z \cdot \kappa(\gamma, z) \cdot d.$$

Let

$$J_\rho(\gamma, z) = \rho(\kappa(\gamma, z)^{-1})$$

Then Vector valued automorphic function of weight ρ , level N and a character χ modular N :

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix}, \Phi(\gamma z) = \chi(d) J_\rho(\gamma, z) \Phi(z)$$

for any $\gamma = \begin{pmatrix} A & B \\ C & d \end{pmatrix} \in \Gamma_0(N)$, $z \in \mathbb{H}^n$.

Atrong approximation

$$K_0(p^f) := \left\{ (g_2, \dots, \begin{pmatrix} A_p & B_p \\ C_p & d_p \end{pmatrix}, \dots) \middle| \begin{array}{l} g_v \in GL(n, \mathbb{Q}_v), v \neq p, A_p \in GL(n-1, \mathbb{Z}_p), \\ C_p \in (p^f \mathbb{Z}_p)^{n-1}, d_p \in \mathbb{Z}_p \end{array} \right\} \\ \subseteq GL(n, \mathbb{A}_{\text{finite}})$$

Then for any $g \in GL(n, \mathbb{A}_\mathbb{Q})$,

$$g = i_{\text{diag}}(\gamma) \cdot i_\infty(g_\infty) \cdot k_{\text{finite}}$$

for some $\gamma \in GL(n, \mathbb{Q})$, $g_\infty \in GL(n, \mathbb{R})$, $k_{\text{finite}} \in K_0(p^f)$.

Definition 3.11.

$$\Phi_{\text{adelic}}(g) := \tilde{\chi}_{\text{idelic}}(k_{\text{finite}}) \cdot (\Phi|_\rho g_\infty)(I_n)$$

where $\chi_{\text{idelic}}(k_{\text{finite}}) = \chi_p(d_p)$ for $k_{\text{finite}} = (g_2, \dots, \begin{pmatrix} A_p & B_p \\ C_p & d_p \end{pmatrix}, \dots)$.

Do adelic automorphic forms exist?

$GL(1)$ yes, because Dirichlet characters exist

$GL(2)$ yes, because moderate forms exist

$GL(3)$ Miller showed infinitely many automorphic forms exist for $GL(3, \mathbb{A}_{\mathbb{Q}})$ (K_{∞} -invariant, with level 1)

$GL(n)$ Lapid-Müller got asymptotic for number of classical forms (K -inv, level N), for $n \geq 2$

G Lindenstrauss-Venkatesh (K -invariant)

3.7 Cusp forms for $GL(n, \mathbb{A}_{\mathbb{Q}})$

For $n = 2$ and level $N = 2$, if f is a cusp form, then it means that

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$$

then $a_0 = 0$, and it is equivalent to say $\int_0^1 f(z+u) du = 0$. But for the general level $N > 1$, and character χ , requires more complicated definition.

$$\text{cusps} = \{\infty\} \cup \mathbb{Q}$$

for any cusp \mathfrak{a} , we have $\sigma \in GL(2, \mathbb{Q})$ such that $\sigma \mathfrak{a} = \infty$. Then

$$(f|_k \sigma^{-1})(z) = \sum_{n=0}^{\infty} a_{\mathfrak{a}}(n) e^{2\pi i n(z+\mu_{\mathfrak{a}})}$$

and f is a cusp form if $a_{\mathfrak{a}}(0) = 0$ for all cusps \mathfrak{a} with $\mu_{\mathfrak{a}} = 0$.

Parabolic subgroups

Definition 3.12. A subgroup of $GL(n)$, for $n = n_1 + \dots + n_r$,

$$\begin{pmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix}, \quad A_i \in GL(n_i), \text{ for } i = 1, \dots, r$$

is called a standard parabolic subgroup.

Definition 3.13. A subgroup of $GL(n)$ of the form

$$\begin{pmatrix} A_1 & & & \\ & A_2 & 0 & \\ & & \ddots & \\ & & & A_r \end{pmatrix}, \quad A_i \in GL(n_i), \text{ for } i = 1, \dots, r, \quad n = n_1 + \dots + n_r$$

is called a standard Levi subgroup.

Definition 3.14 (Unipotent radical). is a subgroup of $GL(n)$ of the form

$$\begin{pmatrix} I_{n_1} & & & * \\ & I_{n_2} & & \\ & & \ddots & \\ & & & I_{n_r} \end{pmatrix}, \quad I_l = l \times l \text{ identity matrix } n_1 + n_2 + \dots + n_r = n.$$

Definition 3.15 (Adelic automorphic cusp forms). An adelic automorphic form

$$\phi : GL(n, \mathbb{Q}) \backslash GL(n, \mathbb{A}_{\mathbb{Q}}) \rightarrow \mathbb{C}$$

with central character $\omega : \mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}}^{\times} \rightarrow \mathbb{C}^{\times}$ (i.e., $\phi(zg) = \omega(z)\phi(g)$, $\forall z \in Z(GL(n, \mathbb{A}_{\mathbb{Q}}))$, $g \in GL(n, \mathbb{A}_{\mathbb{Q}})$) is called a cusp form for every unipotent radical $U(\mathbb{A}_{\mathbb{Q}})$ of $GL(n, \mathbb{A}_{\mathbb{Q}})$ we have

$$\int_{U(\mathbb{Q}) \backslash U(\mathbb{A}_{\mathbb{Q}})} \phi(ug) du \equiv 0 \quad (3.5)$$

Remark. It is enough to have (3.5) had for standard unipotent.

Lecture 7

4 Automorphic Representations

4.1 Review

Let $\omega : \mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{C}}^{\times} \rightarrow \mathbb{C}^{\times}$ be a character, i.e.,

- $\omega(ab) = \omega(a)\omega(b)$
- $\omega(\alpha b) = \omega(b)$, $\forall b \in \mathbb{A}_{\mathbb{Q}}^{\times}$, $\alpha \in \mathbb{Q}^{\times}$.

We studied the space $A_{\omega}(G(n, \mathbb{A}_{\mathbb{Q}}))$ of all adelic automorphic forms

$$\phi : GL(n, \mathbb{Q}) \backslash GL(n, \mathbb{A}_{\mathbb{Q}}) \rightarrow \mathbb{C}$$

with central character ω satisfying the followings:

- (1) Modularity condition: $\phi(\gamma g) = \phi(g)$, $\gamma \in GL(n, \mathbb{Q})$, $g \in GL(n, \mathbb{A}_{\mathbb{Q}})$
- (2) $\phi(zg) = \omega(\delta)\phi(g)$, $\forall z \in Z(GL(n, \mathbb{A}_{\mathbb{Q}}))$, $z = \begin{pmatrix} \delta & & \\ & \ddots & \\ & & \delta \end{pmatrix}$, $g \in GL(n, \mathbb{A}_{\mathbb{Q}})$.
- (3) ϕ is right K -finite

$$\iff \left\{ \phi(gk) \mid k \in K = O(n, \mathbb{R}) \cdot \prod_p GL(n, \mathbb{Z}_p) \right\} \text{ is finite dimensional}$$

- (4) ϕ is $\mathcal{Z}(\mathcal{U}_{\mathfrak{g}})$ -finite

$$\iff \{D\phi(g) \mid D \in \mathcal{Z}(\mathcal{U}_{\mathfrak{g}})\} \text{ is finite dimensional}$$

- (5) ϕ has moderate growth.

4.2 Definition for adelic automorphic representations

Rough Definition An adelic automorphic representations with central character ω is a vector space $V \subseteq A_{\omega}(GL(n, \mathbb{A}_{\mathbb{Q}}))$ with three actions: π_{finite} , $\pi_{K_{\infty}}$, $\pi_{\mathfrak{g}}$.

- (1) Action of finite adeles by right translation:

Let $a_{\text{finite}} = (I_n, a_2, \dots, a_p, \dots)$ with $a_p \in GL(n, \mathbb{Q}_p)$ and $a_p \in GL(n, \mathbb{Z}_p)$ for almost all p . For any $\phi \in V$, define

$$\pi_{\text{finite}}(a_{\text{finite}}) \cdot \phi(g) := \phi(g \cdot a_{\text{finite}})$$

(2) Action by K_∞ by right translation:

Let $k = (k_\infty, I_n, \dots, I_n, \dots) \in K_\infty$ with $k_\infty \in O(n, \mathbb{R})$. For any $\phi \in V$ define

$$\pi_{K_\infty}(k) \cdot \phi(g) := \phi(g \cdot k)$$

(3) Action by differential operators:

Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. For $\alpha \in \mathfrak{g}$ and $\phi \in V$, we have

$$D_\alpha \phi(g) = \left. \frac{d}{dt} \phi(g \exp(\alpha t)) \right|_{t=0}$$

Let

$$\begin{aligned} \mathcal{Z}(\mathcal{U}(\mathfrak{g})) &= \text{center of the algebra spanned by all such } D_\alpha \\ &= \text{generated by the Casimir differential operators} \end{aligned}$$

For any $D \in \mathcal{U}(\mathfrak{g})$ define

$$\pi_{\mathfrak{g}}(D) \cdot \phi(g) = D\phi(g).$$

Remark. (1) π_{finite} commutes with π_{K_∞} , i.e.,

$$\pi_{\text{finite}} \circ \pi_{K_\infty} = \pi_{K_\infty} \circ \pi_{\text{finite}}$$

(2) $\pi_{\mathfrak{g}} \circ \pi_{\text{finite}} = \pi_{\text{finite}} \circ \pi_{\mathfrak{g}}$

(3) π_{K_∞} does not commute with $\pi_{\mathfrak{g}}$.

Lemma 4.1. For any $\alpha \in \mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $k = (k_\infty, I_n, \dots) \in K_\infty$, $k_\infty \in O(n, \mathbb{R})$

$$\pi_{\mathfrak{g}}(D_\alpha) \circ \pi_{K_\infty}(k) = \pi_{K_\infty}(k) \circ \pi_{\mathfrak{g}}(D_{k^{-1}\alpha k})$$

Proof. Let $\phi \in V$, $g \in GL(n, \mathbb{A}_\mathbb{Q})$

$$\begin{aligned} \pi_{\mathfrak{g}}(D_\alpha) \cdot (\pi_{K_\infty}(k) \phi(g)) &= \pi_{\mathfrak{g}}(D_\alpha) \phi(gk) = \pi_{\mathfrak{g}}(D_\alpha(\phi(g \cdot (k_\infty, I_n, \dots)))) \\ &= \left. \frac{d}{dt} \phi(g(e^{\alpha t} k_\infty, I_n, \dots)) \right|_{t=0} \\ &= \left. \frac{d}{dt} \phi(gk(k_\infty^{-1} e^{\alpha t} k_\infty, I_n, \dots)) \right|_{t=0} \\ &= \pi_{K_\infty}(k) \cdot (\pi_{\mathfrak{g}}(k_\infty^{-1} \alpha k_\infty)) \cdot \phi(g) \end{aligned}$$

□

Definition 4.2 ((\mathfrak{g}, K_∞) -module, Harish-Chandra). Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $K_\infty = \{(k_\infty, I_n, \dots, I_n, \dots) | k_\infty \in O(n, \mathbb{R})\}$. Then a (\mathfrak{g}, K_∞) -module is a vector space V with 2 actions.

(1) $\pi_{\mathfrak{g}} : \mathcal{U}(\mathfrak{g}) \rightarrow \text{End}(V)$

(2) $\pi_{K_\infty} : K_\infty \rightarrow GL(V)$ where for any $v \in V$, with K_∞ is finite dimensional.

Furthermore, the actions $\pi_{\mathfrak{g}}$, π_{K_∞} satisfy the relation:

$$\pi_{\mathfrak{g}}(D_\alpha) \cdot \pi_{K_\infty}(k) = \pi_{K_\infty}(k) \cdot \pi_{\mathfrak{g}}(D_{k^{-1}\alpha k})$$

for any $k \in K_\infty$, $\alpha \in \mathfrak{g}$.

Also,

$$\pi_{\mathfrak{g}}(D_\alpha) \cdot v = \lim_{t \rightarrow 0} \frac{1}{t} (\pi_{K_\infty}(\exp(\alpha t)) \cdot v - v)$$

where $\alpha \in \mathfrak{k} = \text{Lie algebra of } K_\infty$.

Definition 4.3 ($(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$ -modules). Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, $K_\infty = \{(k_\infty, I_n, I_n, \dots) \mid k_\infty \in O(n, \mathbb{R})\}$. Let $\mathbb{A}_{\text{finite}} = \{(I_n, a_2, \dots, a_p, \dots)\}$ be the finite adeles. A $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$ module is a vector space V with three actions:

- $\pi_{\mathfrak{g}} : \mathcal{U}(\mathfrak{g}) \rightarrow \text{End}(V)$
- $\pi_{K_\infty} : K_\infty \rightarrow GL(V)$
- $\pi_{\text{finite}} : \mathbb{A}_{\text{finite}} \rightarrow GL(V)$

In addition, we require, that V is a (\mathfrak{g}, K_∞) -module with respect to $\pi_{\mathfrak{g}}$, π_{K_∞} and also satisfy the relations.

$$\begin{aligned}\pi_{\mathfrak{g}} \cdot \pi_{\text{finite}} &= \pi_{\text{finite}} \cdot \pi_{\mathfrak{g}} \\ \pi_{K_\infty} \cdot \pi_{\text{finite}} &= \pi_{\text{finite}} \cdot \pi_{K_\infty}\end{aligned}$$

Notation: $\pi = (\pi_{\mathfrak{g}}, \pi_{K_\infty}, \pi_{\text{finite}})$. We call (π, V) a $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$ module.

Definition 4.4 (Smooth $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$ -module). Let (π, V) with $\pi = (\pi_{\mathfrak{g}}, \pi_{K_\infty}, \pi_{\text{finite}})$ be a $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$ -module. Then (π, V) is smooth if every $v \in V$ is fixed by some open compact subgroup of $GL(n, \mathbb{A}_{\text{finite}})$ i.e., there exists $H \subset GL(n, \mathbb{A}_{\text{finite}})$, such that $\pi(h).v = v$ for any $h \in H$.

Exercise: Prove that $A_\omega(GL(n, \mathbb{A}_\mathbb{Q})) = \text{space of all adelic automorphic forms with central character } \omega$ is a smooth $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$ -module.

Intertwining Maps Let (π, V) , (π', V') be representations of G . An intertwining map is a linear map $L : V \rightarrow V'$ such that

$$L(\pi(g).v) = \pi'(g).L(v)$$

for any $g \in G$ and $v \in V$. If L is isomorphic (bijective map) then we say $(\pi, V) \cong (\pi', V')$.

Subquotient of a $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$ -module Let (π, V) with $\pi = (\pi_{\mathfrak{g}}, \pi_{K_\infty}, \pi_{\text{finite}})$ be a $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$ -module. Assume there exist $W' \subset W \subset V$ where W, W' are both invariant under the action π . Then W/W' (vector space quotient) is again a $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$ -module.

Let $w + W' \in W/W'$ for $w \in W$. Just define,

$$\begin{aligned}\pi_{\mathfrak{g}}(D).(w + W') &:= \pi_{\mathfrak{g}}(D).w + W' \\ \pi_{K_\infty}(k).(w + W') &:= \pi_{K_\infty}(k).w + W' \\ \pi_{\text{finite}}(a_{\text{finite}}).(w + W') &:= \pi_{\text{finite}}(a_{\text{finite}}).w + W'\end{aligned}$$

Definition 4.5 (Adelic automorphic representation of $GL(n, \mathbb{A}_\mathbb{Q})$). An adelic automorphic representation with central character ω is a $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$ -module, whose vector space is a subquotient of the space of adelic automorphic forms $A_\omega(GL(n, \mathbb{A}_\mathbb{Q}))$.

Lecture 8

Last time Global Automorphic representations for $GL(n, \mathbb{A}_\mathbb{Q})$ is a vector space V of adelic

automorphic forms with three actions: $\pi = \left(\underbrace{\pi_{\mathfrak{g}} \quad , \quad \pi_{K_\infty} \quad , \quad \pi_{\text{finite}}}_{(\mathfrak{g}, K_\infty)} \right)$.

$\underbrace{\pi_{\mathfrak{g}}}_{\text{action by differential operators at } \infty}$, $\underbrace{\pi_{K_\infty}}_{\text{action by right translation of } K_\infty}$, $\underbrace{\pi_{\text{finite}}}_{\text{action by right translation of finite adeles}}$

4.3 Examples of Automorphic representations

(1) $A_\omega(GL(n, \mathbb{A}_\mathbb{Q})) =$ vector space of all adelic automorphic forms with central character $\omega : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \rightarrow \mathbb{C}^\times$. Not irreducible!

This space decomposes into cuspidal automorphic representations and Eisenstein series.

(2) Let $f(z)$ be a classical automorphic form on the upper half plane \mathbb{H}^n . Let f_{adelic} be its adelic lift, i.e., $f_{\text{adelic}} \in A_\omega(GL(n, \mathbb{A}_\mathbb{Q}))$ for some central character ω .

Studying with the classical automorphic form f , we will construct a vector space $V_f \subset A_\omega(GL(n, \mathbb{A}_\mathbb{Q}))$ and three actions to get an automorphic representation.

$$V_f := \text{Span} \left\{ \sum_{D \in \mathcal{U}(\mathfrak{g})} \sum_{k_\infty \in K_\infty} \sum_{a_{\text{finite}} \in GL(n, \mathbb{A}_{\text{finite}})} \pi_{\mathfrak{g}}(D) \cdot \pi_{K_\infty}(k_\infty) \cdot f_{\text{adelic}}(g) \right\}$$

If f is a newform, then (π, V_f) will be irreducible.

For $n = 2$, let

$$\begin{aligned} f(z) &= \text{Im}(z)^6 \Delta(z) = \sum_{n=1}^{\infty} \tau(n) y^6 e^{2\pi i n z} \\ &= \text{Im}(z)^6 e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^{24} \end{aligned}$$

then for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$,

$$f\left(\frac{az + b}{cz + d}\right) = \left(\frac{cz + d}{|cz + d|}\right)^{12} f(z).$$

What is the Ramanujan automorphic representation (π, V_Δ) ?

- $\mathcal{U}(\mathfrak{g}) =$ Maass type Differential operators
- $\pi_{\text{finite}} \left(I_2, \dots, \begin{pmatrix} p^{-e} & 0 \\ 0 & 1 \end{pmatrix}, I_2, \dots \right) \cdot \Delta_{\text{adelic}}(z) = \Delta(p^e z)_{\text{adelic lift}}$

So, V_Δ will be spanned by all holomorphic modular forms (old forms) of weights 12, 14, 16, 18, 20, 22, ...

4.4 Cuspidal adelic automorphic representations

Let

$$U := \left\{ \begin{pmatrix} I_{r_1} & & * \\ & \ddots & \\ & & I_{r_k} \end{pmatrix}, r_1 + \dots + r_k = n \right\}.$$

If $f \in A_\omega(GL(n, \mathbb{A}_\mathbb{Q}))$ i.e., the adelic automorphic form with the central character ω . Then f is cuspidal if

$$\int_{U(\mathbb{Q}) \backslash U(\mathbb{A}_\mathbb{Q})} f(ug) du = 0$$

for any U .

Definition 4.6. An automorphic representation is cuspidal if its space V is a subquotient of the space of all adelic cusp forms.

4.5 Fourier expansion of adelic automorphic cusp forms

Classical For $f(x+1) = f(x)$

$$f(x) = \sum_{n \in \mathbb{Z}} \int_0^1 f(u+x) e^{-2\pi i n u} du$$

Theorem 4.7 (Piatetski-Shapiro, Shalika independently). *Let ϕ be an adelic automorphic cusp form for $GL(n, \mathbb{A}_{\mathbb{Q}})$. Then for any $g \in GL(n, \mathbb{A}_{\mathbb{Q}})$,*

$$\phi(g) = \sum_{\gamma \in U_{n-1}(\mathbb{Q}) \backslash GL(n-1, \mathbb{Q})} W_{\phi} \left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \cdot g \right)$$

where

$$W_{\phi}(g) := \int_{U_n(\mathbb{Q}) \backslash U_n(\mathbb{A}_{\mathbb{Q}})} \phi(ug) \psi(u)^{-1} du,$$

$U_n = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$ and ψ is the additive character.

Exponential of $\mathbb{A}_{\mathbb{Q}}$ For $a = (a_{\infty}, a_2, \dots, a_p, \dots) \in \mathbb{A}_{\mathbb{Q}}$, define

$$\text{Exp}(a) := e^{2\pi i a_{\infty}} \prod_p e^{-2\pi i \{a_p\}}.$$

For $a_p \in \mathbb{Q}_p$, then

$$a_p = \alpha_{-l} p^{-l} + \alpha_{-l+1} p^{-l+1} + \dots + \alpha_{-1} p^{-1} + \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \dots$$

and

$$\{a_p\} := \alpha_{-l} p^{-l} + \dots + \alpha_{-1} p^{-1}.$$

Periodic: $f(\alpha+x) = f(x)$ for any $\alpha \in \mathbb{Q}$,

$$f(x) = \sum_{\alpha \in \mathbb{Q}} \int_{\mathbb{Q} \backslash \mathbb{A}} f(u+x) \overline{\text{Exp}(\alpha u)} du.$$

Definition of the character ψ which is a generalization of $e^{2\pi i x}$ Let $u = \begin{pmatrix} 1 & u_1 & * & & \\ & 1 & u_2 & & \\ & & \ddots & \ddots & \\ & & & & u_{n-1} \\ & & & & & 1 \end{pmatrix} \in$

$U_n(\mathbb{A}_{\mathbb{Q}})$ (super diagonal).

Definition 4.8.

$$\psi(u) = \text{Exp}(u_1) \cdot \text{Exp}(u_2) \cdots \text{Exp}(u_{n-1})$$

Lemma 4.9.

$$\psi(u \cdot u') = \psi(u) \cdot \psi(u')$$

Example for $n = 2$ The Fourier expansion of $GL(2)$ takes the most familiar form

$$\phi(g) = \sum_{\gamma \in \mathbb{Q}^{\times}} W_{\phi} \left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \cdot g \right)$$

Whittaker function W_ϕ is called a (global) Whittaker function. It is characterized by the following properties.

- (1) $W_\phi(u \cdot g) = \psi(u)W_\phi(g)$, $\forall u \in U_n(\mathbb{A})$, $g \in GL(n, \mathbb{A})$
- (2) $W_\phi(g)$ is smooth and of moderate growth.
- (3) $\{DW_\phi(g) \mid D \in \mathcal{Z}(\mathcal{U}(\mathfrak{g}))\}$ is finite dimensional.
- (4) $\{W_\phi(gk) \mid k \in K\}$ is finite dimensional.

4.6 Whittaker Model

If you have two isomorphic representations

$$(\pi, V) \cong (\pi', V')$$

then we say V' is a model for V

$$\iff \text{isomorphism } L : V \rightarrow V'$$

intertwining, i.e., $L(\pi(g).v) = \pi'(g).L(v)$.

Let (π, V) be an adelic automorphic representation of $GL(n, \mathbb{A}_{\mathbb{Q}})$, and $V = \{f : GL(n, \mathbb{Q}) \backslash GL(n, \mathbb{A}_{\mathbb{Q}}) \rightarrow \mathbb{C}\}$. Let

$$\begin{aligned} L : V &\hookrightarrow \mathcal{W} = \text{Whittaker space} \\ &= \text{space of all Whittaker functions} \end{aligned}$$

with

$$L(f(g)) := W_f(g) = \int_{U_n(\mathbb{Q}) \backslash U_n(\mathbb{A})} f(ug) \text{Exp}(-u) du.$$

$L : V \rightarrow$ spanned by all linear combinations of $W_f(g)$ with $f \in V =: W_V$, then

$$(\pi, V) \cong (\pi, W_V)$$

and W_V is called a Whittaker model for (π, V) .

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4.7 L -functions associated to an irreducible adelic cuspidal automorphic representations (π, V) of $GL(n, \mathbb{A}_{\mathbb{Q}})$

Folklore conjecture Every L -function which has an Euler product

$$L(s) = \prod_p \prod_{i=1}^n (1 - \alpha_{p,i} p^{-s})^{-1}$$

and has holomorphic continuation in $s \in \mathbb{C}$ and satisfies a functional equation

$$G(s)L(s) = \tilde{G}(1-s)\tilde{L}(1-s)$$

for $G(s)$ is the product of Gamma functions, is of the above type: associated to an irreducible adelic automorphic representations (π, V) of $GL(n, \mathbb{A}_{\mathbb{Q}})$

L -functions discovered:

- $\zeta(s)$
- $L(s, \chi)$, Dirichlet
- $L(s, \psi)$, Hecke of number fields
- $L(s, A)$, Hasse-Weil L -function associated to an algebraic variety A . (NOT proved)
- Artin L -functions, etc (NOT proved)

Classically, people attached L -function to automorphic forms. For $GL(2)$, we have

$$\phi(g) = \sum_{\gamma \in \mathbb{Q}^\times} W_\phi \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right)$$

then

$$L_\phi(s) = \int_0^\infty \phi \left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) y^s \frac{dy}{y}.$$

But for $n > 2$, this integral doesn't work. For $GL(n)$,

$$\phi(g) = \sum_{\gamma \in U_{n-1}(\mathbb{Q}) \backslash GL(n-1, \mathbb{Q})} W_\phi \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right).$$

For example, for $n = 3$, let

$$\int_0^\infty \int_0^\infty \phi \left(\begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix} \right) \left(\det \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix} \right)^s \frac{dy_1}{y_1} \frac{dy_2}{y_2} \neq L_\phi(s).$$

There are several ways to defined an L -function. We can use the method used in (Automorphic ..., by Goldfeld), Rankin-Selberg method, or using representations.

History

- (1) K. Hey in 1929 and Eichler in 1938, defined zeta function on $GL(n)$
- (2) In 1950, Tate-Iwasawa constructed all L -functions for $GL(1)$
- (3) In 1958, Godement suggested a generalization of Tate-Iwasawa to $GL(n)$
- (4) In 1963, Tamagaa worked out Godement suggestion for abelian ... found Euler product
- (5) In 1972, Godement-Jacquet use matrix coefficients of automorphic representations.

Definition 4.10. Let $R = \text{ring}$. Let $n \geq 1$.

$$M(n, R) := \text{ring of } n \times n \text{ matrices with coefficients in } R.$$

Definition 4.11 (Schwartz-Bruhat function on $M(n, \mathbb{A}_\mathbb{Q})$).

$$\phi : M(n, \mathbb{A}_\mathbb{Q}) \rightarrow \mathbb{C}$$

which is a finite sum of products

$$\prod_{v \leq \infty} \phi_v(m_v)$$

where $m = (m_\infty, m_2, \dots, m_p, \dots) \in \mathbb{A}_\mathbb{Q}$ which satisfies the followings

- $\phi_\infty(m_\infty)$ is smooth and has rapid decay at ∞ . i.e., for $m_\infty = \begin{pmatrix} m_\infty^{1,1} & \dots & m_\infty^{1,n} \\ \vdots & \dots & \vdots \\ m_\infty^{n,1} & \dots & m_\infty^{n,n} \end{pmatrix}$ then $\phi_\infty(m_\infty)$ is smooth in all the variables $m_\infty^{i,j}$.
- $\phi_p(m_p)$ is locally constant, compactly supported
- $\phi_p(m_p)$ is the characteristic function of $M(n, \mathbb{Z}_p)$ for almost all p

Definition 4.12 (Global matrix coefficients for (π, V)). Let $f_1, f_2 \in V$. A matrix coefficient associated to f_1, f_2 ,

$$\omega : GL(n, \mathbb{A}_\mathbb{Q}) \rightarrow \mathbb{C}$$

given by

$$\omega(g) := \int_{\mathcal{Z}(GL(n, \mathbb{A}_\mathbb{Q}))GL(n, \mathbb{A}_\mathbb{Q}) \backslash GL(n, \mathbb{A}_\mathbb{Q})} f_1(hg) \overline{f_2(h)} dh \quad (4.1)$$

for any $g \in GL(n, \mathbb{A}_\mathbb{Q})$.

Remark. (i) Since (π, V) is irreducible, then all matrix coefficients for any $f_1, f_2 \in V$ are the same up to a constant factor. (Not completely trivial to prove).

(ii) For $g = I_n$, then $\omega(I_n) =$ Pettersson inner product.

Definition 4.13 (L -function associated to (π, V)). Let $\phi : M(n, \mathbb{A}_\mathbb{Q}) \rightarrow \mathbb{C}$ be a Schwartz-Bruhat function. Let $s \in \mathbb{C}$ with $\Re(s) \gg 1$. Let $\omega : GL(n, \mathbb{A}_\mathbb{Q}) \rightarrow \mathbb{C}$ as in (4.1). Then

$$Z(s, \phi, \omega) := \int_{GL(n, \mathbb{A}_\mathbb{Q})} \omega(g) \phi(g) |\det g|^{s+\frac{1}{2}} d^\times g = L(s, \pi) \cdot G(\pi, s).$$

Theorem 4.14 (Godement-Jacquet). *The zeta function $Z(s, \phi, \omega)$ has holomorphic continuation to all $s \in \mathbb{C}$ and satisfies a functional equation*

$$Z(s, \phi, \omega) = Z(1-s, \widehat{\phi}, \check{\omega})$$

where $\widehat{\phi} =$ Fourier transform of ϕ and $\check{\omega}(g) := \omega(g^{-1})$.

Proof. Poisson summation formula □

Tensor product theorem

$$\begin{aligned} \pi &= \otimes'_{v \leq \infty} \pi_v \\ \Rightarrow \omega(g) &= \prod_{v \leq \infty} \omega_v(g_v) \end{aligned}$$

where $\omega_v : GL(\mathbb{Q}_v) \rightarrow \mathbb{C}$ are local matrix coefficients.

Local L -functions

$$Z(s, \phi_v, \omega_v) := \int_{GL(\mathbb{Q}_v)} \omega_v(g) \phi_v(g) |\det(g)|^{s+\frac{1}{2}} ds$$

$$Z(s, \phi, \omega) = \prod_{v \leq \infty} Z(s, \phi_v, \omega_v) \Rightarrow \text{Euler product}$$

5 Local Theory: Representations of $GL(n, \mathbb{Q}_p)$

5.1 Representations for $GL(n, \mathbb{Q}_p)$

Let $r \geq 0$ and

$$K_r := \{k \in GL(n, \mathbb{Z}_p) \mid k - I_n \in p^r M(n, \mathbb{Z}_p)\}.$$

(open compact nbhd for I_n ?)

Definition 5.1. Let $\pi : GL(n, \mathbb{Q}_p) \rightarrow GL(V)$ be a representation for $V = \infty$ -dimensional vector space. We say (π, V) is smooth if the map $g \mapsto \pi(g).v$ for $g \in GL(n, \mathbb{Q}_p)$ and $v \in V$ is locally constant.

Definition 5.2. (π, V) is admissible if

$$V^{K_r} = \{v \in V \mid \pi(k).v = v, k \in K_r\}$$

is finite dimensional for any K_r with $r \geq 0$.

We want to classify smooth, admissible and irreducible representation of $GL(n, \mathbb{Q}_p)$.

Lemma 5.3 (Dixmier-Schur). Let (π, V) be a smooth and irreducible representation of $GL(n, \mathbb{Q}_p)$. Let $T : V \rightarrow V$ be a linear map, satisfying

$$T(\pi(g).v) = \pi(g).T(v)$$

for any $g \in GL(n, \mathbb{Q}_p)$, $v \in V$ (Intertwining map). Then there exists $c \in \mathbb{C}$ such that $Tv = cv$ for any $v \in V$.

Proof. Fix $v \in V$. Claim the vector space is spanned by

$$\{\pi(g).v \mid g \in GL(n, \mathbb{Q}_p)\}$$

is countable.

For fixed $g \in GL(n, \mathbb{Q}_p)$ the space

$$\{\pi(gk).v \mid k \in GL(n, \mathbb{Z}_p)\}$$

is finite because π is smooth so there exists K_r for some $r \geq 0$ such that $\pi(gk).v = \pi(g).v$ for any $k \in K_r$. Also by the Iwasawa decomposition, $\mathbb{Q}_p^\times \backslash GL(n, \mathbb{Q}_p) / GL(n, \mathbb{Z}_p)$ is countable. So this claim is proved.

Claim 2: Let $c \in \mathbb{C}$, $I.v = v$, identity transform. Then $(T - cI) \cdot \pi = \pi \cdot (T - cI)$ (commuting operators).

Assume that $T - cI \neq 0$, for all $c \in \mathbb{C}$ then $(T - cI)^{-1} : V \rightarrow V$ exists and is a bijective linear map. for any fixed $v \in V$,

$$V := \text{Span} \{\pi(g).v \mid g \in GL(n, \mathbb{Q}_p)\}$$

by irreducibility. Fix $v \in V$, consider

$$\{(T - cI)^{-1}.v \mid c \in \mathbb{C}\} = \text{uncountable set of vectors} \subseteq V.$$

Then there exists a linear relation

$$\sum_{i=1}^r a_i (T - c_i I)^{-1}.v = 0, \text{ for some } a_i, c_i \in \mathbb{C}. \quad (5.1)$$

Now write the algebraic relation

$$\sum_{i=1}^l \frac{a_i}{X - c_i} = \frac{Q(x)}{\prod_{i=1}^l (X - c_i)}, \quad Q(X) = \sum_{i=1}^d b_i X^i$$

$$\Rightarrow Q(x) = \left(\sum_{i=1}^l \frac{a_i}{X - c_i} \right) \cdot \prod_{i=1}^l (X - c_i)$$

Define an operator $V \rightarrow V$ as

$$Q(T) := \sum_{i=1}^d b_i \underbrace{T \circ \cdots \circ T}_{i\text{-times}}$$

Multiplying (5.1) on the left by $\underbrace{T \circ \cdots \circ T}_{l\text{-times}}$ then

$$\Rightarrow Q(T) = 0$$

But we can factor $Q(X) = \alpha \prod_{i=1}^d (X - \alpha_i)$ then $(T - \alpha_i)^{-1}$ does not exist for some α_i . Contradiction. So the assumption $T - cI \neq 0$ for any $c \in \mathbb{C}$ is false. \square

Lecture 10: 2010-2-18

Reference notes: Prasad, Raghuram, [Representation theory of $GL(n)$ over non-archimidean local fields] in cauchy.math.okstate.edu/araghur

Last time $\pi : GL(n, \mathbb{Q}_p) \rightarrow GL(V)$ for some complex vector space V . π is smooth $\Rightarrow g \mapsto \pi(g).v$ is locally constant for any fixed $v \in V$.

Dixmier-Schur Lemma: If $T : V \rightarrow V$, $T\pi(g).v = \pi(g).Tv$ and (π, V) is irreducible then T acts by scalars. i.e., $Tv = cv$ for some $c \in \mathbb{C}$ and all $v \in V$.

Corollary 5.4. Let (π, V) be a smooth irreducible representation of $GL(n, \mathbb{Q}_p)$. Then there exists a character $\omega_\pi : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$ such that

$$\pi(aI_n).v = \omega_\pi(a)v$$

for any $a \in \mathbb{Q}_p^\times$, $v \in V$ and $I_n = n \times n$ identity matrix and $aI_n \in \mathcal{Z}(GL(n, \mathbb{Q}_p))$. The character ω_π is called the central character associated to π .

Proof. Let $z \in \mathcal{Z}(GL(n, \mathbb{Q}_p)) \Rightarrow \pi(zg) = \pi(z)\pi(g) = \pi(g)\pi(z)$, for any $g \in GL(n, \mathbb{Q}_p)$. For every fixed $z \in \mathcal{Z}(GL(n, \mathbb{Q}_p))$ it is clear that $\pi(z)$ is an intertwining operator. So $\pi(z) : V \rightarrow V$ and it commutes with everything else in $GL(n, \mathbb{Q}_p)$. So by Dixmier-Schur's lemma, $\pi(z)$ acts by a scalar $\omega_\pi(z)$ for each $z \in \mathcal{Z}(GL(n, \mathbb{Q}_p))$. Since π is smooth and has multiplicative, ω_π is a character. \square

Let (π, V) be an irreducible smooth representation of $GL(n, \mathbb{Q}_p)$. Then there are two important cases.

- (1) $\dim(V) < \infty$: It is not so interesting. Because we will prove that $\dim(V) = 1$ and every representation is a character.
- (2) $\dim(V) = \infty$

Proposition 5.5. *Let (π, V) be an irreducible and smooth representation of $GL(n, \mathbb{Q}_p)$. If $\dim V < \infty$ then $\dim V = 1$ and every representation is a character.*

Proof. (for $n = 2$) Let $x \in \mathbb{Q}_p$. Then since (π, V) is smooth then there exists $b \in \mathbb{Q}_p$ such that $|xb|_p$ is sufficiently small. So that $\begin{pmatrix} 1 & xb \\ 0 & 1 \end{pmatrix} \in \text{Ker}(\pi)$, i.e., $\pi\left(\begin{pmatrix} 1 & xb \\ 0 & 1 \end{pmatrix}\right).v = v$. (using $\dim V < \infty$)

$$\Rightarrow \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & xb \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \in \text{Ker}(\pi)$$

$$(\because \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right).v = \pi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)^{-1} \pi\left(\begin{pmatrix} 1 & xb \\ 0 & 1 \end{pmatrix}\right) \cdot (\pi\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right).v) = v)$$

Similarly, we can show

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \in \text{Ker}(\pi), \quad \forall y \in \mathbb{Q}_p.$$

But the matrices $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$ for $x, y \in \mathbb{Q}_p$ generate $SL(2, \mathbb{Q}_p)$.

$$\Rightarrow SL(2, \mathbb{Q}_p) \subset \text{Ker}(\pi)$$

But $\forall g_1, g_2 \in GL(2, \mathbb{Q}_p)$ the matrices $g_1 g_2 g_1^{-1} g_2^{-1} \in SL(2, \mathbb{Q}_p)$.

$$\Rightarrow \pi(g_1 g_2 g_1^{-1} g_2^{-1}).v = v$$

$$\Rightarrow \pi(g_1).(\pi(g_2).v) = \pi(g_2).(\pi(g_1).v), \quad \forall g_1, g_2 \in GL(2, \mathbb{Q}_p)$$

$$\Rightarrow \text{every } \pi(g) \text{ acts by a scalar and } \dim(V) = 1$$

□

From now on we will only consider (π, V) with $\dim V = \infty$.

5.2 Contragredient representations

Let (π, V) be a smooth irreducible representation of $GL(n, \mathbb{Q}_p)$. We will now define another representation, denoted by $(\tilde{\pi}, \tilde{V})$ of $GL(n, \mathbb{Q}_p)$ called the contragredient representation.

- $\tilde{V} =$ subspace of $\hat{V} = \{l : V \rightarrow \mathbb{C}, l = \text{linear map}\}$, such that for any $l \in \tilde{V}$ and for any fixed $v \in V$, $l(\pi(k).v) = l(v)$ for any $k \in$ compact open subset of $GL(n, \mathbb{Z}_p)$ depends on v .
- We define an action $\tilde{\pi}$ on \tilde{V} as follows: for $l \in \tilde{V}, v \in V$ and $g \in GL(n, \mathbb{Q}_p)$,

$$\tilde{\pi}(g).l(v) := l(\pi(g)^{-1}.v).$$

Once we have defined the contragredient representations, we can construct a bilinear pairing $\langle \cdot, \cdot \rangle : V \times \tilde{V} \rightarrow \mathbb{C}$ as follows:

Definition 5.6. For any $v \in V$ and $l \in \tilde{V}$, then

$$\langle v, l \rangle := l(v).$$

The pairing $\langle \cdot, \cdot \rangle : V \times \tilde{V} \rightarrow \mathbb{C}$ is invariant under the action of $GL(n, \mathbb{Q}_p)$ in the sense that

$$\langle \pi(g).v, \tilde{\pi}(g).l \rangle = \langle v, l \rangle$$

for any $g \in GL(n, \mathbb{Q}_p), v \in V$ and $l \in \tilde{V}$.

$$(\because \langle \pi(g).v, \tilde{\pi}(g).l \rangle = \tilde{\pi}(g).l(\pi(g).v) = l(\pi(g)^{-1}\pi(g).v) = l(v))$$

Matrix Coefficient for smooth irreducible representations for $GL(n, \mathbb{Q}_p)$ Fix $v \in V$ and $\tilde{v} \in \tilde{V}$. Then the map

$$g \mapsto \langle \pi(g).v, \tilde{v} \rangle, \forall g \in GL(n, \mathbb{Q}_p)$$

is called a matrix coefficient.

Fix a smooth infinite dimensional irreducible representation (π, V) of $GL(n, \mathbb{Q}_p)$ with contragredient $(\tilde{\pi}, \tilde{V})$. Let $\langle \cdot, \cdot \rangle : V \times \tilde{V} \rightarrow \mathbb{C}$ be the canonical invariant bilinear pairing.

Definition 5.7 (Local L -function associated to π). Fix $v \in V, \tilde{v} \in \tilde{V}$.

$$Z_\pi(s, \phi) = \int_{GL(n, \mathbb{Q}_p)} \phi(g) \langle \pi(g).v, \tilde{v} \rangle |\det g|^{s+\frac{1}{2}} dg$$

for $\phi =$ locally constant compactly supported function $\pi : GL(n, \mathbb{Q}_p) \rightarrow \mathbb{C}$. Define

$$L_\pi(s) := \text{greatest common division or all} \\ \{Z_\pi(s, \phi) \mid \phi \text{ locally constant compactly supported}\}$$

(Godement-Jacquet)

Question:

- (1) How to prove the integral $Z_\pi(s, \phi)$ converges? Need to growth of matrix coefficients.
- (2) Can we classify smooth irreducible representation of $GL(n, \mathbb{Q}_p)$? \Rightarrow There will be only finitely many possibilities.
- (3) $\langle \cdot, \cdot \rangle_{\text{global}} = \prod_v \langle \cdot, \cdot \rangle_v$

For $GL(2, \mathbb{Q}_p)$ we will only the following types of $L_\pi(s)$:

- $(1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1}$
- $(1 - \alpha_p p^{-s})$
- $1 \iff$ supercuspidal representations

and α_p, β_p are called Langlands parameters. For $GL(3, \mathbb{Q}_p)$, we have

- $(1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1} (1 - \gamma_p p^{-s})^{-1}$
- $(1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1}$
- $(1 - \alpha_p p^{-s})$
- 1

Lecture 11: 2010-2-23

Let (π, V) be a representation of $GL(n, \mathbb{Q}_p)$. We have the dual space

$$V^* = \{\ell : V \rightarrow \mathbb{C} \mid \ell = \text{linear map}\}.$$

Then $(\tilde{\pi}, \tilde{V})$ is the contragredient where $\tilde{V} \subset V^*$ with the action

$$\tilde{\pi}(g).\ell(v) := \ell(\pi(g)^{-1}.v), \forall \ell \in \tilde{V}, v \in V$$

and ℓ is smooth,

$$\tilde{\pi}(k).\ell = \ell$$

for some compact subgroup $K, k \in K$.

We want to construct $(\tilde{\pi}, \tilde{V})$ in an explicitly way if (π, V) is irreducible.

Theorem 5.8. *Let (π_1, V_1) and (π_2, V_2) be two smooth and irreducible representations for $GL(n, \mathbb{Q}_p)$. Assume there exists a (bilinear and nontrivial) pairing*

$$\langle, \rangle : V_1 \times V_2 \rightarrow \mathbb{C}$$

satisfying

$$\langle \pi_1(g).v_1, \pi_2(g).v_2 \rangle = \langle v_1, v_2 \rangle$$

for all $g \in GL(n, \mathbb{Q}_p)$, $v_1 \in V_1$ and $v_2 \in V_2$. (invariant pairing) Then

$$(\pi_2, V_2) \cong \underbrace{(\widetilde{\pi}_1, \widetilde{V}_1)}_{\text{contragredient of } (\pi_1, V_1)}.$$

Remark. • If we can construct such a pairing, we can construct the contragredient.

- If $V_1 = V_2$ and \langle, \rangle is Hermitian and positive definite then we say that (π_1, V_1) is unitary.

Interlude For automorphic forms, we have Petersson inner product

$$\langle f_1, f_2 \rangle = \int_{GL(n, \mathbb{Q}) \backslash GL(n, \mathbb{A})} f_1(g) \overline{f_2(g)} dg$$

and in the global case, the complex conjugate gives the contragredient. It is Hermitian and positive definite, so global representation should be unitary. By tensor product theory, global representations factor into local representations, so local representations (from the global representations) should be unitary.

Proof.

Lemma 5.9. *The contragredient of a smooth irreducible representations of $GL(n, \mathbb{Q}_p)$ is again a smooth irreducible representation of $GL(n, \mathbb{Q}_p)$.*

(will prove shortly)

Fix $v_2 \in V_2$. Associated to v_2 we can construct a linear map

$$L_{v_2} : v_1 \mapsto \langle v_1, v_2 \rangle =: L_{v_2}(v_1)$$

for any $v_1 \in V_1$. Want to prove

$$(\widetilde{\pi}_1, \widetilde{V}_1) \cong (\pi_2, V_2)$$

\iff there exists an isomorphism (intertwining map) $T : V_2 \rightarrow \widetilde{V}_1$ satisfying

$$T(\pi_2(g).v_2) = \widetilde{\pi}_1(g).T(v_2) \tag{5.2}$$

for any $g \in GL(n, \mathbb{Q}_p)$ and $v_2 \in V_2$. For any $v_2 \in V_2$, define

$$T(v_2) := L_{v_2} = \{v_1 \mapsto \langle v_1, v_2 \rangle\}.$$

We check that (5.2) holds

$$\begin{aligned} T(\pi_2(g).v_2) &= \{v_1 \mapsto \langle v_1, \pi_2(g).v_2 \rangle\} \\ &= \{v_1 \mapsto \langle \pi_1(g^{-1}).v_1, v_2 \rangle\} \\ &= \widetilde{\pi}_1(g).(T(v_2)). \end{aligned}$$

Since $(\widetilde{\pi}_1, \widetilde{V}_1)$ and (π_2, V_2) are irreducible, by Diximer-Schur's lemma we are done. \square

Proof. of Lemma.

Assume that $(\tilde{\pi}, \tilde{V})$ is not irreducible, then there exists $\tilde{W} \subset \tilde{V}$ with $\tilde{W} \neq \{0\}$, \tilde{V} where \tilde{W} is an invariant subspace of $(\tilde{\pi}, \tilde{V})$. Define

$$W = \left\{ v \in V \mid \ell(v) = 0, \forall \ell \in \tilde{W} \right\}$$

It is enough to prove that $W \neq \{0\}$ or V . This gives a contradiction. Exercise: complete proof. \square

5.3 Generic Representation of $GL(n, \mathbb{Q}_p)$

Example of representation of $GL(n, \mathbb{Q}_p)$ Let $f(g)$ be an automorphic adelic cusp form for $GL(n, \mathbb{A}_{\mathbb{Q}})$ with central character ω . Then f satisfies

- $f(\gamma g) = f(g)$, $\forall \gamma = (\alpha, \alpha, \dots)$, $\alpha \in GL(n, \mathbb{Q})$, $g \in GL(n, \mathbb{A}_{\mathbb{Q}})$
- $f(zg) = \omega(z)f(g)$, $\forall z \in \mathcal{Z}(GL(n, \mathbb{A}_{\mathbb{Q}}))$, $g \in GL(n, \mathbb{A})$

Define

$$g_p := \left(I_n, \dots, I_n, \underbrace{g_p}_{\in GL(n, \mathbb{Q}_p)}, I_n, \dots \right) \in GL(n, \mathbb{A})$$

and

$$V_f := \left\{ \sum_{i=1}^r c_i f(gh_i), c_i \in \mathbb{C}, h_i \in GL(n, \mathbb{Q}_p) \right\}$$

Then (π, V_f) is a representation of $GL(n, \mathbb{Q}_p)$ with the action

$$\pi(h).F(g) = F(gh), F \in V_f.$$

Intertwining Integral:

$$\begin{aligned} W(g) &:= \int_{U_n(\mathbb{Q}) \backslash U_n(\mathbb{A})} F(ug) \text{Exp}(-u) du = \text{First Fourier coeff. of } F \\ &= \text{Whittaker function} \end{aligned}$$

$$W(ug) = \text{Exp}(u) \cdot W(g)$$

and $\pi(u).W = \text{Exp}(u)W$ for any $u \in U_n(\mathbb{A})$. Then

$$(\pi, V_f) \cong (\pi, W_f)$$

is a Whittaker model for (π, V_f) where W_f is a space of Whittaker functions.

Definition 5.10. A representation (π, V) of $GL(n, \mathbb{Q}_p)$ is said to be generic if it has a Whittaker model i.e.,

$$(\pi, V) \cong (\pi, \mathcal{W})$$

where \mathcal{W} is a space with the property that $\pi(u).W = \exp_p(u)W$, $\forall u \in U_n(\mathbb{Q}_p)$, $W \in \mathcal{W}$.

Remark. Not every representation of $GL(n, \mathbb{Q}_p)$ is generic.

Theorem 5.11 (Gelfand-Kazhdan 1974, Multiplicity One Theorem). *Let (π, V) be a smooth irreducible representation of $GL(n, \mathbb{Q}_p)$. Then either (π, V) is not generic or (π, V) is generic and there is at most one space of Whittaker functions.*

The contragredient of a generic representation of $GL(n, \mathbb{Q}_p)$ is again generic.

Gelfand-Kazhdan show that if (π, V) is a smooth and irreducible representation of $GL(n, \mathbb{Q}_p)$ then for the contragredient $(\tilde{\pi}, \tilde{V})$ we may take $V = \tilde{V}$ and the action $\tilde{\pi}(g) = \pi({}^t g^{-1})$.

Definition 5.12 (Ramified and Unramified). A smooth and irreducible representation (π, V) of $GL(n, \mathbb{Q}_p)$ is unramified if there exists $v_0 \in V$ such that $\pi(k).v_0 = v_0$ for all $k \in GL(n, \mathbb{Z}_p)$. If there is no such $v_0 \in V$ then we say (π, V) is ramified.

Motivation: If we have an adelic automorphic form $f(g)$ and $f(gk) = f(g)$ for any $k \in K$.

Lecture 12: 2009-2-25

Let (π_1, V_1) and (π_2, V_2) are smooth and irreducible representations of $GL(n, \mathbb{Q}_p)$. If there exists a invariant pairing $\langle, \rangle : V_1 \times V_2 \rightarrow \mathbb{C}$ where invariant means

$$\langle \pi_1(g).v_1, \pi_2(g).v_2 \rangle = \langle v_1, v_2 \rangle$$

for any $g \in GL(n, \mathbb{Q}_p)$, $v_1 \in V_1$ and $v_2 \in V_2$.

$$\Rightarrow (\pi_2, V_2) = \text{contragredient of } (\pi_1, V_1)$$

5.4 Example of a principal series representations of $GL(2, \mathbb{Q}_p)$

where we will explicitly construct the contragredient by explicitly creating a suitable pairing.

Fix two characters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$.

$$\chi_1(x) = \omega_1(x) \cdot |x|_p^{s_1}, \quad \chi_2(x) = \omega_2(x) |x|_p^{s_2}$$

where $\omega_1, \omega_2 : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$ and $s_1, s_2 \in \mathbb{C}$ with $|\omega_1(x)| = |\omega_2(x)| = 1$ for any $x \in \mathbb{Q}_p^\times$ and $\omega_1(p) = \omega_2(p) = 1$.

Definition 5.13 (The vector space $B(\chi_1, \chi_2)$).

$$B(\chi_1, \chi_2) := \left\{ \begin{array}{l} f : GL(2, \mathbb{Q}_p) \rightarrow \mathbb{C} \\ \text{smooth} \end{array} \left| \begin{array}{l} f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \left|\frac{a}{d}\right|_p^{1/2} \chi_1(a) \chi_2(d) f(g) \\ \forall a, d \in \mathbb{Q}_p^\times, b \in \mathbb{Q}_p, g \in GL(2, \mathbb{Q}_p) \end{array} \right. \right\}.$$

Definition 5.14 (The principal series representation associated to χ_1, χ_2). The principal series representation associated to χ_1, χ_2 is defined to be $(\pi, B(\chi_1, \chi_2))$ where

$$\pi : GL(2, \mathbb{Q}_p) \rightarrow GL(B(\chi_1, \chi_2))$$

is the action by right translation, i.e., $\pi(h).f(g) := f(gh)$ for any $g, h \in GL(2, \mathbb{Q}_p)$ and $f \in B(\chi_1, \chi_2)$.

Remark. If $(\pi, B(\chi_1, \chi_2))$ is not irreducible then one can prove that it decomposes into two pieces and one of them is called the special representation.

Our aim today is to construct explicitly the contragredient representations of a principal series representations (assuming the principal series representation is irreducible).

Theorem 5.15.

$$\begin{aligned} \left(\tilde{\pi}, \widetilde{B(\chi_1, \chi_2)} \right) &= \text{contragredient of } (\pi, B(\chi_1, \chi_2)) \\ &= (\pi, B(\chi_1^{-1}, \chi_2^{-1})). \end{aligned}$$

Idea of proof Let $f_1 \in B(\chi_1, \chi_2)$, $f_2 \in \widetilde{B(\chi_1, \chi_2)} = B(\chi_1^{-1}, \chi_2^{-1})$. Then

$$\int_{GL(2, \mathbb{Z}_p)} f_1(k) f_2(k) dk = \text{invariant pairing}$$

where invariant means

$$\int_{GL(2, \mathbb{Z}_p)} f_1(kg) f_2(kg) dk = \int_{GL(2, \mathbb{Z}_p)} f_1(k) f_2(k) dk$$

for all $g \in GL(2, \mathbb{Q}_p)$.

Let $f : GL(2, \mathbb{Q}_p) \rightarrow \mathbb{C}$ be smooth. By Iwasawa decomposition $g = \begin{pmatrix} a_1 & x \\ & a_2 \end{pmatrix} \cdot k$ for $k \in GL(2, \mathbb{Z}_p)$.

$$\int_{GL(2, \mathbb{Q}_p)} f(g) dg := \int_{a_1 \in \mathbb{Q}_p^\times} \int_{a_2 \in \mathbb{Q}_p^\times} \int_{x \in \mathbb{Q}_p} \int_{\substack{k \in K \\ = GL(2, \mathbb{Z}_p)}} f\left(\begin{pmatrix} a_1 & x \\ & a_2 \end{pmatrix} k\right) dk dx d^\times a_1 d^\times a_2$$

if it is convergent. (Haar measure)

Lemma 5.16. Let $\phi : GL(2, \mathbb{Q}_p) \rightarrow \mathbb{C}$ be locally constant compactly supported and define

$$\phi_0(g) := \int_{\mathbb{Q}_p^\times} \int_{\mathbb{Q}_p^\times} \int_{\mathbb{Q}_p} \phi\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx d^\times a_1 d^\times a_2.$$

Then

$$\phi_0\left(\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) = \left|\frac{b_1}{b_2}\right|_p \phi_0(g) \quad (5.3)$$

for any $b_1, b_2 \in \mathbb{Q}_p^\times$, $u \in \mathbb{Q}_p$ and $g \in GL(2, \mathbb{Q}_p)$. Further any locally constant function satisfying (5.3) is equal to ϕ_0 for some ϕ .

Proof.

$$\begin{aligned} \phi_0\left(\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) &= \int_{\mathbb{Q}_p^\times} \int_{\mathbb{Q}_p^\times} \int_{\mathbb{Q}_p} \phi\left(\begin{pmatrix} a_1 b_1 & 0 \\ 0 & a_2 b_2 \end{pmatrix} \begin{pmatrix} 1 & \frac{b_2}{b_1} x + u \\ 0 & 1 \end{pmatrix} g\right) dx d^\times a_1 d^\times a_2 \\ &= \left|\frac{b_1}{b_2}\right|_p \phi_0(g) \end{aligned}$$

Finally, we must show that any function satisfying (5.3) can be constructed in the way. Assume $f\left(\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) = \left|\frac{b_1}{b_2}\right|_p f(g)$ for all $b_1, b_2 \in \mathbb{Q}_p^\times$, $g \in GL(2, \mathbb{Q}_p)$.

Define

$$\phi(g) := \begin{cases} f(g) & \text{if } g \in GL(2, \mathbb{Z}_p) \\ 0 & \text{otherwise} \end{cases}$$

Let's compute ϕ_0

$$\phi_0(k) = \int_{\mathbb{Z}_p^\times} \int_{\mathbb{Z}_p^\times} \int_{\mathbb{Z}_p} f\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k\right) dx d^\times a_1 d^\times a_2$$

for all $k \in GL(2, \mathbb{Z}_p)$.

$$\Rightarrow \phi_0(k) = f(k)$$

$$\Rightarrow \phi_0(g) = f(g), \forall g \in GL(2, \mathbb{Q}_p)$$

□

Corollary 5.17. Let $F : GL(2, \mathbb{Q}_p) \rightarrow \mathbb{C}$ be locally constant and satisfy (5.3), i.e.,

$$F \left(\begin{pmatrix} b_1 & \\ & b_2 \end{pmatrix} \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} g \right) = \left| \frac{b_1}{b_2} \right|_p F(g)$$

for any b_1, b_2, g . Then

$$\int_{GL(2, \mathbb{Z}_p)} F(kg) dk = \int_{GL(2, \mathbb{Z}_p)} F(k) dk$$

for any $g \in GL(2, \mathbb{Q}_p)$.

Proof. By lemma, $F = \phi_0$ for some ϕ , compactly supported.

$$\begin{aligned} F(g) &= \int_{\mathbb{Q}_p^\times} \int_{\mathbb{Q}_p^\times} \int_{\mathbb{Q}_p} \phi \left(\begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g \right) dx d^\times a_1 d^\times a_2 \\ \Rightarrow \int_K F(kg) dk &= \int_K \int_{\mathbb{Q}_p^\times} \int_{\mathbb{Q}_p^\times} \int_{\mathbb{Q}_p} \phi \left(\begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} kg \right) d^\times a_1 d^\times a_2 dx dk \\ &= \int_{GL(2, \mathbb{Q}_p)} \phi(hg) dh = \int_{GL(2, \mathbb{Q}_p)} \phi(h) dh \\ &= \int_K F(k) dk \end{aligned}$$

□

Theorem 5.18. Let $f \in B(\chi_1, \chi_2)$ and $\tilde{f} \in B(\chi_1^{-1}, \chi_2^{-1})$. Then

$$\langle f, \tilde{f} \rangle := \int_K f(k) \tilde{f}(k) dk$$

is an invariant pairing.

Proof. Since $f(g) \cdot \tilde{f}(g)$ for any $g \in GL(2, \mathbb{Q}_p)$ satisfies (5.3), corollary implies this theorem. □

5.5 Principal series for $GL(n, \mathbb{Q}_p)$

For $i = 1, \dots, n$, let $\chi_i : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$ be a character and $\chi_i(x) = \omega_i(x) |x|_p^{s_i}$ where $|\omega_i(x)| = 1$, $\omega_i(p) = 1$ and $s_i \in \mathbb{C}$. Define

$$\begin{aligned} B(\chi_1, \chi_2, \dots, \chi_n) & \tag{5.4} \\ &= \left\{ \begin{array}{l} f : GL(n, \mathbb{Q}_p) \rightarrow \mathbb{C} \\ \text{smooth} \end{array} \left| \begin{array}{l} f \left(\begin{pmatrix} a_1 & & * \\ & a_2 & \\ & & \ddots \\ & & & a_n \end{pmatrix} g \right) = \chi_1(a_1) \chi_2(a_2) \cdots \chi_n(a_n) f(g), \\ a_1, \dots, a_n \in \mathbb{Q}_p^\times \end{array} \right. \right\} \end{aligned}$$

Parabolic induction.

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5.6 Parabolic induction: general

Restricted and induced representations Let (π, V) be a representation of G i.e., $\pi : G \rightarrow GL(V)$. Let $H \subset G$ be a subgroup of G . Then π restricts to a representation of H , which we shall call

$$\pi_H := \text{Res}_H^G(\pi)$$

where $\pi_H : H \rightarrow GL(V)$ is defined by

$$\pi_H(h).v := \pi(h).v, \forall h \in H, v \in V$$

and (π_H, V) is called the restricted representation.

Question: Starting with a representation (π, W) where $\pi : H \rightarrow GL(W)$ and $H \subset G$ can we construct an induced representation

$$\pi^G := \text{Ind}_H^G(\pi), \pi^G : G \rightarrow GL(V)$$

for some other space V ?

Definition 5.19. Let

$$W^G := V := \{f : G \rightarrow \text{smooth } W \mid f(hg) = \pi(h).f(g)\}.$$

We define

$$\pi^G := \text{Ind}_H^G(\pi)$$

to be the representation $\pi^G : G \rightarrow GL(W^G)$ with the action by

$$\pi^G(g').f(g) := f(gg')$$

for all $g' \in G, f \in W^G$. Then (π^G, W^G) is called the induced representation. Then (π^G, W^G) is called the induced representation.

Frobenius Reciprocity

$$\text{Hom}_G(V, W^G) \cong \text{Hom}_H(V_H, W)$$

5.7 Parabolic induction for $GL(n, \mathbb{Q}_p)$

Fix $n \geq 1$. Then $k = k_1 + \cdots + k_r = n$, partition. We have

- Parabolic subgroup

$$P_k := \left\{ \begin{pmatrix} A_1 & & * \\ & A_2 & \\ & & \ddots \\ & & & A_r \end{pmatrix} \mid A_i \in GL(k_i), i = 1, \dots, r \right\}$$

- Levi subgroup

$$M_k := \left\{ \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ & & A_r \end{pmatrix} \right\}$$

- Unipotent

$$U_k := \left\{ \begin{pmatrix} I_{k_1} & & * \\ & \ddots & \\ & & I_{k_r} \end{pmatrix} \right\}$$

and we have the Langlands Decomposition:

$$P_k = M_k \cdot U_k.$$

It is easy to construct representations on M_k . We want to induce these representations up to $GL(n, \mathbb{Q}_p)$. (The simplest case is $k_1 = \dots = k_r = 1$ then each representation are one-dimensional(i.e., characters).) In the literature it is more common to induce from P_k . We need to introduce natural projection.

Natural projection

$$\text{pr} : P_k \rightarrow M_k$$

where

$$\text{pr} \left(\begin{pmatrix} A_1 & & * \\ & \ddots & \\ & & A_r \end{pmatrix} \right) := \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ & & A_r \end{pmatrix}.$$

Let $\pi : M_k \rightarrow GL(V)$ be a representation of $GL(n, \mathbb{Q}_p)$.

Trivial Extension

$$\pi^{\text{TE}} : P_k \rightarrow GL(V)$$

where $\pi^{\text{TE}}(\rho) = \pi(\text{pr}(\rho))$ for $\rho \in P_k$.

Key point Let (π, V) be a representation of M_k , $\pi : M_k \rightarrow GL(V)$. To induce up, we introduce the space

$$V^{GL(n, \mathbb{Q}_p)} := \left\{ f : GL(n, \mathbb{Q}_p) \rightarrow V \mid \begin{array}{l} f(umg) = \pi(m)f(g) \\ \text{locally constant} \quad \mid \quad u \in U_k, m \in M_k \end{array} \right\}$$

Definition 5.20 (Modular Quasi character).

$$\delta_{P_k} \left(\begin{pmatrix} A_1 & & * \\ & \ddots & \\ & & A_r \end{pmatrix} \right) := \prod_{i=1}^r |\det A_i|_p^{\sum_{j=i+1}^r k_j - \sum_{j=1}^{i-1} k_j}$$

Remark. If we don't have $\delta_{P_k}^{\frac{1}{2}}(m)$ in the definition of $V^{GL(n, \mathbb{Q}_p)}$ then its called non-normalized induction. Otherwise, normalized induction. Then

$$\pi^{GL(n, \mathbb{Q}_p)} : GL(n, \mathbb{Q}_p) \rightarrow GL \left(V^{GL(n, \mathbb{Q}_p)} \right)$$

where the action is defined by right translation

$$\pi^{GL(n, \mathbb{Q}_p)} = \text{Ind}_{M_k(\mathbb{Q}_p)}^{GL(n, \mathbb{Q}_p)}(\pi).$$

Principle series representation Let $k = 1 + \dots + 1 = n$.

$$M_k = \left\{ \left(\begin{array}{ccc} t_1 & & 0 \\ & \ddots & \\ & & t_n \end{array} \right) \middle| t_i \in \mathbb{Q}_p^\times, i = 1, \dots, n \right\}.$$

Let $\chi_i : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$ be characters for $i = 1, 2, \dots, n$. These are also 1-dimensional representations. Then

$$\begin{aligned} V^{GL(n, \mathbb{Q}_p)} &:= \left\{ f : GL(n, \mathbb{Q}_p) \rightarrow \mathbb{C} \mid \begin{array}{l} f(utg) = \chi_1(t_1) \cdots \chi_n(t_n) f(g) \\ \text{locally constant} \quad \left| \quad \begin{array}{l} u \in U_n(\mathbb{Q}_p), t \in M_k(\mathbb{Q}_p) \end{array} \right. \end{array} \right\} \\ &= B(\chi_1, \chi_2, \dots, \chi_n). \end{aligned}$$

So the induced representation is the principle series $V^{GL(n, \mathbb{Q}_p)}$ with right action.

If we want the contragredient of $B(\chi_1, \dots, \chi_n) := B(\chi_1, \dots, \chi_n)$, to $B(\chi_1^{-1}, \dots, \chi_n^{-1})$ then we need to do normalized induction with modular character δ .

In the 1960's Mautner showed that even for $GL(2, \mathbb{Q}_p)$ not every irreducible smooth representation is principal series or a subquotient of a principle series representations. This was very surprising.

5.8 Supercuspidal representation of $GL(n, \mathbb{Q}_p)$

Definition 5.21. Let (π, V) be a smooth irreducible representation of $GL(n, \mathbb{Q}_p)$. If $\forall k = k_1 + \dots + k_r = n$ and every $v \in V$, there exists $m \in \mathbb{Z}$ such that

$$\int_{U_k(p^m \mathbb{Z}_p)} \pi(u).v \, du = 0$$

then (π, V) is called supercuspidal. Here

$$U_k(p^m \mathbb{Z}_p) = \left\{ \left(\begin{array}{ccc} I_1 & & b_{ij} \\ & \ddots & \\ & & I_r \end{array} \right) \middle| b_{ij} \in p^m \mathbb{Z}_p \right\}.$$

Remark. Supercuspidal representation must be infinite dimensional. (otherwise, it is one-dimensional, then action by character, so $\int_{U_k(p^m \mathbb{Z}_p)} \pi(u).v \, du = \text{constant} \cdot v = 0$ and this implies that the space is 0)

Question: Why do supercuspidal representation exist? How to classify them? (book: Bushnell-Kutzko, theory of types)

Supercuspidal representation for $GL(2, \mathbb{Q}_p)$ (π, V) is a supercuspidal representation of $GL(2, \mathbb{Q}_p)$ if $\forall v \in V$ there exists $m \in \mathbb{Z}$ such that

$$\int_{u \in p^m \mathbb{Z}_p} \pi \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right).v \, du = 0.$$

Theorem 5.22. (π, V) is supercuspidal if $\forall v \in V$, there exists $n \in \mathbb{Z}$ such that

$$\int_{u \in p^{-m} \mathbb{Z}_p} \pi \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right).v \, du = 0$$

for all $m \geq n$.

Proof. Since (π, V) is supercuspidal, then there exists $n \in \mathbb{Z}$ such that $\int_{u \in p^{-n}\mathbb{Z}_p} \pi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \cdot v \, du = 0$. For all $m \geq n$, let $S =$ set of representations for $p^{-n}\mathbb{Z}_p$ in $p^{-m}\mathbb{Z}_p$. Then

$$\begin{aligned} \int_{p^{-m}\mathbb{Z}_p} \pi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \cdot v \, du &= \sum_{s \in S} \int_{p^{-n}\mathbb{Z}_p} \pi\left(\begin{pmatrix} 1 & u+s \\ 0 & 1 \end{pmatrix}\right) \cdot v \, du \\ &= \sum_{s \in S} \pi\left(\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\right) \cdot \int_{p^{-n}\mathbb{Z}_p} \pi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \cdot v \, du = 0 \end{aligned}$$

□

Jacquet found a new definition of supercuspidal. He introduced what is now called the Jacquet Module.

Definition 5.23 (Jacquet Module). Let (π, V) be a smooth representation of $GL(2, \mathbb{Q}_p)$.

$$N := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{Q}_p \right\}$$

Define

$$V(N) := \text{Span} \{ \pi(\eta) \cdot v - v \mid \eta \in N, v \in V \}$$

and

$$V_N = V / V(N).$$

So the Jacquet module is V_N with an action.

Theorem 5.24.

$$v \in V(N) \iff \int_{p^{-n}\mathbb{Z}_p} \pi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \cdot v \, du = 0 \text{ for some } n \in \mathbb{Z}.$$

So

$$\text{supercuspidal} \iff V = V(N) \iff V_N = 0.$$

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Jacquet Module

Definition 5.25 (Finitely generated representation). Let (π, V) be a representation of $GL(n, \mathbb{Q}_p)$. We say (π, V) is finitely generated if there exist v_1, \dots, v_m and every $v \in V$ can be written as

$$v = \sum_{i=1}^m \sum_{j=1}^r c_{ij} \pi(g_{ij}) \cdot v_i, \quad c_{ij} \in \mathbb{C}, \quad g_{ij} \in GL(n, \mathbb{Q}_p)$$

Lemma 5.26. Let (π, V) be a finitely generated smooth representation of $GL(n, \mathbb{Q}_p)$. Then (π, V) has an irreducible subquotient. This means that there exists $V' \subset V$ where V' is invariant under the action of π and V/V' is irreducible.

Proof. Let

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \dots \subset V \tag{5.5}$$

be a proper ascending series of $GL(n, \mathbb{Q}_p)$ -invariant subspaces $V_i, i = 1, 2, 3, \dots$

Claim: $V' := \bigcup_{i=1}^{\infty} V_i$ is again a proper invariant subspace (maximal). (We are using really John's lemma).

Assume that $V' = V$. We want to show this can't happen.

Let v_1, \dots, v_m be generators of V . If $V' = V$, this implies that each $v_i \in V_{j_i}$ in (5.5). Then there exists $M \geq 1$ such that $v_1, \dots, v_m \in V_M$. This implies that $V_M = V$, and it is a contradiction. □

Let

$$N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Q}_p) \right\}.$$

Definition 5.27. Let (π, V) be a smooth representation of $\mathrm{GL}(2, \mathbb{Q}_p)$. Then

$$V(N) := \mathrm{Span} \{ \pi(\eta).v - v \mid \eta \in N, v \in V \}$$

and

$$V_N := V/V(N).$$

Interlude: H -module Let (π, V) be a smooth representation of G . We also call (π, V) a smooth G -module. If $H \subset G$, then H -submodule.

We want to make $V_N \cong V/V(N)$ into a module so we need to find a group that acts on it.

Definition 5.28 (Torus).

$$T = \left\{ \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \mid t_1, t_2 \in \mathbb{Q}_p^\times \right\}$$

Claim: T acts on V_N .

Proof. We must show that $V(N)$ is invariant under the action of T . Let $t \in T$, then

$$\pi(t).(\pi(\eta).v - v) = \pi(t).\pi(\eta).\pi(t)^{-1}.v' - v' \in V(N)$$

where $v' = \pi(t).v$ (since $t\eta t^{-1} \in N$). □

Definition 5.29 (Jacquet Module for $\mathrm{GL}(2, \mathbb{Q}_p)$). The Jacquet module is the vector space $V/V(N) = V_N$ with an action of the torus T , i.e.,

$$\pi : T \rightarrow \mathrm{GL}(V_N)$$

defined by

$$\pi(t).(v + V(N)) = \pi(t).v + V(N).$$

Definition 5.30 (Supercuspidal). A representation (π, V) of $\mathrm{GL}(2, \mathbb{Q}_p)$ is supercuspidal if $V_N = \{0\} \iff V = V(N)$.

Theorem 5.31. Let (π, V) be a smooth infinite dimensional irreducible representation of $\mathrm{GL}(2, \mathbb{Q}_p)$. Assume that (π, V) is not supercuspidal, then it must be a principle series representation or a (sub)quotient of a principle series representation (special representation).

Remark. We're going to prove this theorem using Jacquet module. in Jacquet-Langlands, they proved this theorem using Kirillov module. Using Jacquet module, we can easily generalize this theorem to $\mathrm{GL}(n, \mathbb{Q}_p)$.

Proof. Since we assume that (π, V) is not supercuspidal, then $V_N \neq \{0\}$. Since (π, V) is irreducible then it is generated by some $v \in V$. Fix v . Since π is smooth, then there exist compact subgroups $K_n = \{k \in \mathrm{GL}(2, \mathbb{Z}_p) \mid k - I_2 \in M_2(p^n \mathbb{Z}_p)\}$ such that

$$\pi(k).v = v, \forall k \in K_n$$

and K_n has a finite index in $K = \mathrm{GL}(2, \mathbb{Z}_p)$. By Iwasawa decomposition,

$$\mathrm{GL}(2, \mathbb{Q}_p) = B(\mathbb{Q}_p) \cdot K$$

where $B = \begin{pmatrix} * & * \\ & * \end{pmatrix}$. It follows that V is finitely generated as a B -module.

Let v_1, \dots, v_m be generators of V as a B -module. Then their images in $V_N \cong V/V(N)$ generate V_N as a T -module, for $T = \begin{pmatrix} * & \\ & * \end{pmatrix}$.

By lemma 5.26, V_N has an irreducible quotient as a T -module. Let (π_N, V_N) with $\pi_N : T \rightarrow GL(V_N)$ denote the Jacquet module. So there exists $V' \subset V_N$ such that V_N/V' is irreducible and π_N acts on V' by the rule

$$\pi(t).(v + V') = \pi(t).v + V'$$

for any $v \in V_N$, $t \in T$. Therefore, for any fixed $t \in T$, the linear map

$$\pi(t) : V_N/V' \rightarrow V_N/V'$$

is an intertwining map. By Schur-Dixmier lemma, this implies that $\pi(t)$ acts by a scalar. $\Rightarrow (\pi_N, V_N/V')$ is a one-dimensional representation defined by a character of the torus. $\Rightarrow \exists \chi_1, \chi_2 : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$ and a linear form

$$\theta : V_N/V' \rightarrow \mathbb{C}$$

such that

$$\theta \left(\pi_N \left(\begin{pmatrix} t_1 & b \\ 0 & t_2 \end{pmatrix} \right) .v \right) = \chi_1(t_1)\chi_2(t_2)\theta(v)$$

for all $t_1, t_2 \in \mathbb{Q}_p^\times$, $b \in \mathbb{Q}_p$ and $v \in V_N/V'$.

Consider

$$V \rightarrow V_N \rightarrow V_N/V'$$

By composing θ with the above maps, we get a linear form $\theta : V \rightarrow \mathbb{C}$ satisfying

$$\theta \left[\pi_N \left(\begin{pmatrix} t_1 & b \\ 0 & t_2 \end{pmatrix} \right) .v \right] = \chi_1(t_1)\chi_2(t_2)\theta(v)$$

This shows that $(\pi_N, V_N/V')$ can be realized as a principal series representation.

This is the same thing as an element of $\text{Hom}_B(V_B, W)$, $B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$, and W is just $\cong \mathbb{C}$ regarding as a B -module with action

$$\pi' \left(\begin{pmatrix} t_1 & b \\ 0 & t_2 \end{pmatrix} \right) .z = \chi_1(t_1)\chi_2(t_2)z$$

then we induce up to $GL(2, \mathbb{Q}_p)$.

i.e., for the fixed $v \in V$ (generates all V with the action of $GL(2, \mathbb{Q}_p)$) and define a function of $GL(2, \mathbb{Q}_p)$ as

$$f_v(g) := \theta(\pi(g).v)$$

for any $g \in GL(2, \mathbb{Q}_p)$. Then

$$\text{Span} \{ \pi(h).f_v \mid h \in GL(2, \mathbb{Q}_p) \}$$

is isomorphic to the principle series or special representation. \square

5.9 Rough classification of smooth irreducible representations of $GL(n, \mathbb{Q}_p)$

Theorem 5.32. Fix $n \geq 1$ and a prime p . Let (π, V) be a smooth irreducible representation of $GL(n, \mathbb{Q}_p)$. Then either

- (π, V) is supercuspidal
- there exists $\kappa = (\kappa_1, \dots, \kappa_r)$, $\kappa_1 + \dots + \kappa_r = n$ and an irreducible smooth representation (π', V') of $M_\kappa(\mathbb{Q}_p)$ such that $(\pi, V) \cong$ a subrepresentation of $\text{Ind}_{P_\kappa(\mathbb{Q}_p)}^{GL(n, \mathbb{Q}_p)}(\pi')$.

Here

$$M_\kappa = \begin{pmatrix} A_{\kappa_1} & & * \\ & \ddots & \\ & & A_{\kappa_r} \end{pmatrix}, \quad A_{\kappa_i} \in GL(\kappa_i, \mathbb{Q}_p).$$

Bernstein-Zelevinsky found a better way to classify the smooth irreducible representation of $GL(n, \mathbb{Q}_p)$.