Topics in Number theory

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Contents

Lecture 1

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1 Introduction

Topic: Automorphic representations for $GL(n, \mathbb{A})$, focus when mainly $n > 2$.

Recommended prior knowledge

- modular forms
- adeles
- representation theory

References

- Automorphic forms and L-function for $GL(n, \mathbb{R})$, Goldfeld
- Representation theroy for $GL(n)$ over non-archimedean local fields, D.Prasad, A.Raghuram
- Zeta functions for simple algebras, Godement, Jacquet

Topics:

1. Classical automorphic forms for $GL(n, \mathbb{R})$
2. Automorphic representations for $GL(n, \mathbb{A})$ for $\mathbb{A}$, an adele ring over $\mathbb{Q}$
3. One-to-one correspondence between automorphic representations and classical automorphic forms
4. Adelic automorphic forms, Fourier-Whittaker expansions, and L-functions
5. Admissible representations of $GL(n, \mathbb{Q}_p)$ and $GL(n, \mathbb{R})$
6. Parabolic induction, Jacquet Modules
7. Supercuspidal representations
8. Bernstein-Zelavinskin classification
9. Matrix coefficients
10. Classification of representations by growth of matrix coefficients
11. Local-Global L-functions as integrals of matrix coefficients (Godement-Jacquet)

For $1 \sim 4$, we have global theory, and after 5 we have local theory.

1.1 Automorphic forms

Reference: paper by Borel

Let $G =$ group and $X =$ topological space, and G acts on X, for $g \in G,\ x \in X$,

$$g.x \in X$$

where . is an action. Assume that there exists a function, $\psi : G \times X \to \mathbb{C}$. We look at the space of automorphic functions with multiolier ψ:

$$\{f : X \to \mathbb{C},\ f(g.x) = \psi(g, x)f(x)\}.$$

$$\begin{aligned} f(g_1 g_2.x) &= \psi(g_1 g_2, x) f(x) \\ = f(g_1.(g_2.x)) &= \psi(g_1, g_2.x) f(g_2.x) = \psi(g_1, g_2.x)\psi(g_2, x) f(x) \end{aligned}$$

$$\Rightarrow \quad \text{Cocycle relation: } \psi(g_1 g_2, x) = \psi(g_1, g_2.x)\psi(g_2, x), \ f(x) \neq 0$$

(Cohomology: cocycle relation)

Example: $G = GL(2, \mathbb{R})$ and $X = \mathbb{H} = \{z = x + iy \in \mathbb{C},\ x, y \in \mathbb{R},\ y > 0\}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az + b}{cz + d}, \ \text{linear fractional action}$$

one-cocycle: $j\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z\right) := cz + d$
We have famous classical modular forms,

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z)$$

for $k = 2, 4, \ldots,\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$.

1.2 Automorphic representations

Gelfand-Gruer, P.Shapiro, 1970 Jacquet-Langlands

Representation = vector space V with a group G acting on V.

Roughly, If $V =$ space of automorphic function then we have an automorphic representations

$$V = \{f : X \to \mathbb{C} \mid f(g.x) = \psi(g, x) f(x)\}$$

with the right action,

$$h.f(g) = f(gh),\ g, h \in G$$

1.3 Adelic automorphic forms

We'll be looking at the group

$$\begin{aligned} GL(n, \mathbb{A}) = \ &\text{group of } n \times n \text{ matrices, } \det \neq 0 \\ &\text{with coefficients in } \mathbb{A} \end{aligned}$$

where

$$\mathbb{A} = \left\{ \begin{matrix} \{a_\infty, a_2, \ldots, a_p, \ldots\},\ a_\infty \in \mathbb{R},\ a_p \in \mathbb{Q}_p, \\ a_p \in \mathbb{Z}_p \text{ for almost all } p \end{matrix} \right\}$$

- $g \in GL(n, \mathbb{Q})$ can be embedded diagonally in $GL(n, \mathbb{A})$

$$\{g, g, \ldots,\} \in GL(n, \mathbb{A})$$

- $GL(n, \mathbb{Q}) = \{\{g, g, \ldots,\},\ g \in GL(n, \mathbb{Q})\}$.
- $GL(n, \mathbb{Q})$ acts on $GL(n, \mathbb{A})$. For $\gamma = \{\alpha, \alpha, \ldots\},\ \alpha \in GL(n, \mathbb{Q})$ and $g \in GL(n, \mathbb{A})$ then we have the action:

$$\gamma.g = \gamma g,\ \text{matrix multiplication}$$

Adelic Automorphic functions

$$f : GL(n,\mathbb{A}) \to \mathbb{C}$$

$$f(\gamma.g) = f(g),\ \forall\gamma \in GL(n,\mathbb{Q}),\ g \in GL(n,\mathbb{A}).$$

Forms vs. functions: Forms are eigenfunctions for certain Differential operators for Lie. At place ∞ (real place), we need a theory for $(\mathfrak{g}, K)$-modules.

Automorphic forms: automorphic functions with

- growth condition
- eigenfunction of certain differential operators

1.4 Local theory

We consider the vector spaces of functions

$$V_p = \{f : GL(n,\mathbb{Q}_p) \to \mathbb{C}\}$$

with the right action,

$$h.f(g) = f(gh),\ g, h \in GL(n,\mathbb{Q}_p).$$

Then we have local representations. (This is a group representation.) We want to classify the representations. At ∞ it is different.

$$V_\infty = \{f : GL(n,\mathbb{R}) \to \mathbb{C}\}$$

with an action by differential operator,

$$D.f = Df$$

Here D acts as a differential operator. (This is not a group representation. In a group, every element has inverse, but for differential operators, we don't have inverses. But it is an algebraic representation.)

1.5 Parabolic inductions

You can look at this form

$$P := \begin{pmatrix} m_1 & & & \\ & m_2 & * & \\ & & \ddots & \\ & & & m_r \end{pmatrix},\ m_i \in GL(n_i),\ n_1 + \cdots + n_r = n$$

Parabolic subgroups.

$$\mathrm{Ind}_{\mathrm{P}}^{\mathrm{GL(n,\mathbb{R})}} \text{ powerful way of classifying local representations}$$

1.6 Matrix coefficients

Globally from a classical point of view

- Non-holomorphic modular forms:

$$V := \left\{ f : \mathbb{H} \to \mathbb{C}, \left| \begin{array}{c} f\left(\frac{az+b}{cz+d}\right) = f(z),\ \forall z \in \mathbb{H},\ \begin{pmatrix} a & b\, c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \\ \Delta f = \lambda f \end{array} \right. \right\}$$

  where $\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$.

- Petersson Inner Product: $f_1,\ f_2 \in V$

$$\langle f_1, f_2 \rangle := \iint_{SL(2,\mathbb{Z})\backslash\mathbb{H}} f_1(z)\overline{f_2(z)}\frac{dxdy}{y^2}$$

  Hilbert space, Unitary representation.

Let

$$\pi : G \to GL(V)$$

where

$$\pi(h).f(g) := f(gh)$$

for all $g, h \in G$. Then the inner product satisfies the following:

$$\langle \pi(h).f_1, \pi(h)f_2 \rangle = \langle f_1, f_2 \rangle \,.$$

Once you have such a thing, then the matrix coefficient:

$$g \mapsto \langle \pi(g).f_1, f_2 \rangle \,.$$

Harish-Chandra discovered that local and global automorphic representations can be classified by growth of matrix coefficients. If the matrix coefficients is L^2, $f_1, f_2 \in V$, fixed,

$$\int_G |\langle \pi(g).f_1, f_2 \rangle|^2 \, dg < \infty$$

then the representation is called square integrable by some people but other people say the representation is discrete series. For any $\epsilon > 0$,

$$\int_G |\langle \pi(g).f_1, f_2 \rangle|^{2+\epsilon} \, dg < \infty$$

then the representation is tempered.

1.7 L-functions

Let V = vector space of automorphic functions (local or global) and G = group. Usually, $G = GL(n, \mathbb{A}),\ GL(n, \mathbb{Q}_p)$ or $GL(n, \mathbb{R})$. Fix $v_1,\ v_2 \in V$ and look at matrix coefficients.

$$g \mapsto \langle \pi(g).v_1, v_2 \rangle$$

and integrate over G:

$$L(s; H, \pi, v_1, v_2) := \int_G \langle \pi(g).v_1, v_2 \rangle \, H(g) \, |\det g|^s \, dg$$

where H is a function with a rapid decay. This function is an L-function.

Globally the Pettersson inner product is Hermitian and we have unitary representations. It is not hard to show that all the representations. The global L-function is factored into local L-functions.

Last remark If you are working with modular forms, then

$$f(z) = f\left(-\frac{1}{z}\right), \ f(iy) = f\left(\frac{i}{y}\right)$$

Traditionally this is used to obtain the function equation. But if we work in the above way, then we don't need the relations. The proof is different.

Lecture 2

2010-1-21

2 Classical automorphic forms for $GL(n, \mathbb{R})$

2.1 Review of classical automorphic forms for $GL(2, \mathbb{R})$

Hecke.

$$\mathbb{H} = \{x + iy,\ x \in \mathbb{R},\ y > 0\}$$

Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function that satisfies the following condition:

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z),\ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ then $f(z+1) = f(z)$

$$\Rightarrow\ f(z) = \sum_{n \in \mathbb{Z}} a(n) e^{2\pi i n z} = \sum_{n \in \mathbb{Z}} a(n) e^{2\pi i n x} e^{-2\pi n y}$$

In this case if n is negative, then $e^{-2\pi n y}$ blows up! So we need Moderate Growth Assumption

Moderate Growth Assumption There exists $C, B > 0$ such that

$$|f(z)| \le C|z|^B, \text{ as } |z| \to \infty$$

$$\Rightarrow\ f(z) = \sum_{n=0}^{\infty} a(n) e^{2\pi i n z}$$

Definition 2.1 (Holomorphic modular form of weight k for $SL(2, \mathbb{Z})$)**.** A function $f : \mathbb{H} \to \mathbb{C}$ is a holomorphic modular form of weight k if f satisfies the following conditions:

(1) $f : \mathbb{H} \to \mathbb{C}$ holomorphic

(2) $|f(z)| \le C|z|^B$ as $|z| \to \infty$ (moderate growth)

(3) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z),\ \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$

Later Hecke generalize the definition: Fix an integer $N \ge 1$. Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})\, | c \equiv 0 \pmod N \right\}.$$

Fix a Dirichlet character

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times.$$

Definition 2.2 (Holomorphic modular form of weight k, level N, character χ modular N)**.** A function $f : \mathbb{H} \to \mathbb{C}$ is a holomorphic modular form of weight k, level N, character χ modular N if f satisfies the following conditions:

(1) $f : \mathbb{H} \to \mathbb{C}$ holomorphic

(2) f has moderate growth

(3) $f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z),\ \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$

The space of these functions is the vector space and it is a Hilbert space with the Petersson inner product.

There is no reason that these functions should be holomorphic. In 1940, Maass introduced non-holomorphic forms.

Maass 1940 Introduced non-holomorphic automorphic forms.

Definition 2.3 (Maass form of weight k, level N and character χ modular N)**.** A function $f : \mathbb{H} \to \mathbb{C}$ is a Maass form of weight k if f satisfies the following conditions:

(1) $f : \mathbb{H} \to \mathbb{C}$ smooth and satisfies $\Delta_k f = \lambda f$ where

$$\Delta_k = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \frac{\partial}{\partial x} \text{ (elliptic partial differential operator)}$$

and $\lambda \in \mathbb{R}$

(2) Moderate growth: There exists $B, C > 0$ such that

$$|f(z)| \le C|z|^B, \text{ as } |z| \to \infty.$$

(3) $f\left(\frac{az+b}{cz+d}\right) = \chi(d) \left(\frac{cz+d}{|cz+d|}\right)^k f(z), \ \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$

Aside: Δ_k satisfies the following condition: if f satisfies (3) then $\Delta_k f$ also satisfies (3).

The third conditions for Maass forms and holomorphic modular forms are equivalent. Let $z = x + iy$ and $\mathrm{Im}(z) = y$. Then

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{y}{|cz+d|^2}, \text{ if } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$$

Let f be a holomorphic modular form and

$$F(z) := f(z)\mathrm{Im}(z)^{\frac{k}{2}}$$

then $F\left(\frac{az+b}{cz+d}\right) = \chi(d) \left(\frac{cz+d}{|cz+d|}\right)^k F(z)$.

Let's look at the case when the weight $k = 0$.

$$\Delta(z) := \Delta_0(z) = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

Lemma 2.4.

$$\Delta\left(\frac{az+b}{cz+d}\right) = \Delta(z)$$

i.e., Δ does not change when $z \mapsto \frac{az+b}{cz+d}$ for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$

Proof.

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$$

Then

$$\Delta(z) = -4\mathrm{Im}(z)^2 \frac{\partial}{\partial z} \circ \frac{\partial}{\partial \overline{z}}$$

$$z \mapsto \frac{az+b}{cz+d}, \ \partial z \mapsto (cz+d)^2 \partial z, \ \partial \overline{z} \mapsto \left(\overline{cz+d}\right)^2 \partial \overline{z} \text{ and } \mathrm{Im}(z) \mapsto \frac{\mathrm{Im}(z)}{|cz+d|^2}$$

□

We want to generalize Maass forms to $GL(n, \mathbb{R})$ for all $n \ge 2$.

2.2 Iwasawa decomposition

Every $g \in GL(n, \mathbb{R})$ can be uniquely written in the form

$$g = \widetilde{g} \cdot k \cdot d$$

where $k \in O(n, \mathbb{R}) = \{h \in GL(n, \mathbb{R}) \mid h \cdot^t h = I_n\}$, d is a diagonal matrix, $d = \begin{pmatrix} \delta & & \\ & \ddots & \\ & & \delta \end{pmatrix}$ for $\delta \neq 0$ and $\widetilde{g} \in \mathbb{H}^n$.

Definition 2.5.

$$\mathbb{H}^n = \left\{ x \cdot y \,\middle|\, x = \begin{pmatrix} 1 & x_{ij} & \\ & \ddots & \\ & & 1 \end{pmatrix},\ y = \begin{pmatrix} y_1 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix} \text{ and } x_{ij} \in \mathbb{R},\ y_i > 0 \right\}$$

Proof. for Iwasawa decomposition.
Let $g \in GL(n, \mathbb{R})$. Consider $g \cdot^t g =$ positive definite non-singular matrix. Claim that there exists an upper triangular matrix $u = \begin{pmatrix} 1 & * & \\ & \ddots & \\ & & 1 \end{pmatrix}$, and lower triangular matrix $l = \begin{pmatrix} 1 & & \\ * & \ddots & \\ & & 1 \end{pmatrix}$ and diagonal matrix $d = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{pmatrix}$ for $\delta_i \neq 0$ such that

$$(*)\ u \cdot g \cdot^t g = l \cdot d.$$

For $n = 2$: $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, ${}^t g = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$, $g^t g = \left(\begin{smallmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g^t g = \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right)$.

$$(*) \Rightarrow\ g^t g = u^{-1} l d =^t (u^{-1} l d) = d^t l^t u^{-1}$$

$$\Rightarrow\ (**)\ l d^t u = u d^t l$$

Then the LH is lower triangular and RH is upper triangular so they are diagonal. So

$$(**)\ l d^t u = u d^t l = d$$

$$\Rightarrow\ (***) l d = d^t u^{-1}$$

$$(***) \text{ into } (*) \Rightarrow\ u g^t g = d^t u^{-1}\ \Rightarrow\ u g^t g^u = d = a^{-1t} a^{-1}$$

$$\Rightarrow\ a u g \cdot^t (a u g) = I_n$$

$$\Rightarrow\ a u g = k \in O(n, \mathbb{R}) \text{ Iwasawa decomposition}$$

□

2.3 Maass forms of weight 0 for $SL(n, \mathbb{Z})$

Let $n \geq 2$.

(1) $f : \mathbb{H}^n \to \mathbb{C}$ smooth and satisfy the set of differential equations $\Delta_i = \lambda_i f$ for $i = 1, 2, \ldots, n-1$. Here $\Delta_i(z) = \Delta_i(\gamma z)$ for all $z \in \mathbb{H}^n$ $\gamma \in SL(n, \mathbb{R})$.

(2) Moderate growth condition: there exists $C > 0,\ B_1, B_2, \ldots, B_{n-1} > 0$ such that

$$|f(z)| \leq C y_1^{B_1} \cdots y_{n-1}^{B_{n-1}}$$

as $y_1, \ldots, y_{n-1} \to \infty$.

(3) $f(\gamma z) = f(z)$ for all $\gamma \in SL(n,\mathbb{Z})$ and

$$f(zkd) = f(z),\ \forall z \in \mathbb{H}^n,\ k \in K_n = O(n,\mathbb{R}),\ d \in Z_n = \text{center}(GL(n,\mathbb{R})).$$

Let $K = O(n,\mathbb{R}) =$ orthogonal group.

$$f : GL(n,\mathbb{R})/O(n,\mathbb{R}) \cdot \mathbb{R}^\times \to \mathbb{C}$$

These Maass forms of weight 0 are K-invariant under right multiplication by $k \in K$ correspond to unramified representation.

Lecture 3

2010-01-26

Review

Iwasawa decomposition For any $g \in GL(n,\mathbb{R})$ we have unique

$$x = \begin{pmatrix} 1 & x_{ij} & \\ 0 & \ddots & \\ & & 1 \end{pmatrix},\ y = \begin{pmatrix} y_1 \cdots y_{n-1} & & \\ & \ddots & \\ & & 1 \end{pmatrix},\ k \in O(n,\mathbb{R})$$

and $d = \begin{pmatrix} \delta & & \\ & \ddots & \\ & & \delta \end{pmatrix}$ such that

$$g = xykd.$$

Using Iwasawa decomposition, we can define generalized upper half plane:

$$\mathbb{H}^n = \{xy\}$$

For example, for $n = 2$,

$$\mathbb{H}^2 = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix},\ x \in \mathbb{R},\ y > 0 \right\}.$$

Automorphic functions on $\mathbb{H}^n$ of weight 0 for $GL(n,\mathbb{Z})$

$$\phi : \mathbb{H}^n \to \mathbb{C},\ \phi(\gamma z) = \phi(z),\ \forall \gamma \in GL(n,\mathbb{Z}),\ z \in \mathbb{H}^n.$$

For example we have Ramanujan weight 12 function

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z)$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$.

2.4 Generalize level and weights

Definition 2.6 (Jacquet,Shalika,Piatetski-Shapiro)**.** Fix an integer $N \geq 1$. Fix $n \geq 2$. We define

$$\Gamma_0(N) := \left\{ \begin{pmatrix} A & B \\ C & d \end{pmatrix} \in GL(n,\mathbb{Z}) \middle| \begin{array}{c} A \in GL(n-1,\mathbb{Z}), \\ C = (c_1, c_2, \ldots, c_{n-1}) \equiv (0,0,\ldots,0) \operatorname{mod} N,\ d \in \mathbb{Z} \end{array} \right\} \tag{2.1}$$

Let $\gamma \in GL(n,\mathbb{R})^+$ and $z \in \mathbb{H}^n$. By the Iwasawa decomposition,

$$\gamma z = \widetilde{\gamma g} \cdot \kappa(\gamma, z) \cdot d$$

where $\widetilde{\gamma g} \in \mathbb{H}^n$ and $\kappa(\gamma, z) \in O(n,\mathbb{R})^+ =: K$ and $d = \begin{pmatrix} \delta & & \\ & \ddots & \\ & & d \end{pmatrix}$ for $\delta > 0$. i.e.,

$$\kappa : GL(n,\mathbb{R})^+ \times \mathbb{H}^n \to K.$$

For example, $n = 2$, $z = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{cx+d}{|cz+d|} & -\frac{cy}{|cz+d|} \\ \frac{cy}{|cz+d|} & \frac{cx+d}{|cz+d|} \end{pmatrix} \begin{pmatrix} |cz+d| & 0 \\ 0 & |cz+d| \end{pmatrix}$$

where $\left(\begin{smallmatrix} y' & x' \\ 0 & 1 \end{smallmatrix}\right) \in \mathbb{H}^2$ and $\begin{pmatrix} \frac{cx+d}{|cz+d|} & -\frac{cy}{|cz+d|} \\ \frac{cy}{|cz+d|} & \frac{cx+d}{|cz+d|} \end{pmatrix} \in SO(2,\mathbb{R})$.

Lemma 2.7. $\kappa(\gamma, z)$ *is a one-cocycle, i.e.,*

$$\kappa(\gamma\gamma', z) = \kappa(\gamma, \widetilde{\gamma' z}.$$

Proof.

$$\gamma\gamma' z = \widetilde{\gamma\gamma' z} \cdot \kappa(\gamma\gamma', z) \cdot d$$

and

$$\begin{aligned} \gamma(\gamma' z) &= \gamma \left(\widetilde{\gamma' z} \kappa(\gamma', z) d' \right) \\ &= \widetilde{\gamma\widetilde{\gamma' z}} \cdot \kappa(\gamma, \widetilde{\gamma' z}) \cdot \kappa(\gamma', z) d'' \\ &= \widetilde{\gamma\gamma' z} \kappa(\gamma, \widetilde{\gamma' z}) \kappa(\gamma', z) \cdot d \end{aligned}$$

by the uniqueness of the Iwasawa decomposition, we are done. ☐

Fix $n \geq 2$ and $r \geq 1$. Let

$$\rho : SO(n,\mathbb{R}) \to GL(r,\mathbb{C})$$

be an irreducible representation. Then ρ will be the analogue of the weight k.

Definition 2.8.

$$J_\rho(\gamma, z) = \rho\left(\kappa(\gamma, z)^{-1}\right) \tag{2.2}$$

for $z \in \mathbb{H}^n$, $\gamma \in GL(n,\mathbb{R})^+$. Furthermore,

$$J_\rho(\gamma\gamma', z) = J_\rho(\gamma, \widetilde{\gamma' z}) \cdot J_\rho(\gamma', z).$$

Example for $n = 2$:

$$SO(2,\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid 0 \leq \theta \leq 2\pi \right\}$$

is an abelian group. The only representation of $SO(2,\mathbb{R})$ are characters, i.e., Define

$$\rho\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right) := (\cos\theta + i\sin\theta)^k$$

for some integer $k \in \mathbb{Z}$. This is a one-dimensional representation of $SO(2,\mathbb{R})$.

$$J_\rho(\gamma, z) = \rho(\kappa(\gamma, z)^{-1}) = \rho\left(\begin{pmatrix} \frac{cx+d}{|cz+d|} & \frac{cy}{|cz+d|} \\ -\frac{cy}{|cz+d|} & \frac{cx+d}{|cz+d|} \end{pmatrix} \right) = \left(\frac{cz+d}{|cz+d|} \right)^k$$

2.5 Classical vector valued automorphic functions

$\Phi:\mathbb{H}^n\to\mathbb{C}^r$ and $\phi_i:\mathbb{H}^n\to\mathbb{C}$ for $i=1,2,\dots,r$ such that

$$\Phi(z)=\begin{pmatrix}\phi_1(z)\\ \vdots\\ \phi_r(z)\end{pmatrix}$$

Definition 2.9 (Slash operator)**.**

$$(\Phi\,|_\rho)\,(z):=J_\rho(\gamma,z)^{-1}\Phi(\gamma,z)$$

Definition 2.10 (Classical vector valued automorphic functions of arbitrary weight $\rho:SO(n,\mathbb{R})\to GL(r,\mathbb{C})$, level $N\geq 1$ and character $\chi \mathrm{mod} N$)**.** Fix integers $n\geq 2$, $r\geq 1$. Fix irreducible representation

$$\rho:SO(n,\mathbb{R})\to GL(r,\mathbb{C}).$$

Let $\Phi=\begin{pmatrix}\phi_1\\ \vdots\\ \phi_r\end{pmatrix}:\mathbb{H}^n\to\mathbb{C}^r$. Fix $N\geq 1$ and a Dirichlet character $\chi\mathrm{mod}N$. Then Φ is a classical automorphic function of weight ρ, level N and character χ if

$$(\Phi\,|_\gamma)\,(z)=\chi(d)\Phi(z)$$

for any $z\in\mathbb{H}^n$ and $\gamma=\left(\begin{smallmatrix}A & B\\ C & d\end{smallmatrix}\right)\in\Gamma_0(N)$.

Recall that an automorphic form is an automorphic function satisfying 2 extra conditions:

(1) Moderate growth

(2) Satisfy some differential equations

Definition 2.11 (Moderate growth)**.** A smooth function $f:\mathbb{H}^n\to\mathbb{C}$ is of moderate growth if for each fixed $\sigma\in GL(n,\mathbb{Q})$ there exists $C>0,\ B>0$ such that

$$|f(\sigma z)|\leq C|y_1y_2\cdots y_{n-1}|^B$$

for $z=xy,\ y=\begin{pmatrix}y_1y_2\cdots y_{n-1} & & \\ & \ddots & \\ & & 1\end{pmatrix}$.

We'll work for $GL(n,\mathbb{R})^+$. Why? Explain by example below:

Lift from upper half-plane $\mathbb{H}^2=\{x+iy,\ x\in\mathbb{R},\ y>0\}$ **to** $GL(2,\mathbb{R})^+$ Let $f:\mathbb{H}^2\to\mathbb{C}$, $g=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$

$$(f|_k g)\,(z):=\left(\frac{cz+d}{|cz+d|}\right)^{-k}f\left(\frac{az+b}{cz+d}\right).$$

If $g\in GL(2,\mathbb{R})^+$, by Iwasawa

$$g=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\delta & 0\\ 0 & \delta\end{pmatrix}$$

for $0\leq\theta\leq 2\pi,\ \delta>0$.

Definition 2.12.

$$\widetilde{f}(g)=(f|_k g)\,(i)=e^{ik\theta}f(x+iy)$$

Then

$$\widetilde{f}:GL(2,\mathbb{R})^+\to\mathbb{C}.$$

If we use $GL(2,\mathbb{R})$ then $\widetilde{f}$ will be the combination of weight k and weight $-k$.
Let Φ be automorphic for $GL(n,\mathbb{R})$

Definition 2.13 (Lift to $GL(n,\mathbb{R})^+$)**.**

$$\widetilde{\Phi}(g):=(\Phi|_\rho g)\,(I_n),\ I_n=n\times n\text{identity matrix.}$$

Differential operators Let $\alpha \in \mathfrak{gl}(n, \mathbb{R})$ and $f : GL(n, \mathbb{R})^+ \to \mathbb{C}$.

$$D_\alpha f(g) := \lim_{t \to 0} \frac{1}{t} \left[f(g \cdot \exp(t\alpha)) - f(g) \right]$$

(Lie derivative) We are looking for a differential operator which takes automorphic functions to automorphic functions

Casimir

$$E_{ij} = n \times n \text{ matrix which is 1 at position } (i,j) \text{ and 0 every where else}$$

Let $D_{ij} := D_{E_{ij}}$.

$$\sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_m=1}^{m} D_{i_1,i_2} \circ \cdots D_{i_m,i_1}$$

for $1 \leq m \leq n$. We have n different Casimir operators. The Casimir operator generate $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ =center of the universal enveloping algebra, where $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$.

Definition 2.14 (Maass form for $GL(n, \mathbb{R})$)**.** Fix $n \geq 2,\ r \geq 1,\ \rho : SO(n, \mathbb{R}) \to GL(r, \mathbb{C}),\ N \geq 1$ and $\chi \bmod N$. Let

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix} : \mathbb{H}^n \to \mathbb{C}^r$$

Then Φ is a vector valued Masss form of weight ρ, level N and character χ. if

(1) $\left(\Phi|_\rho \gamma\right)(z) = \chi(d)\Phi(z),\ \forall \gamma = \begin{pmatrix} A & B \\ C & d \end{pmatrix} \in \Gamma_0(N),\ z \in \mathbb{H}^n$

(2) Φ has moderate growth.

(3) For any Casimir differential operator $D \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})),\ \exists \lambda_D \in \mathbb{C}$ such that

$$D\Phi = \lambda_D \Phi$$

(4) L^2 condition.

Lecture 4

2010-01-28

2.6 Casimir differential operators on $GL(n, \mathbb{R})$

For $\alpha \in \mathfrak{gl}(n, \mathbb{R})$ and $f : GL(n, \mathbb{R}) \to \mathbb{C}$ we define Lie Derivative

$$D_\alpha f(g) := \lim_{t \to 0} \frac{1}{t} \left[f(g \cdot \exp(t\alpha)) - f(g) \right].$$

For $m = 1, 2, \ldots, n$ we have the Casimir operators

$$D_m := \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} D_{i_1,i_2} \circ D_{i_2,i_3} \circ \cdots \circ D_{i_m,i_1} \tag{2.3}$$

where $E_{i,j} = n \times n$ matrix with 1 at position (i,j), zeros everywhere else and $D_{i,j} = D_{E_{i,j}}$.

Example for $n = 3$

$$z = x \cdot y = \begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} y_1 y_2 & x_2 y_1 & x_3 \\ 0 & y_1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{H}^3$$

$$E_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

then

$$\exp(tE_{2,3}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} + \text{ higher powers}$$

Let's compute

$$\begin{aligned} D_{2,3} f(z) &= \lim_{t \to 0} \frac{1}{t} \left[f\left(z \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \right) - f(z) \right] \\ &= \frac{\partial}{\partial t} f\left(\begin{pmatrix} y_1 y_2 & x_2 y_1 & x_2 y_1 t + x_3 \\ 0 & y_1 & y_1 t + x_1 \\ 0 & 0 & 1 \end{pmatrix} \right) \Bigg|_{t=0} \\ &= x_2 y_1 \frac{\partial}{\partial x_3} f(z) + y_1 \frac{\partial}{\partial x_1} f(z) \end{aligned}$$

Then $D_{2,3} = x_2 y_1 \frac{\partial}{\partial x_3} + y_1 \frac{\partial}{\partial x_1}$.

Prove "Invariant" Let V =vector space of all smooth functions $f : GL(n, \mathbb{R}) \to \mathbb{C}$ and

$$\pi : GL(n, \mathbb{R}) \to \mathrm{End}(V)$$

given by right translation, i.e., for any $f \in V$, and for any $h \in GL(n, \mathbb{R})$ define the action

$$\pi(h).f(g) := f(gh).$$

Now we want to prove the following theorem:

Theorem 2.15. *Fix $n \geq 2$ and for $1 \leq m \leq n$ let D_m be the Casimir differential operator defined in (2.3). Then*

$$(\pi(h) \circ D_m) .f(g) = (D_m \circ \pi(h)) .f(g)$$

for any $g, h \in GL(n, \mathbb{R})$ and any $f \in V$.

Proof. Define

$$M = \begin{pmatrix} D_{1,1} & \cdots & D_{1,n} \\ \vdots & \cdots & \vdots \\ D_{n,1} & \cdots & D_{n,n} \end{pmatrix}$$

Claim: $\mathrm{Trace}(M^m) = D_m$.
Need to prove that $\pi(h) \circ D_m \circ \pi(h)^{-1} = D_m$
Claim: $\pi(h) \circ D_\alpha \circ \pi(h)^{-1} = D_{h\alpha h^{-1}}$
Because

$$\begin{aligned} h \exp(\alpha) h^{-1} = \exp(h\alpha h^{-1}) &= \sum_{k=0}^{\infty} \frac{\left(h\alpha h^{-1}\right)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{h\alpha^k h^{-1}}{k!} \end{aligned}$$

π and D are linear operators and

$$\text{Trace}\left(\pi(h)M^m\pi(h)^{-1}\right) = \text{Trace}\left(M^m\right)$$

□

Easy to show

$$D_m \circ D_{i,j} = D_{i,j} \circ D_m, \forall i,j,m$$

$\Rightarrow D_m$ commutes with all differential operators.

Definition 2.16 (Maass forms for $GL(n,\mathbb{R})$ of weight ρ, level N and character χ)**.** Smooth function $f : \mathbb{H}^n \to \mathbb{C}^r$, $\mathbb{H}^n = GL(n,\mathbb{R})/O(n,\mathbb{R}) \cdot \mathbb{R}^\times$.

$$\rho : O(n,\mathbb{R}) \to GL(r,\mathbb{C}),\ \text{irreducible representation}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} A & B \\ C & d \end{pmatrix} \middle| \begin{matrix} A \in GL(n-1,\mathbb{Z}), \\ C = (c_1,\dots,c_{n-1}) \equiv (0,\dots,0) \bmod N,\ d \in \mathbb{Z} \end{matrix} \right\}$$

(1) $\left(f\,|_\rho \gamma\right)(z) = \chi(d) f(z),\ \forall \gamma \in \Gamma_0(N)$

(2) Moderate growth

(3) $\lambda_1,\dots,\lambda_n \in \mathbb{C}$ such that $D_m f = \lambda_m f$ where $D_m = m$th Casimir operator.

(4) $\int_{\Gamma_0(N)\backslash\mathbb{H}^n} |f_i(z)|^2\, d^*z < \infty$ where $f(z) = \begin{pmatrix} f_1(z) \\ \vdots \\ f_r(z) \end{pmatrix}$.

2.7 Cusp forms

Definition 2.17. Fix $n \geq 2$, $1 \leq m < n$. Let $\mathbb{H}^n = GL(n,\mathbb{R})/O(n,\mathbb{R}) \cdot \mathbb{R}^\times$. Define $\mathrm{X}^n_m \subset \mathbb{H}^n$ by

$$\mathrm{X}^n_m := \left\{ \begin{pmatrix} I_m & * \\ 0 & I_{n-m} \end{pmatrix} \right\}$$

where $I_k = k \times k$ identity matrix.

$$\mathrm{Y}^n_m := \left\{ \begin{pmatrix} z'd & 0 \\ 0 & z'' \end{pmatrix} \middle| z' \in \mathbb{H}^m,\ z'' \in \mathbb{H}^{m-n},\ d = \begin{pmatrix} \delta & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \delta \end{pmatrix},\ \delta > 0 \right\}.$$

Definition 2.18 (Constant terms of a classical automorphic form)**.** Fix $n \geq 2,\ r > 1,\ N \geq 1,\ \chi \mathrm{mod} N$ and $\rho : O(n,\mathbb{R}) \to GL(r,\mathbb{C})$(irreducible). Let

$$\Phi := \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix}$$

be a vector valued Maass form of weight ρ, level N and chracter χ. For $1 \leq m < n$ and $\gamma \in GL(n,\mathbb{Z})$ let $D(m,N,\gamma)$ be a fundamental domain for $\mathrm{X}^n_m(\mathbb{Z}) \cap \left(\gamma^{-1}\Gamma_0(N)\gamma\right)$ acting on $\mathrm{X}^n_m(\mathbb{R})$. i.e.,

$$D(m,N,\gamma) := \left(\mathrm{X}^n_m(\mathbb{Z}) \cap \gamma^{-1}\Gamma_0(N)\gamma\right) \backslash \mathrm{X}^n_m(\mathbb{R})$$

Then we define

$$\Phi_{m,\gamma}(y) := \int_{D(m,N,\gamma)} \left(\Phi\,|_\rho \gamma\right)(xy)dx = \begin{matrix}\text{constant terms along} \\ \mathrm{X}^n_m \text{ for } \gamma^{-1}\Gamma_0(N)\gamma\end{matrix}$$

Definition 2.19 (Maass cusp form for $GL(n,\mathbb{R})$)**.** A vector valued Mass form $\Phi : \mathbb{H}^n \to \mathbb{C}^r$ is said to be a cusp form of

$$\Phi_{m,\gamma}(y) = 0$$

for all $1 \leq m < n$ and all $\gamma \in GL(n,\mathbb{Z})$.

Example for $n = 2$ In this case $m = 1$. Then

$$\mathrm{X}_1^2 = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{X}_1^2(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix},\ l \in \mathbb{Z} \right\}$$

Then

$$\mathrm{X}_1^2(\mathbb{Z}) \cap \gamma^{-1}\Gamma_0(N)\gamma = \text{cyclic group generated by } \begin{pmatrix} 1 & m_{\mathfrak{a}} \\ 0 & 1 \end{pmatrix}$$

where $\mathfrak{a} \in \mathbb{Q} \cup \{\infty\}$ is a cusp, $\mathfrak{a} = \gamma\infty$. So

$$\int_{\left(\begin{smallmatrix} 1 & m_{\mathfrak{a}} \\ 0 & 1 \end{smallmatrix}\right) \backslash \left(\begin{smallmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{smallmatrix}\right)} = \int_0^{m_{\mathfrak{a}}}$$

3 Adelic automorphic forms for $GL(n)$

3.1 Adelic rings

Adelie ring

$$\mathbb{A}_{\mathbb{Q}} = \{(a_\infty, a_2, \ldots, a_p, \ldots)\}$$

where $a_\infty \in \mathbb{R},\ a_p \in \mathbb{Q}_p$ and $a_p \in \mathbb{Z}_p$ for all but finitely many p.

$$K := O(n, \mathbb{R}) \prod_p GL(n, \mathbb{Z}_p) \subset GL(n, \mathbb{A}_{\mathbb{Q}})$$

is the maximal compact subgroup.
$GL(n, \mathbb{Q})$ can be diagonally embedded in $GL(n, \mathbb{A})$ if $\alpha \in GL(n, \mathbb{Q})$ then $(\alpha, \alpha, \ldots,) \in GL(n, \mathbb{A})$.

3.2 Adelic automorphic functions

- $f : GL(n, \mathbb{A}) \to \mathbb{C}$
- $f(\gamma g) = f(g),\ \forall \gamma \in GL(n, \mathbb{Q}),\ \forall g \in GL(n, \mathbb{A})$
- Smooth conditions: $f\left((g_\infty, g_2, \ldots, g_p, \ldots)\right)$ should be differentiable at ∞, and at p, a locally constant function.

Lecture 5

2010-2-2

3.3 Adelic automorphic forms for $GL(n, \mathbb{A}_{\mathbb{Q}})$

$$GL(n, \mathbb{A}_{\mathbb{Q}}) = \text{ restricted product } GL(n, \mathbb{R}) \cdot \prod_p GL(n, \mathbb{Q}_p) \tag{3.1}$$

where restricted is with respected to $GL(n, \mathbb{Z}_p)$, i.e., if $g = \{g_\infty, g_2, \ldots, g_p, \ldots\} \in GL(n, \mathbb{A}_{\mathbb{Q}})$ then $g_p \in GL(n, \mathbb{Z}_p)$ for almost all finitely many p. In order to define automorphic forms, we need several conditions.

An Adelic Automorphic Form

$$f : GL(n, \mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$$

with central character $\omega : \mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}}^{\times} \to \mathbb{C}^{\times}$(i.e., $\omega(\alpha z) = \omega(z)$ for any $z \in \mathbb{A}_{\mathbb{Q}}^{\times}$ and $\alpha \in \mathbb{Q}^{\times}$), satisfies several conditions.

(1) f is smooth (i.e., f is smooth for ∞ place, and locally constant for finite places).

(2) $f(\gamma g) = f(g), \ \forall \gamma \in GL(n, \mathbb{Q}), \ g \in GL(n, \mathbb{A}_{\mathbb{Q}})$ where $\gamma = (\alpha, \alpha, \ldots,)$ for $\alpha \in GL(n, \mathbb{Q})$.

(3) $f(zg) = \omega(z) f(g), \ \forall z \in \mathbb{A}_{\mathbb{Q}}^{\times}, \ g \in GL(n, \mathbb{A}_{\mathbb{Q}})$

(4) f is right K-finite, for $K = O(n, \mathbb{R}) \cdot \prod_p GL(n, \mathbb{Z}_p)$.

(5) f is $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$-finite where $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}), \ \mathcal{U}(\mathfrak{g})$ = universal enveloping algebra generated by $D_{\alpha}, \ \alpha \in \mathfrak{g}$ and $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ = center of $\mathcal{U}(\mathfrak{g})$.

(6) f is of moderate growth.

We're going to explain the conditions.

For $g \in GL(n, \mathbb{A}_{\mathbb{Q}})$, let $f(g) = f(g_{\infty}, \ldots, g_p, \ldots)$. Fix all g_v except at one place p, then smooth means that the function f restricted to the place p, $GL(n, \mathbb{Q}_p) \to \mathbb{C}$ is locally constant. For $v = \infty$, fix all $v < \infty$ then $f(g_{\infty}) : GL(n, \mathbb{R}) \to \mathbb{C}$ is smooth if this function is infinitely differentiable in all values $g_{i,j} \in \mathbb{R}$ for $g_{\infty} = (g_{i,j})_{1 \le i,j \le n}$.

Definition 3.1 (Right K-finite)**.**

$$K = O(n, \mathbb{R}) \cdot \prod_p GL(n, \mathbb{Z}_p) = \begin{array}{c}\text{maximal compact}\\ \text{subgroup of } GL(n, \mathbb{A}_{\mathbb{Q}})\end{array}$$

We say a function $f : GL(n, \mathbb{Q}) \to \mathbb{C}$ is right K-finite, if the vector space

$$\text{Span}\{f(gk), \ k \in K\}$$

is a finite dimensional vector space.

Definition 3.2 ($\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$-finite)**.** Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ is the polynomial algebra generated by the Casimir differential operator $D_1, \ldots, D_n$ act on g_{∞}.

We say f is $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$-ifnite if the vector space

$$\text{Span}\{Df(g), \ \forall D \in \mathcal{Z}(\mathcal{U}(\mathfrak{g}))\}$$

is finite dimensional.

Norm on $GL(n, \mathbb{Q}_v)$ Let $a \in \mathbb{A}_{\mathbb{Q}}$,

$$||a|| = \prod_{v \le \infty} |a_v|_v \tag{3.2}$$

Let

$$g = \begin{pmatrix} g_{1,1} & \cdots & g_{1,n} \\ \vdots & \ddots & \vdots \\ g_{n,1} & \cdots & g_{n,n} \end{pmatrix} \in GL(n, \mathbb{Q}_v)$$

for $v \leq \infty$. For $v = \infty$, $GL(n, \mathbb{Q}_\infty) = GL(n, \mathbb{R})$. Define

$$||g||_v := \max\left(|g_{i,j}|_v, |\det g|_v\right)_{1 \leq i,j \leq n} \tag{3.3}$$

Let $g = \{g_\infty, g_2, \ldots, g_p, \ldots\} \in GL(n, \mathbb{A}_\mathbb{Q})$ then

$$||g|| = \prod_{v \leq \infty} ||g_v||_v. \tag{3.4}$$

3.4 Adelic lift of a classical automorphic form on $GL(n, \mathbb{R})$

Theorem 3.3. *Fix integers $n \geq 2$, $r, N \geq 1$. Fix irreducible representation $\rho : O(n, \mathbb{R}) \to GL(r, \mathbb{C})$. Fix a character $\chi : (N\mathbb{Z}\backslash\mathbb{Z})^\times \to \mathbb{C}^\times$. Let $\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix}$ be a classical vector valued automorphic form (i.e., $(\Phi\,|_\rho \gamma)(z) = \chi(d) \cdot \Phi(z)$ for all $z \in \mathbb{H}^n \cong GL(n, \mathbb{R})/\mathbb{R}^\times \cdot O(n, \mathbb{R})$ and $\gamma = \left(\begin{smallmatrix} A & B \\ C & d \end{smallmatrix}\right) \in \Gamma_0(N)$). Then there exists r adelic automorphic forms which can be explicitly constructed from $\phi_1, \ldots, \phi_r$ which can be explicitly constructed from $\phi_1, \ldots, \phi_r$. Each $\phi_i \to \phi_{i,\text{adelic}}$.*

The adelic requires strong approximation Two embeddings:

- $i_{\text{diag}}(\gamma) := (\gamma, \ldots, \gamma, \ldots) : \ GL(n, \mathbb{Q}) \to GL(n, \mathbb{A}_\mathbb{Q})$
- $i_\infty(g_\infty) := (g_\infty, I_n, I_n, \ldots) : \ GL(n, \mathbb{R}) \to GL(n, \mathbb{A}_\mathbb{Q})$

Lemma 3.4.

$$i_{\text{diag}}(SL(n, \mathbb{Q})) \cdot i_\infty(SL(n, \mathbb{R}))$$

is dense in $SL(n, \mathbb{A}_\mathbb{Q})$.

Sketch: Let $U_{i,j}$ be a matrix in $SL(n, \mathbb{A}_\mathbb{Q})$, 1 on the diagonal and $u_{i,j} \in \mathbb{A}_\mathbb{Q}^\times$ for (i, j) place, and otherwise 0. Then $SL(n, \mathbb{A}_\mathbb{Q})$ is generated by $U_{i,j}$. Use Tate Thesis for $n = 1$. Then we are done. ☐

$$\mathbb{A}_{\text{finite}} = ((1)?, a_2, \ldots, a_p, \ldots)\,, \ a_p \in \mathbb{Q}_p \text{ and } a_p \in \mathbb{Z}_p \text{ for almost all } p$$

Definition 3.5. Let $N \geq 1$, $N = \prod_{i=1}^{l} p_i^{e_i}$. We define the compact subgroup $K_0(N) \subseteq GL(n, \mathbb{A}_\mathbb{Q})$ as follows:

$$K_0(N) \subseteq \prod_p GL(n, \mathbb{Z}_p) \subset GL(n, \mathbb{A}_{\text{finite}})$$

(for $k \in \prod_p GL(n, \mathbb{Z}_p)$, $k = (k_2, \ldots, k_p, \ldots)$) We define

$$K_0(N) = \left\{ k \in \prod_p GL(n, \mathbb{Z}_p) \;\middle|\; \begin{array}{c} k_{p_i} = \begin{pmatrix} A_i & B_i \\ C_i & d_i \end{pmatrix},\ A_i \in GL(n-1, \mathbb{Z}_{p_i}), \\ C_i \in \left(p_i^{e_i}\mathbb{Z}_{p_i}\right)^{n-1},\ d \in \mathbb{Z}_{p_i},\ \text{for } i = 1, \ldots, l,\ \text{otherwise } k_p \in GL(n, \mathbb{Z}_p) \end{array} \right\}$$

(it is a compact open subset.)

Theorem 3.6 (Stronger approximate)**.** *Fix $N \geq 1$ and $n \geq 2$. Let $g \in GL(n, \mathbb{A}_\mathbb{Q})$. Then there exist $\gamma \in GL(n, \mathbb{Q})$ and $g_\infty(GL(n, \mathbb{R}))$, $k_N \in K_0(N)$ such that*

$$g = i_{\text{diag}}(\gamma) i_\infty(g_\infty) k_N$$

(this decomposition may not be unique!)

Sketch of the proof. We first consider $g \in SL(n, \mathbb{A}_{\mathbb{Q}})$. Let U_∞ be an open neighborhood of I_n in $SL(n, \mathbb{R})$. consider the set

$$g \cdot (i_\infty(U_\infty) \cdot K_0(N)) = \text{ open set}$$

By strong approximate, this set contains $i_{\text{diag}}(\gamma) \cdot i_\infty(g'_\infty)$ for some $\gamma \in SL(n, \mathbb{Q})$ and $g'_\infty \in SL(n, \mathbb{R})$. Then we are done. □

Definition 3.7 (Idelic lifting of a classical Dirichlet character)**.** Fix $N \geq 1$, $\chi : (N\mathbb{Z}\backslash\mathbb{Z})^\times \to \mathbb{C}^\times$, $N = \prod_{i=1}^{l} p_i^{e_i}$. Let $\chi_{p_i} : (p_i\mathbb{Z}\backslash\mathbb{Z})^\times \to \mathbb{C}^\times$ such that

$$\chi(d) = \prod_{i=1}^{l} \chi_i(d)^{e_i}$$

Let $k \in K_0(N)$ and $k = (k_2, \ldots, k_p, \ldots)$ and $k_p = \begin{pmatrix} A_b & B_p \\ C_p & d_p \end{pmatrix}$ for $p = p_i$. Then

$$\widetilde{\chi}_{idelic} : K_0(N) \to \mathbb{C}^\times$$

is defined by

$$\widetilde{\chi}_{idelic}(k) := \prod_{i=1}^{l} \chi_i(d_{p_i})^{-e_i}$$

Let $\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix}$ be a classical automorphic form of weight $\rho : O(n, \mathbb{R}) \to GL(n, \mathbb{C})$ with level N, character χ.

Definition 3.8.

$$\Phi_{\text{adelic}}\left(i_{\text{diag}}(\gamma) \cdot i_\infty(g_\infty) \cdot k_{\text{finite}}\right) := \widetilde{\chi}_{\text{idelic}}(k_{\text{finite}}) \cdot \left(\Phi \mid_\rho g_\infty\right)(I_n)$$

Lecture 6

slight mistakes last time. The "lifting" also works for $GL(1)$.

3.5 Adelic automorphic forms for $GL(1, \mathbb{A}_{\mathbb{Q}})$

$$GL(1, \mathbb{A}_{\mathbb{Q}}) = \mathbb{A}_{\mathbb{Q}}^\times$$

Definition 3.9 (Adelic automorphic forms for $GL(1, \mathbb{A}_{\mathbb{Q}})$)**.** A smooth function

$$\phi : \mathbb{Q}^\times \backslash GL(1, \mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$$

is an adelic automorphic form with central character $\omega : \mathbb{Q}^\times \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$ if

(1) $\phi(\gamma g) = \phi(g)$, for all $\gamma \in \mathbb{Q}^\times$ and $g \in GL(1, \mathbb{A}_{\mathbb{Q}})$

(2) $\phi(zg) = \omega(z)\phi(g)$, for all $z \in$ center of $GL(1, \mathbb{A}_{\mathbb{Q}}) = \mathbb{A}_{\mathbb{Q}}^\times$

(3) ϕ is of moderate growth.

Examples Consider a Dirichlet character modular p^f for a prime p and $f \geq 1$,

$$\chi : \left(p^f\mathbb{Z}\backslash\mathbb{Z}\right)^\times \to \mathbb{C}^\times$$

such that $\chi(ab) = \chi(a)\chi(b)$ and $|\chi(a)| = 1$ for all $a \in \left(p^f\mathbb{Z}\backslash\mathbb{Z}\right)^\times$

Definition 3.10 (Adelic lift of Dirighlet series)**.** Let χ be a Dirichlet character $\bmod\ p^f$ for a prime p and $f \geq 1$. We defined an adelic lift of χ:

$$\chi_{\text{idelic}}(g) := \chi_\infty(g_\infty)\chi_2(g_2)\cdots\chi_p(g_p)\cdots$$

where

$$\chi_\infty(g_\infty) = \begin{cases} 1, & \chi(-1) = 1 \\ 1, & \chi(-1) = -1,\ g_\infty > 0 \\ -1 & \chi(-1) = -1,,\ g_\infty < 0 \end{cases}$$

and for $v < \infty$,

$$\chi_v(g_v) := \begin{cases} \chi(v)^m, & g_v \in v^m\mathbb{Z}_v^\times, & (v \neq p) \\ \chi(j)^{-1}, & g_v \in p^k\left(j + p^f\mathbb{Z}_p\right),\ j,k \in \mathbb{Z}, (j,p) = 1, & (v = p) \end{cases}$$

Then χ_{idelic} satisfies

$$\chi_{\text{idelic}}(ab) = \chi_{\text{idelic}}(a) \cdot \chi_{\text{idelic}}(b),\ \forall a, b \in \mathbb{A}_\mathbb{Q}^\times$$

$$\chi_{\text{idelic}}(ab) = \chi_{idelic}(b),\ \forall b \in \mathbb{A}_\mathbb{Q}^\times,\ \gamma \in \mathbb{Q}^\times$$

$$\chi_{\text{idelic}}(\alpha) = 1,\ \text{if } \alpha \in \mathbb{Q}^\times$$

On $GL(1, \mathbb{A}_\mathbb{Q})$ every automorphic form is a character.

Characters for $\mathbb{Q}_p^\times$ Characters $\psi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, $\psi(ab) = \psi(a)\psi(b)$.

Case 1 unramified

$$\psi(u) = 1,\ \forall u \in \mathbb{Z}_p^\times$$

Then $\psi(u)$ determines all possible character.

Case 2 ramified

$$\psi(u) \neq 1,\ \text{for some } u \in \mathbb{Z}_p^\times$$

In this case there exists an open subgroup $U \subset \mathbb{Z}_p^\times$ such that for any $u \in U$, $\psi(u) = 1$. So $\psi(p)$ and $\psi(u)$ for $u \in \mathbb{Z}_p^\times/U$ determine ψ.

3.6 Adelic lifts to $GL(n, \mathbb{A}_\mathbb{Q})$

For real place, we have generalized upper half plane

$$\mathbb{H}^n = GL(n, \mathbb{R})/O(n, \mathbb{R}) \cdot \mathbb{R}^\times.$$

Fix $r \geq 1$, irreducible representation

$$\rho : O(n, \mathbb{R}) \to GL(r, \mathbb{C})$$

Fix $N \geq 1$,

$$\Gamma_0(N) = \left\{ \begin{pmatrix} A & B \\ C & d \end{pmatrix} \middle| \begin{array}{c} A \in GL(n-1, \mathbb{Z}), \\ C = (c_1, \ldots, c_{n-1}) \equiv (0, \ldots, 0) \bmod N, d \in \mathbb{Z} \end{array} \right\}$$

Then for any $\gamma \in \Gamma_0(N)$ and $z \in \mathbb{H}^n$, there exists $\widetilde{\gamma z} \in \mathbb{H}^n$, $\kappa(\gamma, z) \in O(n,\mathbb{R})$ and $d = \begin{pmatrix} \delta & & \\ & \ddots & \\ & & \delta \end{pmatrix}$, $\delta \neq 0$ such that

$$\gamma z = \widetilde{\gamma z} \cdot \kappa(\gamma, z) \cdot d.$$

Let

$$J_\rho(\gamma, z) = \rho\left(\kappa(\gamma, z)^{-1}\right)$$

Then Vector valued automorphic function of weight ρ, level N and a character χ modular N:

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix}, \; \Phi(\gamma z) = \chi(d) J_\rho(\gamma, z)\, \Phi(z)$$

for any $\gamma = \left(\begin{smallmatrix} A & B \\ C & d \end{smallmatrix}\right) \in \Gamma_0(N),\; z \in \mathbb{H}^n$.

Atrong approximation

$$K_0(p^f) := \left\{ (g_2, \ldots, \begin{pmatrix} A_p & B_p \\ C_p & d_p \end{pmatrix}, \ldots) \;\middle|\; \begin{array}{c} g_v \in GL(n,\mathbb{Q}_v),\; v \neq p,\; A_p \in GL(n-1,\mathbb{Z}_p), \\ C_p \in (p^f\mathbb{Z}_p)^{n-1},\; d_p \in \mathbb{Z}_p \end{array} \right\}$$
$$\subseteq GL(n, \mathbb{A}_{\text{finite}})$$

Then for any $g \in GL(n, \mathbb{A}_\mathbb{Q})$,

$$g = i_{\text{diag}}(\gamma) \cdot i_\infty(g_\infty) \cdot k_{\text{finite}}$$

for some $\gamma \in GL(n,\mathbb{Q})$, $g_\infty \in GL(n,\mathbb{R})$, $k_{\text{finite}} \in K_0(p^f)$.

Definition 3.11.

$$\Phi_{\text{adelic}}(g) := \widetilde{\chi}_{\text{idelic}}(k_{\text{finite}}) \cdot \left(\Phi\,|_\rho g_\infty\right)(I_n)$$

where $\chi_{\text{idelic}}(k_{\text{finite}}) = \chi_p(d_p)$ for $k_{\text{finite}} = \left(g_2, \ldots, \left(\begin{smallmatrix} A_p & B_p \\ C_p & d_p \end{smallmatrix}\right), \ldots\right)$.

Do adelic automorphic forms exist?

- $GL(1)$ yes, because Dirichlet characters exist
- $GL(2)$ yes, because moderate forms exist
- $GL(3)$ Miller showed infinitely many automorphic forms exist for $GL(3, \mathbb{A}_\mathbb{Q})$ (K_∞-invariant, with level 1)
- $GL(n)$ Lapid-Müller got asymptotic for number of classical forms (K-inv, level N), for $n \geq 2$
- G Lindenstrauss-Venkatesh (K-invariant)

3.7 Cusp forms for $GL(n, \mathbb{A}_\mathbb{Q})$

For $n = 2$ and level $N = 2$, if f is a cusp form, then it means that

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$$

then $a_0 = 0$, and it is equivalent to say $\int_0^1 f(z+u)du = 0$. But for the general level $N > 1$, and character χ, requires more complicated definition.

$$cusps = \{\infty\} \cup \mathbb{Q}$$

for any cusp $\mathfrak{a}$, we have $\sigma \in GL(2, \mathbb{Q})$ such that $\sigma\mathfrak{a} = \infty$.Then

$$\left(f|_k \sigma^{-1}\right)(z) = \sum_{n=0}^{\infty} a_{\mathfrak{a}}(n) e^{2\pi i n(z+\mu_{\mathfrak{a}})}$$

and f is a cusp form if $a_{\mathfrak{a}}(0) = 0$ for all cusps $\mathfrak{a}$ with $\mu_{\mathfrak{a}} = 0$.

Parabolic subgroups

Definition 3.12. A subgroup of $GL(n)$, for $n = n_1 + \cdots + n_r$,

$$\begin{pmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix}, \ A_i \in GL(n_i), \text{ for } i = 1, \ldots, r$$

is called a standard parabolic subgroup.

Definition 3.13. A subgroup of $GL(n)$ of the form

$$\begin{pmatrix} A_1 & & & \\ & A_2 & 0 & \\ & & \ddots & \\ & & & A_r \end{pmatrix}, \ A_i \in GL(n_i), \text{ for } i = 1, \ldots, r, \ n = n_1 + \cdots + n_r$$

is called a standard Levi subgroup.

Definition 3.14 (Unipotent radical)**.** is a subgroup of $GL(n)$ of the form

$$\begin{pmatrix} I_{n_1} & & * & \\ & I_{n_2} & & \\ & & \ddots & \\ & & & I_{n_r} \end{pmatrix}, \ I_l = l \times l \text{ identity matrix } n_1 + n_2 + \cdots + n_r = n.$$

Definition 3.15 (Adelic automorphic cusp forms)**.** An adelic automorphic form

$$\phi : GL(n, \mathbb{Q}) \backslash GL(n, \mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$$

with central character $\omega : \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$ (i.e., $\phi(zg) = \omega(z)\phi(g), \ \forall z \in Z(GL(n, \mathbb{A}_{\mathbb{Q}})), \ g \in GL(n, \mathbb{A}_{\mathbb{Q}})$) is called a cusp form for every unipotent radical $U(\mathbb{A}_{\mathbb{Q}})$ of $GL(n, \mathbb{A}_{\mathbb{Q}})$ we have

$$\int_{U(\mathbb{Q}) \backslash U(\mathbb{A}_{\mathbb{Q}})} \phi(ug) du \equiv 0 \tag{3.5}$$

Remark. It is enough to have (3.5) had for standard unipotent.

Lecture 7

4 Automorphic Representations

4.1 Review

Let $\omega : \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{C}}^\times \to \mathbb{C}^\times$ be a character, i.e.,

- $\omega(ab) = \omega(a)\omega(b)$

- $\omega(\alpha b) = \omega(b), \ \forall b \in \mathbb{A}_{\mathbb{Q}}^{\times}, \ \alpha \in \mathbb{Q}^{\times}.$

We studied the space $A_\omega\left(G(n, \mathbb{A}_{\mathbb{Q}})\right)$ of all adelic automorphic forms

$$\phi : GL(n, \mathbb{Q})\backslash GL(n, \mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$$

with central character ω satisfying the followings:

(1) Modularity condition: $\phi(\gamma g) = \phi(g), \ \gamma \in GL(n, \mathbb{Q}), \ g \in GL(n, \mathbb{A}_{\mathbb{Q}})$

(2) $\phi(zg) = \omega(\delta)\phi(g), \ \forall z \in \mathcal{Z}\left(GL(n, \mathbb{A}_{\mathbb{Q}})\right), \ z = \begin{pmatrix} \delta & & \\ & \ddots & \\ & & \delta \end{pmatrix}, \ g \in GL(n, \mathbb{A}_{\mathbb{Q}}).$

(3) ϕ is right K-finite

$$\iff \left\{ \phi(gk) \,\middle|\, k \in K = O(n, \mathbb{R}) \cdot \prod_p GL(n, \mathbb{Z}_p) \right\} \text{ is finite dimensional}$$

(4) ϕ is $\mathcal{Z}\left(\mathcal{U}_{\mathfrak{g}}\right)$-finite

$$\iff \{ D\phi(g) \mid D \in \mathcal{Z}(\mathcal{U}_{\mathfrak{g}}) \} \text{ is finite dimensional}$$

(5) ϕ has moderate growth.

4.2 Definition for adelic automorphic representations

Rough Definition An adelic automorphic representations with central character ω is a vector space $V \subseteq A_\omega\left(GL(n, \mathbb{A}_{\mathbb{Q}})\right)$ with three actions: $\pi_{\text{finite}}, \pi_{K_\infty}, \pi_{\mathfrak{g}}$.

(1) Action of finite adeles by right translation:
Let $a_{\text{finite}} = (I_n, a_2, \ldots, a_p, \ldots)$ with $a_p \in GL(n, \mathbb{Q}_p)$ and $a_p \in GL(n, \mathbb{Z}_p)$ for almost all p. For any $\phi \in V$, define

$$\pi_{\text{finite}}(a_{\text{finite}}).\phi(g) := \phi(g \cdot a_{\text{finite}})$$

(2) Action by K_∞ by right translation:
Let $k = (k_\infty, I_n, \ldots, I_n, \ldots) \in K_\infty$ with $k_\infty \in O(n, \mathbb{R})$. For any $\phi \in V$ define

$$\pi_{K_\infty}(k).\phi(g) := \phi(g \cdot k)$$

(3) Action by differential operators:
Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. For $\alpha \in \mathfrak{g}$ and $\phi \in V$, we have

$$D_\alpha \phi(g) = \frac{d}{dt}\phi\left(g \exp(\alpha t)\right)|_{t=0}$$

Let

$$\begin{aligned} \mathcal{Z}(\mathcal{U}(\mathfrak{g})) &= \text{ center of the algebra spanned by all such } D_\alpha \\ &= \text{ generated by the Casimir differential operators} \end{aligned}$$

For any $D \in \mathcal{U}(\mathfrak{g})(?)$ define

$$\pi_{\mathfrak{g}}(D).\phi(g) = D\phi(g).$$

Remark. (1) π_{finite} commutes with π_{K_∞}, i.e.,

$$\pi_{\text{finite}} \circ \pi_{K_\infty} = \pi_{K_\infty} \circ \pi_{\text{finite}}$$

(2) $\pi_{\mathfrak{g}} \circ \pi_{\text{finite}} = \pi_{\text{finite}} \circ \pi_{\mathfrak{g}}$

(3) π_{K_∞} does not commute with $\pi_{\mathfrak{g}}$.

Lemma 4.1. *For any* $\alpha \in \mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ *and* $k = (k_\infty, I_n, \ldots) \in K_\infty$, $k_\infty \in O(n, \mathbb{R})$

$$\pi_{\mathfrak{g}}(D_\alpha) \circ \pi_{K_\infty}(k) = \pi_{K_\infty}(k) \circ \pi_{\mathfrak{g}}(D_{k^{-1}\alpha k})$$

Proof. Let $\phi \in V,\ g \in GL(n, \mathbb{A}_{\mathbb{Q}})$

$$\begin{aligned}
\pi_{\mathfrak{g}}(D_\alpha).(\pi_{K_\infty}(k)\phi(g)) &= \pi_{\mathfrak{g}}(D_\alpha)\phi(gk) = \pi_{\mathfrak{g}}(D_\alpha(\phi(g \cdot (k_\infty, I_n, \ldots)) \\
&= \frac{d}{dt}\phi(g\left(e^{\alpha t}k_\infty, I_n, \ldots\right))|_{t=0} \\
&= \frac{d}{dt}\phi\left(gk\left(k_\infty^{-1}e^{\alpha t}k_\infty, I_n, \ldots\right)\right)|_{t=0} \\
&= \pi_{K_\infty}(k).\left(\pi_{\mathfrak{g}}(k_\infty^{-1}\alpha k_\infty)\right).\phi(g)
\end{aligned}$$

□

Definition 4.2 ($(\mathfrak{g}, K_\infty)$-module, Harish-Chandra)**.** Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $K_\infty = \{k_\infty, I_n, \ldots, I_n, \ldots)|k_\infty \in O(n, \mathbb{R})\}$. Then a $(\mathfrak{g}, K_\infty)$-module is a vector space V with 2 actions.

(1) $\pi_{\mathfrak{g}} : \mathcal{U}(\mathfrak{g}) \to \text{End}(V)$

(2) $\pi_{K_\infty} : K_\infty \to GL(V)$ where for any $v \in V$, with K_∞ is finite dimensional.

Furthermore, the actions $\pi_{\mathfrak{g}},\ \pi_{K_\infty}$ satisfy the relation:

$$\pi_{\mathfrak{g}}(D_\alpha).\pi_{K_\infty}(k) = \pi_{K_\infty}(k).\pi_{\mathfrak{g}}(D_{k^{-1}\alpha k})$$

for any $k \in K_\infty,\ \alpha \in \mathfrak{g}$.
Also,

$$\pi_{\mathfrak{g}}(D_\alpha).v = \lim_{t \to 0} \frac{1}{t}\left(\pi_{K_\infty}\left(\exp(\alpha t)\right).v - v\right)$$

where $\alpha \in \mathfrak{k}$ =Lie algebra of K_∞.

Definition 4.3 ($(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$-modules)**.** Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, $K_\infty = \{(k_\infty, I_n, I_n, \ldots) \mid k_\infty \in O(n, \mathbb{R})\}$. Let $\mathbb{A}_{\text{finite}} = \{(I_n, a_2, \ldots, a_p, \ldots)\}$ be the finite adeles. A $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$ module is a vector space V with three actions:

- $\pi_{\mathfrak{g}} : \mathcal{U}(\mathfrak{g}) \to \text{End}(V)$
- $\pi_{K_\infty} : K_\infty \to GL(V)$
- $\pi_{\text{finite}} : \mathbb{A}_{\text{finite}} \to GL(V)$

In addition, we require, that V is a $(\mathfrak{g}, K_\infty)$-module with resprect to $\pi_{\mathfrak{g}},\ \pi_{K_\infty}$ and also satisfy the relations.

$$\pi_{\mathfrak{g}}.\pi_{\text{finite}} = \pi_{\text{finite}}.\pi_{\mathfrak{g}}$$

$$\pi_{K_\infty}.\pi_{\text{finite}} = \pi_{\text{finite}}.\pi_{K_\infty}$$

Notation: $\pi = (\pi_{\mathfrak{g}}, \pi_{K_\infty}, \pi_finite)$. We call (π, V) a $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$ module.

Definition 4.4 (Smooth $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$-module)**.** Let (π, V) with $\pi = (\pi_{\mathfrak{g}}, \pi_{K_\infty}, \pi_{\text{finite}})$ be a $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$-module. Then (π, V) is smooth if every $v \in V$ is fixed by some open compact subgroup of $GL(n, \mathbb{A}_{\text{finite}})$ i.e., there exists $H \subset GL(n, \mathbb{A}_{\text{finite}})$, such that $\pi(h).v = v$ for any $h \in H$.

Exercise: Prove that $A_\omega(GL(n, \mathbb{A}_{\mathbb{Q}}))$=space of all adelic automorphic forms with central character ω is a smooth $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$-module.

Intertwining Maps Let (π, V), (π', V') be representations of G. An intertwining map is a linear map $L : V \to V'$ such that

$$L(\pi(g).v) = \pi'(g).L(v)$$

for any $g \in G$ and $v \in V$. If $L =$ isomorphic (bijective map) then we say $(\pi, V) \cong (\pi', V')$.

Subquotient of a $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$-module Let (π, V) with $\pi = (\pi_{\mathfrak{g}}, \pi_{K_\infty}, \pi_{\text{finite}})$ be a $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$-module. Assume there exist $W' \subset W \subset V$ where W, W' are both invariant under the action π. Then W/W' (vector space quotient) is again a $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$-module.
Let $w + W' \in W/W'$ for $w \in W$. Just define,

$$\pi_{\mathfrak{g}}(D).(w + W') := \pi_{\mathfrak{g}}(D).w + W'$$

$$\pi_{K_\infty}(k).(w + W') := \pi_{K_\infty}(k).w + W'$$

$$\pi_{\text{finite}}(a_{\text{finite}}).(w + W') := \pi_{\text{finite}}(a_{\text{finite}}).w + W'$$

Definition 4.5 (Adelic automorphic representation of $GL(n, \mathbb{A}_{\mathbb{Q}})$)**.** An adelic automorphic representation with central character ω is a $(\mathfrak{g}, K_\infty) \times GL(n, \mathbb{A}_{\text{finite}})$-module, whose vector space is a subquotient of the space of adelic automorphic forms $A_\omega\left(GL(n, \mathbb{A}_{\mathbb{Q}})\right)$.

Lecture 8

Last time Global Automorphic representations for $GL(n, \mathbb{A}_{\mathbb{Q}})$ is a vector space V of adelic automorphic forms with three actions: $\pi = \left(\underbrace{\underbrace{\pi_{\mathfrak{g}}}_{\substack{\text{action by differential}\\ \text{operators at } \infty}}, \underbrace{\pi_{K_\infty}}_{\substack{\text{action by right}\\ \text{translation of } K_\infty}}}_{(\mathfrak{g}, K_\infty)}, \underbrace{\pi_{\text{finite}}}_{\substack{\text{action by right}\\ \text{translation of finite adeles}}} \right)$.

4.3 Examples of Automorphic representations

(1) $A_\omega\left(GL(n, \mathbb{A}_{\mathbb{Q}})\right) =$ vector space of all adelic automorphic forms with central character $\omega : \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$. Not irreducible!
This space decomposes into cuspidal automorphic representations and Eisenstein series.

(2) Let $f(z)$ be a classical automorphic form on the upper half plane $\mathbb{H}^n$. Let f_{adelic} be its adelic lift, i.e., $f_{\text{adelic}} \in A_\omega\left(GL(n, \mathbb{A}_{\mathbb{Q}})\right)$ for some central character ω.

Studying with the classical automorphic form f, we will construct a vector space $V_f \subset A_\omega\left(GL(n, \mathbb{A}_{\mathbb{Q}})\right)$ and three actions to get an automorphic representation.

$$V_f := \text{Span}\left\{ \sum_{D \in \mathcal{U}(\mathfrak{g})} \sum_{k_\infty \in K_\infty} \sum_{a_{\text{finite}} \in GL(n, \mathbb{A}_{\text{finite}})} \pi_{\mathfrak{g}}(D).\pi_{K_\infty}(k_\infty).f_{\text{adelic}}(g) \right\}$$

If f is a newform, then (π, V_f) will be irreducible.

For $n = 2$, let

$$f(z) = \mathrm{Im}(z)^6 \Delta(z) = \sum_{n=1}^{\infty} \tau(n) y^6 e^{2\pi i n z}$$
$$= \mathrm{Im}(z)^6 e^{2\pi i z} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right)^{24}$$

then for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$,

$$f\left(\frac{az+b}{cz+d}\right) = \left(\frac{cz+d}{|cz+d|}\right)^{12} f(z).$$

What is the Ramanujan automorphic representation (π, V_Δ)?

- $\mathcal{U}(\mathfrak{g}) =$ Maass type Differential operators
- $\pi_{\text{finite}}\left(I_2, \ldots, \left(\begin{smallmatrix} p^{-e} & 0 \\ 0 & 1 \end{smallmatrix}\right), I_2, \ldots\right).\Delta_{\text{adelic}}(z) = \Delta(p^e z)_{\text{adelic lift}}$

So, V_Δ will be spannce by all holomorphic modular forms (old forms) of weights $12, 14, 16, 18, 20, 22, \ldots$.

4.4 Cuspidal adelic automorphic representations

Let

$$U := \left\{ \begin{pmatrix} I_{r_1} & & * \\ & \ddots & \\ & & I_{r_k} \end{pmatrix},\ r_1 + \ldots + r_k = n \right\}.$$

If $f \in A_\omega(GL(n, \mathbb{A}_\mathbb{Q}))$ i.e., the adelic automorphic form with the central character ω. Then f is cuspidal if

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A}_\mathbb{Q})} f(ug)du = 0$$

for any U.

Definition 4.6. An automorphic representation is cuspidal if its space V is a subquotient of the space of all adelic cusp forms.

4.5 Fourier expansion of adelic automorphic cusp forms

Classical For $f(x+1) = f(x)$

$$f(x) = \sum_{n\in\mathbb{Z}} \int_0^1 f(u+x)e^{-2\pi i n u} du$$

Theorem 4.7 (Piatetski-Shapiro, Shalika independently)**.** *Let ϕ be an adelic automorphic cusp form for $GL(n, \mathbb{A}_\mathbb{Q})$. Then for any $g \in GL(n, \mathbb{A}_\mathbb{Q})$,*

$$\phi(g) = \sum_{\gamma \in U_{n-1}(\mathbb{Q})\backslash GL(n-1,\mathbb{Q})} W_\phi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \cdot g\right)$$

where

$$W_\phi(g) := \int_{U_n(\mathbb{Q})\backslash U_n(\mathbb{A}_\mathbb{Q})} \phi(ug)\psi(u)^{-1} du,$$

$U_n = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$ *and ψ is the additive character.*

Exponential of $\mathbb{A}_{\mathbb{Q}}$ For $a=(a_\infty,a_2,\ldots,a_p,\ldots)\in\mathbb{A}_{\mathbb{Q}}$, define

$$\mathrm{Exp}(a):=e^{2\pi i a_\infty}\prod_p e^{-2\pi i\{a_p\}}.$$

For $a_p\in\mathbb{Q}_p$, then

$$a_p=\alpha_{-l}p^{-l}+\alpha_{-l+1}p^{-l+1}+\cdots+\alpha_{-1}p^{-1}+\alpha_0+\alpha_1 p+\alpha_2 p^2+\cdots$$

and

$$\{a_p\}:=\alpha_{-l}p^{-l}+\cdots+\alpha_{-1}p^{-1}.$$

Periodic: $f(\alpha+x)=f(x)$ for any $\alpha\in\mathbb{Q}$,

$$f(x)=\sum_{\alpha\in\mathbb{Q}}\int_{\mathbb{Q}\backslash\mathbb{A}}f(u+x)\overline{\mathrm{Exp}(\alpha u)}du.$$

Definition of the character ψ which is a generalization of $e^{2\pi i x}$ Let $u=\begin{pmatrix}1 & u_1 & & * & \\ & 1 & u_2 & & \\ & & \ddots & \ddots & \\ & & & & u_{n-1}\\ & & & & 1\end{pmatrix}\in$ $U_n(\mathbb{A}_{\mathbb{Q}})$ (super diagonal).

Definition 4.8.

$$\psi(u)=\mathrm{Exp}(u_1)\cdot\mathrm{Exp}(u_2)\cdots\mathrm{Exp}(u_{n-1})$$

Lemma 4.9.

$$\psi(u\cdot u')=\psi(u)\cdot\psi(u')$$

Example for $n=2$ The Fourier expansion of $GL(2)$ takes the most familiar form

$$\phi(g)=\sum_{\gamma\in\mathbb{Q}^\times}W_\phi\left(\left(\begin{smallmatrix}\gamma & 0\\ 0 & 1\end{smallmatrix}\right)\cdot g\right)$$

Whittaker function W_ϕ is called a (global) Whittaker function. It is characterized by the following properties.

(1) $W_\phi(u\cdot g)=\psi(u)W_\phi(g),\ \forall u\in U_n(\mathbb{A}),\ g\in GL(n,\mathbb{A})$

(2) $W_\phi(g)$ is smooth and of moderate growth.

(3) $\{DW_\phi(g)\,|D\in\mathcal{Z}(\mathcal{U}(\mathfrak{g}))\}$ is finite dimensional.

(4) $\{W_\phi(gk)\,|k\in K\}$ is finite dimensional.

4.6 Whittaker Model

If you have two isomorphic representations

$$(\pi,V)\cong(\pi',V')$$

then we say V' is a model for V

$$\iff\quad\text{isomorphism } L:V\to V'$$

intertwining, i.e., $L(\pi(g).v)=\pi'(g).L(v)$.

Let (π, V) be an adelic automorphic representation of $GL(n, \mathbb{A}_{\mathbb{Q}})$, and $V = \{f : GL(n, \mathbb{Q})\backslash GL(n, \mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}\}$. Let

$$\begin{aligned} L : V \hookrightarrow \mathcal{W} = & \text{ Whittaker space} \\ = & \text{ space of all Whittaker functions} \end{aligned}$$

with

$$L(f(g)) := W_f(g) = \int_{U_n(\mathbb{Q})\backslash U_n(\mathbb{A})} f(ug)\text{Exp}(-u)du.$$

$L : V \to$ spanned by all linear combinations of $W_f(g)$ with $f \in V =: W_V$, then

$$(\pi, V) \cong (\pi, W_V)$$

and W_V is called a Whittaker model for (π, V).

Lecture 9: 2010-2-16

4.7 L-functions associated to an irreducible adelic cuspidal automorphic representations (π, V) of $GL(n, \mathbb{A}_{\mathbb{Q}})$

Folklore conjecture Every L-function which has an Euler product

$$L(s) = \prod_p \prod_{i=1}^n \left(1 - \alpha_{p,i} p^{-s}\right)^{-1}$$

and has holomorphic continuation in $s \in \mathbb{C}$ and satisfies a functional equation

$$G(s)L(s) = \widetilde{G}(1-s)\widetilde{L}(1-s)$$

for $G(s)$ is the product of Gamma functions, is of the above type: associated to an irreducible adelic automorphic representations (π, V) of $GL(n, \mathbb{A}_{\mathbb{Q}})$

L-functions discovered:

- $\zeta(s)$
- $L(s, \chi)$, Dirichlet
- $L(s, \psi)$, Hecke of number fields
- $L(s, A)$, Hasse-Weil L-function associated to an algebraic variety A. (NOT proved)
- Artin L-functions, etc (NOT proved)

Classically, people attached L-function to automorphic forms. For $GL(2)$, we have

$$\phi(g) = \sum_{\gamma \in \mathbb{Q}^\times} W_\phi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right)$$

then

$$L_\phi(s) = \int_0^\infty \phi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) y^s \frac{dy}{y}.$$

But for $n > 2$, this integral doesn't work. For $GL(n)$,

$$\phi(g) = \sum_{\gamma \in U_{n-1}(\mathbb{Q})\backslash GL(n-1,\mathbb{Q})} W_\phi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right).$$

For example, for $n = 3$, let

$$\int_0^\infty \int_0^\infty \phi\left(\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}\right)\left(\det\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}\right)^s \frac{dy_1}{y_1}\frac{dy_2}{y_2} \neq L_\phi(s).$$

There are several ways to defined an L-function. We can use the method used in (Automorphic ..., by Goldfeld), Rankin-Selberg method, or using representations.

History

(1) K. Hey in 1929 and Eichler in 1938, defined zeta function on $GL(n)$

(2) In 1950, Tate-Iwasawa constructed all L-functions for $GL(1)$

(3) In 1958, Godement suggested a generalization of Tate-Iwasawa to $GL(n)$

(4) In 1963, Tamagaa worked out Godement suggestion for abelian ... found Euler product

(5) In 1972, Godement-Jacquet use matrix coefficients of automorphic representations.

Definition 4.10. Let $R =$ ring. Let $n \geq 1$.

$$M(n, R) := \text{ ring of } n \times n \text{ matrices with coefficients in } R.$$

Definition 4.11 (Schwartz-Bruhat function on $M(n, \mathbb{A}_\mathbb{Q})$)**.**

$$\phi : M(n, \mathbb{A}_\mathbb{Q}) \to \mathbb{C}$$

which is a finite sum of products

$$\prod_{v \leq \infty} \phi_v(m_v)$$

where $m = (m_\infty, m_2, \ldots, m_p, \ldots) \in \mathbb{A}_\mathbb{Q}$ which satisfies the followings

- $\phi_\infty(m_\infty)$ is smooth and has rapid decay at ∞. i.e., for $m_\infty = \begin{pmatrix} m_\infty^{1,1} & ... & m_\infty^{1,n} \\ \vdots & ... & \vdots \\ m_\infty^{n,1} & ... & m_\infty^{n,n}\end{pmatrix}$ then $\phi_\infty(m_\infty)$ is smooth in all the variables $m_\infty^{i,j}$.
- $\phi_p(m_p)$ is locally constant, compactly supported
- $\phi_p(m_p)$ is the characteristic function of $M(n, \mathbb{Z}_p)$ for almost all p

Definition 4.12 (Global matrix coefficients for (π, V))**.** Let $f_1, f_2 \in V$. A matrix coefficient associated to f_1, f_2,

$$\omega : GL(n, \mathbb{A}_\mathbb{Q}) \to \mathbb{C}$$

given by

$$\omega(g) := \int_{\mathcal{Z}(GL(n,\mathbb{A}_\mathbb{Q}))GL(n,\mathbb{A}_\mathbb{Q})\backslash GL(n,\mathbb{A}_\mathbb{Q})} f_1(hg)\overline{f_2(h)}dh \tag{4.1}$$

for any $g \in GL(n, \mathbb{A}_\mathbb{Q})$.

Remark. (i) Since (π, V) is irreducible, then all matrix coefficients for any $f_1, f_2 \in V$ are the same up to a constant factor. (Not completely trivial to prove).

(ii) For $g = I_n$, then $\omega(I_n) =$Pettersson inner product.

Definition 4.13 (L-function associated to (π, V))**.** Let $\phi : M(n, \mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$ be a Schwartz-Bruhat function. Let $s \in \mathbb{C}$ with $\Re(s) \gg 1$. Let $\omega : GL(n, \mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$ as in (4.1). Then

$$Z(s, \phi, \omega) := \int_{GL(n,\mathbb{A}_{\mathbb{Q}})} \omega(g)\phi(g) \left|\det g\right|^{s+\frac{1}{2}} d^{\times} g = L(s, \pi) \cdot G(\pi, s).$$

Theorem 4.14 (Godement-Jacquet)**.** *The zeta function $Z(s, \phi, \omega)$ has holomorphic continuation to all $s \in \mathbb{C}$ and satisfies a functional equation*

$$Z(s, \phi, \omega) = Z(1 - s, \widehat{\phi}, \check{\omega})$$

where $\widehat{\phi}$ = Fourier transform of ϕ and $\check{\omega}(g) := \omega(g^{-1})$.

Proof. Poisson summation formula □

Tensor product theorem

$$\pi = \otimes'_{v\leq\infty} \pi_v$$
$$\Rightarrow \ \omega(g) = \prod_{v\leq\infty} \omega_v(g_v)$$

where $\omega_v : GL(\mathbb{Q}_v) \to \mathbb{C}$ are local matrix coefficients.

Local L-functions

$$Z(s, \phi_v, \omega_v) := \int_{GL(\mathbb{Q}_v)} \omega_v(g)\phi_v(g) \left|\det(g)\right|^{s+\frac{1}{2}} ds$$

$$Z(s, \phi, \omega) = \prod_{v\leq\infty} Z(s, \phi_v, \omega_v) \Rightarrow \ \text{Euler product}$$

5 Local Theory: Representations of $GL(n, \mathbb{Q}_p)$

5.1 Representations for $GL(n, \mathbb{Q}_p)$

Let $r \geq 0$ and

$$K_r := \{k \in GL(n, \mathbb{Z}_p) \mid k - I_n \in p^r M(n, \mathbb{Z}_p)\}.$$

(open compact nbhd for I_n?)

Definition 5.1. Let $\pi : GL(n, \mathbb{Q}_p) \to GL(V)$ be a representation for $V = \infty$-dimensional vector space. We say (π, V) is smooth if the map $g \mapsto \pi(g).v$ for $g \in GL(n, \mathbb{Q}_p)$ and $v \in V$ is locally constant.

Definition 5.2. (π, V) is admissible if

$$V^{K_r} = \{v \in V \mid \pi(k).v = v, \ k \in K_r\}$$

is finite dimensional for any K_r with $r \geq 0$.

We want to classify smooth, admissible and irreducible representation of $GL(n, \mathbb{Q}_p)$.

Lemma 5.3 (Dixmier-Schur)**.** *Let (π, V) be a smooth and irreducible representation of $GL(n, \mathbb{Q}_p)$. Let $T : V \to V$ be a linear map, satisfying*

$$T(\pi(g).v) = \pi(g).T(v)$$

for any $g \in GL(n, \mathbb{Q}_p)$, $v \in V$ (Intertwining map). Then there exists $c \in \mathbb{C}$ such that $Tv = cv$ for any $v \in V$.

Proof. Fix $v\in V$. Claim the vector space is spanned by

$$\{\pi(g).v \mid g\in GL(n,\mathbb{Q}_p)\}$$

is countable.
For fixed $g\in GL(n,\mathbb{Q}_p)$ the space

$$\{\pi(gk).v\,|k\in GL(n,\mathbb{Z}_p)\}$$

is finite because π is smooth so there exists K_r for some $r\geq 0$ such that $\pi(gk).v=\pi(g).v$ for any $k\in K_r$. Also by the Iwasawa decomposition, $\mathbb{Q}_p^{\times}\backslash GL(n,\mathbb{Q}_p)/GL(n,\mathbb{Z}_p)$ is countable. So this claim is proved.

Claim 2: Let $c\in\mathbb{C}$, $I.v=v$, identity transform. Then $(T-cI).\pi=\pi.(T-cI)$ (commuting operators).

Assume that $T-cI\neq 0$, for all $c\in\mathbb{C}$ then $(T-cI)^{-1}:V\to V$ exists and is a bijective linear map. for any fixed $v\in V$,

$$V:=\text{Span}\{\pi(g).v\mid g\in GL(n,\mathbb{Q}_p)\}$$

by irreducibility. Fix $v\in V$, consider

$$\left\{(T-cI)^{-1}.v\mid c\in C\right\}=\text{unconutable set of vectors }\subseteq V.$$

Then there exists a linear relation

$$\sum_{i=1}^{r}a_i\,(T-c_iI)^{-1}.v=0,\ \text{for some } a_i,c_i\in\mathbb{C}. \tag{5.1}$$

Now write the algebraic relation

$$\sum_{i=1}^{l}\frac{a_i}{X-c_i}=\frac{Q(x)}{\prod_{i=1}^{l}(X-c_i)},\ Q(X)=\sum_{i=1}^{d}b_iX^i$$

$$\Rightarrow\ Q(x)=\left(\sum_{i=1}^{l}\frac{a_i}{X-c_i}\right)\cdot\prod_{i=1}^{l}(X-c_i)$$

Define an operator $V\to V$ as

$$Q(T):=\sum_{i=1}^{d}b_i\underbrace{T\circ\cdots\circ T}_{i\text{-times}}$$

Multiplying (5.1) on the left by $\underbrace{T\circ\cdots\circ T}_{l\text{-times}}$ then

$$\Rightarrow\ Q(T)=0$$

But we can factor $Q(X)=\alpha\prod_{i=1}^{d}(X-\alpha_i)$ then $(T-\alpha_i)^{-1}$ does not exist for some α_i. Contradiction. So the assumption $T-cI\neq 0$ for any $c\in\mathbb{C}$ is false. □

Lecture 10: 2010-2-18

Reference notes: Prasad, Raghuram, [Representation theory of $GL(n)$ over non-archimidean local fields] in cauchy.math.okstate.edu/araghur

Last time $\pi : GL(n, \mathbb{Q}_p) \to GL(V)$ for some complex vector space V. π is smooth $\Rightarrow$ $g \mapsto \pi(g).v$ is locally constatn for any fixed $v \in V$.
Dixmier-Schur Lemma: If $T : V \to V$, $T\pi(g).v = \pi(g).Tv$ and (π, V) is irreducible then T acts by scalars. i.e., $Tv = cv$ for some $c \in \mathbb{C}$ and all $v \in V$.

Corollary 5.4. *Let (π, V) be a smooth irreducible representation of $GL(n, \mathbb{Q}_p)$. The there exists a character $\omega_\pi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ such that*

$$\pi(aI_n).v = \omega_\pi(a)v$$

for any $a \in \mathbb{Q}_p^\times$, $v \in V$ and $I_n = n \times n$ identity matrix and $aI_n \in \mathcal{Z}(GL(n, \mathbb{Q}_p))$. The character ω_π is called the central character associated to π.

Proof. Let $z \in \mathcal{Z}(GL(n, \mathbb{Q}_p)) \Rightarrow \pi(zg) = \pi(z)\pi(g) = \pi(g)\pi(z)$, for any $g \in GL(n, \mathbb{Q}_p)$. For every fixed $z \in \mathcal{Z}(GL(n, \mathbb{Q}_p))$ it is clear that $\pi(z)$ is an intertwining operator. So $\pi(z) : V \to V$ and it commutes with everything else in $GL(n, \mathbb{Q}_p)$. So by Dixmier-Schur's lemma, $\pi(z)$ acts by a scalar $\omega_\pi(z)$ for each $z \in \mathcal{Z}(GL(n, \mathbb{Q}_p))$. Since π is smooth and has multiplicative, ω_π is a character. □

Let (π, V) be an irreducible smooth representation of $GL(n, \mathbb{Q}_p)$. Then there are two important cases.

(1) $\dim(V) < \infty$: It is not so interesting. Because we will prove that $\dim(V) = 1$ and every representation is a character.

(2) $\dim(V) = \infty$

Proposition 5.5. *Let (π, V) be an irreducible and smooth representation of $GL(n, \mathbb{Q}_p)$. If $\dim V < \infty$ then $\dim V = 1$ and every representation is a character.*

Proof. (for $n = 2$) Let $x \in \mathbb{Q}_p$. Then since (π, V) is smooth then there exists $b \in \mathbb{Q}_p$ such that $|xb|_p$ is sufficiently small. So that $\left(\begin{smallmatrix} 1 & xb \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Ker}(\pi)$, i.e., $\pi\left(\left(\begin{smallmatrix} 1 & xb \\ 0 & 1 \end{smallmatrix}\right)\right).v = v$. (using $\dim V < \infty$)

$$\Rightarrow \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} b & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 1 & xb \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} b & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Ker}(\pi)$$

$(\because \pi\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right).v = \pi\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)^{-1} \pi\left(\begin{smallmatrix} 1 & xb \\ 0 & 1 \end{smallmatrix}\right).\left(\pi\left(\begin{smallmatrix} b & 0 \\ 0 & 1 \end{smallmatrix}\right).v\right) = v)$
Similarly, we can show

$$\left(\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right) \in \mathrm{Ker}(\pi), \ \forall y \in \mathbb{Q}_p.$$

But the matrices $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right)$ for $x, y \in \mathbb{Q}_p$ generate $SL(2, \mathbb{Q}_p)$.

$$\Rightarrow SL(2, \mathbb{Q}_p) \subset \mathrm{Ker}(\pi)$$

But $\forall g_1, g_2 \in GL(2, \mathbb{Q}_p)$ the matrices $g_1 g_2 g_1^{-1} g_2^{-1} \in SL(2, \mathbb{Q}_p)$.

$$\Rightarrow \pi\left(g_1 g_2 g_1^{-1} g_2^{-1}\right).v = v$$

$$\Rightarrow \pi(g_1).(\pi(g_2).v) = \pi(g_2).(\pi(g_1).v), \ \forall g_1, g_2 \in GL(2, \mathbb{Q}_p)$$

$$\Rightarrow \text{every } \pi(g) \text{ acts by a scalar and } \dim(V) = 1$$

□

From now on we will only consider (π, V) with $\dim V = \infty$.

5.2 Contragredient representations

Let (π, V) be a smooth irreducible representation of $GL(n,\mathbb{Q}_p)$. We will now define another representation, denoted by $(\widetilde{\pi}, \widetilde{V})$ of $GL(n,\mathbb{Q}_p)$ called the contragredient representation.

- $\widetilde{V}$ = subspace of $\widehat{V} = \{l : V \to \mathbb{C},\ l = \text{ linear map}\}$, such that for any $l \in \widetilde{V}$ and for any fixed $v \in V$, $l\left(\pi(k).v\right) = l(v)$ for any $k \in$ compact open subset of $GL(n,\mathbb{Z}_p)$ depends on v.
- We define an action $\widetilde{\pi}$ on $\widetilde{V}$ as follows: for $l \in \widetilde{V}$, $v \in V$ and $g \in GL(n,\mathbb{Q}_p)$,

$$\widetilde{\pi}(g).l(v) := l\left(\pi(g)^{-1}.v\right).$$

Once we have defined the contragredient representations, we can construct a bilinear pairing $\langle\ ,\ \rangle : V \times \widetilde{V} \to \mathbb{C}$ as follows:

Definition 5.6. For any $v \in V$ and $l \in \widetilde{V}$, then

$$\langle v, l\rangle := l(v).$$

The pairing $\langle\ ,\ \rangle : V \times \widetilde{V} \to \mathbb{C}$ is invariant under the action of $GL(n,\mathbb{Q}_p)$ in the sense that

$$\langle \pi(g).v, \widetilde{\pi}(g).l\rangle = \langle v, l\rangle$$

for any $g \in GL(n,\mathbb{Q}_p)$, $v \in V$ and $l \in \widetilde{V}$.

$$\left(\because\ \langle \pi(g).v, \widetilde{\pi}(g).l\rangle = \widetilde{\pi}(g).l\left(\pi(g).v\right) = l\left(\pi(g)^{-1}\pi(g).v\right) = l(v)\right)$$

Matrix Coefficient for smooth irreducible representations for $GL(n,\mathbb{Q}_p)$ Fix $v \in V$ and $\widetilde{v} \in \widetilde{V}$. Then the map

$$g \mapsto \langle \pi(g).v, \widetilde{v}\rangle,\ \forall g \in GL(n,\mathbb{Q}_p)$$

is called a matrix coefficient.

Fix a smooth infinite dimensional irreducible representation (π, V) of $GL(n,\mathbb{Q}_p)$ with contragredient $(\widetilde{\pi}, \widetilde{V})$. Let $\langle\ ,\ \rangle : V \times \widetilde{V} \to \mathbb{C}$ be the canonical invariant bilinear pairing.

Definition 5.7 (Local L-function associated to π)**.** Fix $v \in V$, $\widetilde{v} \in \widetilde{V}$.

$$Z_\pi\left(s, \phi\right) = \int_{GL(n,\mathbb{Q}_p)} \phi(g)\,\langle \pi(g).v, \widetilde{v}\rangle\,|\det g|^{s+\frac{1}{2}}\,dg$$

for ϕ =locally constant compactly supported function $\pi : GL(n,\mathbb{Q}_p) \to \mathbb{C}$. Define

$$\begin{aligned} L_\pi(s) := &\ \text{greatest common division or all} \\ &\ \{Z_\pi(s,\phi)\,|\phi \text{ locally constant compactly supported}\} \end{aligned}$$

(Godement-Jacquet)

Question:

(1) How to prove the integral $Z_\pi(s,\phi)$ converges? Need to growth of matrix coefficients.

(2) Can we classify smooth irreducible representation of $GL(n,\mathbb{Q}_p)$? $\Rightarrow$ There will be only finitely many possibilities.

(3) $\langle\ ,\ \rangle_{\text{global}} = \prod_v \langle\ ,\ \rangle_v$

For $GL(2, \mathbb{Q}_p)$ we will only the following types of $L_\pi(s)$:

- $(1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1}$
- $(1-\alpha_p p^{-s})$
- $1 \iff$ supercuspidal representations

and α_p, β_p are called Langlands parameters. For $GL(3, \mathbb{Q}_p)$, we have

- $(1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1}(1-\gamma_p p^{-s})^{-1}$
- $(1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1}$
- $(1-\alpha_p p^{-s})$
- 1