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Second moments of GL_2 automorphic L-functions

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Abstract.

1. Introduction

In 1801, in the Disquisitiones Arithmeticae [**Gau01**], Gauss introduced the class number $h(d)$ as the number of inequivalent binary quadratic forms of discriminant d. Gauss conjectured that the average value of $h(d)$ is $\frac{2\pi}{7\zeta(3)}\sqrt{|d|}$ for negative discriminants d. This conjecture was first proved by I. M. Vinogradov [**Vin18**] in 1918. Remarkably, Gauss also made a similar conjecture for the average value of $h(d)\log(\epsilon_d)$, where d ranges over positive discriminants and ϵ_d is the fundamental unit of the real quadratic field $Q(\sqrt{d})$. Of course, Gauss did not know what a fundamental unit of a real quadratic field was, but he gave the definition that $\epsilon_d = \frac{t+u\sqrt{d}}{2}$, where t, u are the smallest positive integral solutions to Pell's equation $t^2 - du^2 = 4$. For example, he conjectured that

$$d \equiv 0 \pmod 4 \rightarrow \sum_{d\le x} h(d)\log(\epsilon_d) \ \sim\ \frac{4\pi^2}{21\zeta(3)}x^{\frac{3}{2}},$$

while

$$d \equiv 1 \pmod 4 \rightarrow \sum_{d\le x} h(d)\log(\epsilon_d) \ \sim\ \frac{\pi^2}{18\zeta(3)}x^{\frac{3}{2}}.$$

These latter conjectures were first proved by C. L. Siegel [**Sar94**] in 1944.

In 1831, Dirichlet introduced his famous L–functions

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where χ is a character $\pmod q$ and $\Re(s) > 1$. The study of moments

$$\sum_q L(s,\chi_q)^m,$$

say, where χ_q is the real character associated to a quadratic field $Q(\sqrt{q})$, was not achieved until modern times. In the special case when $s = 1$ and $m = 1$, the value of the first moment reduces to the aforementioned conjecture of Gauss because of the Dirichlet class number formula (see [**Dav00**], pp. 43-53) which relates the special value of the L–function $L(1,\chi_q)$ with the class number and fundamental unit of the quadratic field $Q(\sqrt{q})$.

Let

$$L(s)=\sum_{n=1}^{\infty} a(n)n^{-s}$$

be the L–function associated to a modular form for the modular group. The main focus of this paper is to obtain meromorphic continuation and growth estimates in the complex variable w of the Dirichlet series

$$\int_1^{\infty}\left|L\left(\frac{1}{2}+it\right)\right|^k t^{-w}.$$

We shall show, by a new method, that it is possible to obtain meromorphic continuation and rather strong growth estimates of the above Dirichlet series for the case $k=2$. It is then possible, by standard methods, to obtain asymptotics, as $T\to\infty$, for the second integral moment

$$\int_0^T |L(\sigma+it)|^2\,dt.$$

In the special case that the modular form is an Eisenstein series this yields asymptotics for the fourth moment of the Riemann zeta-function.

Moment problems associated to the Riemann zeta-function $\zeta(s)=\sum\limits_{n=1}^{\infty} n^{-s}$ were intensively studied in the beginning of the last century. In 1918, Hardy and Littlewood [**HL18**] obtained the second moment

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2\,dt \;\sim\; T\log T,$$

and in 1926, Ingham [**Ing26**], obtained the fourth moment

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4\,dt \;\sim\; \frac{1}{2\pi^2}\cdot T(\log T)^4.$$

Heath-Brown (1979) [**HB81**] obtained the fourth moment with error term of the form

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4\,dt \;\sim\; \frac{1}{2\pi^2}\cdot T\cdot P_4(\log T) \;+\; \mathcal{O}\left(T^{\frac{7}{8}+\epsilon}\right),$$

where $P_4(x)$ is a certain polynomial of degree four.

Let $f(z)=\sum\limits_{n=1}^{\infty} a(n)e^{2\pi inz}$ be a cusp form of weight κ for the modular group with associated L–function $L_f(s)=\sum\limits_{n=1}^{\infty} a(n)n^{-s}$. Anton Good [**Goo82**] made a significant breakthrough in 1982 when he proved that

$$\int_0^T \left|L_f\left(\frac{\kappa}{2}+it\right)\right|^2\,dt \;=\; 2aT(\log(T)+b)+\mathcal{O}\left((T\log T)^{\frac{2}{3}}\right)$$

for certain constants a,b. It seems likely that Good's method can apply to Eisenstein series.

In 1989, Zavorotny [**Zav89**], improved Heath-Brown's 1979 error term to

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4\,dt \;\sim\; \frac{1}{2\pi^2}\cdot T\cdot P_4(\log T) \;+\; \mathcal{O}\left(T^{\frac{2}{3}+\epsilon}\right).$$

Shortly afterwards, Motohashi [**Mot92**], [**Mot93**] slightly improved the above error term to

$$\mathcal{O}\left(T^{\frac{2}{3}}(\log T)^B\right)$$

for some constant $B > 0$. Motohashi introduced the double Dirichlet series [**Mot95**], [**Mot97**]

$$\int_1^\infty \zeta(s+it)^2\zeta(s-it)^2 t^{-w}\, dt$$

into the picture and gave a spectral interpretation to the moment problem.

Unfortunately, an old paper of Anton Good [**Goo86**], going back to 1985, which had much earlier outlined an alternative approach to the second moment problem for $GL(2)$ automorphic forms using Poincaré series has been largely forgotten. Using Good's approach, it is possible to recover the aforementioned results of Zavorotny and Motohashi. It is also possible to generalize this method to more general situations, for instance see [**DG**], where the case of $GL(2)$ automorphic forms over an imaginary quadratic field is considered. Our aim here is to explore Good's method and show that it is, in fact, an exceptionally powerful tool for the study of moment problems.

Second moments of $GL(2)$ Maass forms were investigated in [**Jut97**], [**Jut05**]. Higher moments of L–functions associated to automorphic forms seem out of reach at present. Even the conjectured values of such moments were not obtained until fairly recently (see [**CF00**], [**CG01**], [**CFK$^+$**], [**CG84**], [**DGH03**], [**KS99**], [**KS00**]).

Let $\mathcal{H}$ denote the upper half-plane. A complex valued function f defined on $\mathcal{H}$ is called an automorphic form for $\Gamma = SL_2(\mathbb{Z})$, if it satisfies the following properties:

(1) We have

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^\kappa f(z) \qquad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma;$$

(2) $f(iy) = \mathcal{O}(y^\alpha)$ for some α, as $y \to \infty$;

(3) κ is either an even positive integer and f is holomorphic, or $\kappa = 0$, in which case, f is an eigenfunction of the non-euclidean Laplacian $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ $(z = x+iy \in \mathcal{H})$ with eigenvalue λ. In the first case, we call f a modular form of weight κ, and in the second, we call f a Maass form with eigenvalue λ.

In addition, if f satisfies

$$\int_0^1 f(x+iy)\, dx = 0,$$

then it is called a cusp form.

Let f and g be two cusp forms for Γ of the same weight κ (for Maass forms we take $\kappa = 0$) with Fourier expansions

$$f(z) = \sum_{m\neq 0} a_m\, |m|^{\frac{\kappa-1}{2}}\, W(mz),\ g(z) = \sum_{n\neq 0} b_n\, |n|^{\frac{\kappa-1}{2}}\, W(nz)\ (z = x+iy,\ y > 0).$$

Here, if f, for example, is a modular form, $W(z) = e^{2\pi iz}$, and the sum is restricted to $m \geq 1$, while if f is a Maass form with eigenvalue $\lambda_1 = \frac{1}{4} + r_1^2$,

$$W(z) \; = \; W_{\frac{1}{2}+ir_1}(z) \; = \; y^{\frac{1}{2}} K_{ir_1}(2\pi y) e^{2\pi ix} \qquad (z = x + iy, \; y > 0),$$

where $K_\nu(y)$ is the K–Bessel function. Throughout, we shall assume that both f and g are eigenfunctions of the Hecke operators, normalized so that the first Fourier coefficients $a_1 = b_1 = 1$. Furthermore, if f and g are Maass cusp forms, we shall assume them to be even.

Associated to f and g, we have the L–functions:

$$L_f(s) \; = \; \sum_{m=1}^{\infty} a_m m^{-s}; \qquad L_g(s) \; = \; \sum_{n=1}^{\infty} b_n n^{-s}.$$

In [**Goo86**], Anton Good found a natural method to obtain the meromorphic continuation of multiple Dirichlet series of type

$$\int_1^\infty L_f(s_1 + it) L_g(s_2 - it) \, t^{-w} \, dt, \tag{1.1}$$

where $L_f(s)$ and $L_g(s)$ are the L–functions associated to automorphic forms f and g on $GL(2, \mathbb{Q})$. For fixed g and fixed $s_1, s_2, w \in \mathbb{C}$, the integral (1.1) may be interpreted as the image of a linear map from the Hilbert space of cusp forms to $\mathbb{C}$ given by:

$$f \quad \longrightarrow \quad \int_1^\infty L_f(s_1 + it) L_g(s_2 - it) \, t^{-w} \, dt.$$

The Riesz representation theorem guarantees that this linear map has a kernel. Good computes this kernel explicitly. For example when $s_1 = s_2 = \frac{1}{2}$, he shows that there exists a Poincaré series P_w and a certain function K such that

$$\langle f, \; \bar{P}_w g \rangle = \int_{-\infty}^{\infty} L_f\left(\frac{1}{2} + it\right) \overline{L_g\left(\frac{1}{2} + it\right)} \, K(t, w) \, dt,$$

where $\langle \; , \; \rangle$ denotes the Peterson inner product on the Hilbert space of cusp forms. Remarkably, it is possible to choose P_w so that

$$K(t, w) \sim |t|^{-w}, \qquad (\text{as } |t| \to \infty).$$

Good's approach has been worked out for congruence subgroups in [**Zha**].

There are, however, two serious obstacles in Good's method.

- *Although $K(t, w) \sim |t|^{-w}$ as $|t| \to \infty$ and w fixed, it has a quite different behavior when $t \ll |Im(w)|$. In this case it grows exponentially in $|t|$.*
- *The function $\langle f, \; \bar{P}_w g \rangle$ has infinitely many poles in w, occurring at the eigenvalues of the Laplacian. So there is a problem to obtain polynomial growth in w by the use of convexity estimates such as the Phragmen-Lindelöf theorem.*

In this paper, we introduce novel techniques for surmounting the above two obstacles. The key idea is to use instead another function K_β, instead of K, so that (1.1) satisfies a functional equation $w \to 1 - w$. This allows one to obtain growth estimates for (1.1) in the regions $\Re(w) > 1$ and $-\epsilon < \Re(w) < 0$. In order to apply the Phragmen-Lindelöf theorem, one constructs an auxiliary function with the same poles as (1.1) and which has good growth properties. After subtracting this auxiliary function from (1.1), one may apply the Phragmen-Lindelöf theorem. It appears that the above methods constitute a new technique which may be applied in much greater generality. We will address these considerations in subsequent papers.

For $\Re(w)$ sufficiently large, consider the function $Z(w)$ defined by the absolutely convergent integral

$$(1.2) \qquad Z(w) = \int_1^\infty L_f\left(\frac{1}{2}+it\right) L_g\left(\frac{1}{2}-it\right) t^{-w}\, dt.$$

The main object of this paper is to prove the following.

THEOREM 1.3. *Suppose f and g are two cusp forms of weight $\kappa \geq 12$ for $SL(2,\mathbb{Z})$. The function $Z(w)$, originally defined by (1.2) for $\Re(w)$ sufficiently large, has meromorphic continuation to the half-plane $\Re(w) > -1$, with at most simple poles at*

$$w = 0,\ \frac{1}{2}+i\mu,\ -\frac{1}{2}+i\mu,\ \frac{\rho}{2},$$

where $\frac{1}{4}+\mu^2$ is an eigenvalue of Δ and $\zeta(\rho)=0$; when $f=g$, it has a pole of order two at $w=1$. Furthermore, for fixed $\epsilon>0$, and $\epsilon<\delta<1-\epsilon$, we have the growth estimate:

$$(1.4) \qquad Z(\delta+i\eta) \ll_\epsilon (1+|\eta|)^{2-\frac{3\delta}{4}},$$

provided $|w|, |w-1|, |w\pm\frac{1}{2}-\mu|, \left|w-\frac{\rho}{2}\right| > \epsilon$ with $w=\delta+i\eta$, and for all μ, ρ, as above.

Note that in the special case when $f(z)=g(z)$ is the usual $SL_2(\mathbb{Z})$ Eisenstein series at $s=\frac{1}{2}$ (suitably renormalized), a stronger result is already known (see [**IJM00**]) for $\Re(\delta) > \frac{1}{2}$. It is remarked in [**IJM00**] that their methods can be extended to holomorphic cusp forms, but that obtaining such results for Maass forms is problematic.

2. Poincaré series

To obtain Theorem 1.3, we shall need two Poincaré series, the second one being first considered by A. Good in [**Goo86**]. The first Poincaré series $P(z;v,w)$ is defined by

$$(2.1) \qquad P(z;v,w) = \sum_{\gamma\in\Gamma/Z} (\Im(\gamma z))^v \left(\frac{\Im(\gamma z)}{|\gamma z|}\right)^w \qquad (Z = \{\pm I\}).$$

This series converges absolutely for $\Re(v)$ and $\Re(w)$ sufficiently large. Writing

$$P(z;v,w) = \frac{1}{2}\sum_{\gamma\in SL_2(\mathbb{Z})} y^{v+w}|z|^{-w}\ \Big|\ [\gamma] = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y^{v+w}\cdot \sum_{m=-\infty}^{\infty} |z+m|^{-w}\ \Big|\ [\gamma],$$

and using the well-known Fourier expansion of the above inner sum, one can immediately write

$$P(z;v,w) = \sqrt{\pi}\,\frac{\Gamma\left(\frac{w-1}{2}\right)}{\Gamma\left(\frac{w}{2}\right)}\,E(z,v+1) + 2\,\pi^{\frac{w}{2}}\,\Gamma\Big(\frac{w}{2}\Big)^{-1}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}|k|^{\frac{w-1}{2}}\,P_k\Big(z;v+\frac{w}{2},\frac{w-1}{2}\Big), \tag{2.2}$$

where $\Gamma(s)$ is the usual Gamma function, $E(z,s)$ is the classical non-holomorphic Eisenstein series for $SL_2(\mathbb{Z})$, and $P_k(z;v,s)$ is the classical Poincaré series defined by

$$P_k(z;v,s) = |k|^{-\frac{1}{2}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(\Im(\gamma z))^v\,W_{\frac{1}{2}+s}(k\cdot\gamma z). \tag{2.3}$$

It is not hard to show that $P_k(z;v,s)\in L^2\big(\Gamma\backslash\mathcal{H}\big)$, for $|\Re(s)|+\frac{3}{4} > \Re(v) > |\Re(s)|+\frac{1}{2}$ (see [**Zha**]).

To define the second Poincaré series $P_\beta(z,w)$, let $\beta(z,w)$ be defined for $z\in\mathcal{H}$ and $\Re(w)>0$ by

$$\beta(z,w) = \begin{cases} \frac{1}{i}\int\limits_{-\log z}^{-\log\bar z}\left[\frac{2ye^\xi}{(ze^\xi-1)(\bar z e^\xi-1)}\right]^{1-w}d\xi & \text{if } \Re(z) = x\geq 0 \text{ and } \Re(w)>0\,,\\ \beta(-\bar z,w) & \text{if } x<0,\end{cases} \tag{2.4}$$

where the logarithm takes its principal values, and the integration is along a vertical line segment. It can be easily checked that $\beta(z,w)$ satisfies the following two properties:

$$\beta(\alpha z,w) = \beta(z,w) \qquad (\alpha>0), \tag{2.5}$$

and for z off the imaginary axis,

$$\Delta\beta = w(1-w)\beta. \tag{2.6}$$

If we write $z = re^{i\theta}$ with $r>0$ and $0<\theta<\frac{\pi}{2}$, then by (2.4) and (2.5), we have

$$\begin{aligned}\beta(z,w) &= \beta\big(e^{i\theta},w\big) = \frac{1}{i}\int\limits_{-i\theta}^{i\theta}\left[\frac{2\,e^\xi\sin\theta}{(e^{\xi+i\theta}-1)(e^{\xi-i\theta}-1)}\right]^{1-w}d\xi\\ &= \int\limits_{-\theta}^{\theta}\left[\frac{2\,e^{it}\sin\theta}{(e^{i(t+\theta)}-1)(e^{i(t-\theta)}-1)}\right]^{1-w}dt\\ &= \int\limits_{-\theta}^{\theta}\left(\frac{\sin\theta}{\cos t-\cos\theta}\right)^{1-w}dt\\ &= \sqrt{2\pi\sin\theta}\;\Gamma(w)\,P_{-\frac{1}{2}}^{\frac{1}{2}-w}(\cos\theta),\end{aligned} \tag{2.7}$$

where $P_\nu^\mu(z)$ is the spherical function of the first kind. This function is a solution of the differential equation

$$(1-z^2)\,\frac{d^2u}{dz^2} - 2z\,\frac{du}{dz} + \left[\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right]u = 0 \qquad (\mu,\,\nu\in C). \tag{2.8}$$

There is another linearly independent solution of (2.8) denoted by $Q_\nu^\mu(z)$ and called the spherical function of the second kind. We shall need these functions for real values of $z = x$ and $-1 \le x \le 1$. For these values, one can take as linearly independent solutions the functions defined by

$$P_\nu^\mu(x) = \frac{1}{\Gamma(1-\mu)} \left(\frac{1+x}{1-x}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; 1-\mu; \frac{1-x}{2}\right); \tag{2.9}$$

$$Q_\nu^\mu(x) = \frac{\pi}{2\sin\mu\pi}\left[P_\nu^\mu(x)\cos\mu\pi - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} P_\nu^{-\mu}(x)\right]. \tag{2.10}$$

Here

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)} z^n$$

is the Gauss hypergeometric function. We shall need an additional formula (see [**GR94**], page 1023, 8.737-2) relating the spherical functions, namely

$$P_\nu^\mu(-x) = P_\nu^\mu(x)\cos[(\mu+\nu)\pi] - \frac{2}{\pi} Q_\nu^\mu(x)\sin[(\mu+\nu)\pi]. \tag{2.11}$$

Now, we define the second Poincaré series $P_\beta(z,w)$ by

$$P_\beta(z,w) = \sum_{\gamma\in\Gamma/Z} \beta(\gamma z, w) \qquad (Z = \{\pm I\}). \tag{2.12}$$

It can be observed that the series in the right hand side converges absolutely for $\Re(w) > 1$.

3. Multiple Dirichlet series

Fix two cusp forms f, g of weight κ for $\Gamma = SL(2,\mathbb{Z})$ as in Section 1. Here f, g are holomorphic for $\kappa \ge 12$ and are Maass forms if $\kappa = 0$. Define

$$F(z) = y^\kappa \overline{f(z)} g(z).$$

For compex variables s_1, s_2, w, we are interested in studying the multiple Dirichlet series of type

$$\int_1^\infty L_f(s_1+it)\, L_g(s_2-it)\, t^{-w}\, dt.$$

As was first discovered by Good [**Goo86**], such series can be constructed by considering inner products of F with Poincaré series of the type that we have introduced in Section 2. Good shows that such inner products lead to multiple Dirichlet series of the form

$$\int_0^\infty L_f(s_1+it)\, L_g(s_2-it)\, K(s_1,s_2,t,w)\, dt,$$

with a suitable kernel function $K(s_1,s_2,t,w)$. One of the main difficulties of the theory is to obtain kernel functions K with good asymptotic behavior. The following kernel functions arise naturally in our approach.

First, if f, g are holomorphic cusp forms of weight κ, then we define:

$$K(s;v,w) = 2^{1-w-2v-2\kappa}\, \pi^{-v-\kappa}\, \frac{\Gamma(w+v+\kappa-1)\,\Gamma(s)\,\Gamma(v+\kappa-s)}{\Gamma\left(\frac{w}{2}+s\right)\Gamma\left(\frac{w}{2}+v+\kappa-s\right)}; \tag{3.1}$$

$$K_\beta(t,w) \;=\; 2^{1-\kappa}\,\pi^{-\kappa-1}\left|\Gamma\Big(\frac{\kappa}{2}+it\Big)\right|^2 \int\limits_0^{\frac{\pi}{2}} \beta\big(e^{i\theta},w\big)\sin^{\kappa-2}(\theta)\,\cosh[t(2\theta-\pi)]\;d\theta. \tag{3.2}$$

Also, for $0<\theta<2\pi$, let $\widetilde{W}_{\frac{1}{2}+\nu}\left(e^{i\theta},s\right)$ denote the Mellin transform of $W_{\frac{1}{2}+\nu}\left(ue^{i\theta}\right)$. Then, if f and g are both Maass cusp forms, we define $K(s;v,w)$ and $K_\beta(t,w)$ with $t\ge 0$, by

$$K(s;v,w) \;=\; \sum_{\epsilon_1,\,\epsilon_2=\pm1}\int\limits_0^{\pi} \widetilde{W}_{\frac12+ir_1}\left(\epsilon_1 e^{i\theta},\,s\right)\;\overline{\widetilde{W}_{\frac12+ir_2}\left(\epsilon_2\, e^{i\theta},\,\bar v-\bar s\right)}\,\sin^{v+w-2}(\theta)\;d\theta; \tag{3.3}$$

$$K_\beta(t,w) \;=\; \sum_{\epsilon_1,\,\epsilon_2=\pm1}\int\limits_0^{\pi} \beta\big(e^{i\theta},w\big)\sin^{-2}(\theta)\,\widetilde{W}_{\frac12+ir_1}\left(\epsilon_1 e^{i\theta},\,it\right)\overline{\widetilde{W}_{\frac12+ir_2}\left(\epsilon_2\, e^{i\theta},\,it\right)}\;d\theta. \tag{3.4}$$

We have the following.

Proposition 3.5. *Fix two cusp forms f,g of weight κ for $SL(2,\mathbb{Z})$ with associated L-functions $L_f(s),L_g(s)$. For $\Re(v)$ and $\Re(w)$ sufficiently large, we have*

$$\langle P(*;\,v,w),\,F\rangle = \int\limits_{-\infty}^{\infty} L_f\left(\sigma-\frac{\kappa}{2}+\frac12+it\right)L_g\left(v+\frac{\kappa}{2}+\frac12-\sigma-it\right)K(\sigma+it;v,w)\;dt,$$

and

$$\langle P_\beta(*;\,w),\,F\rangle = \int\limits_0^{\infty} L_f\left(\frac12+it\right)L_g\left(\frac12-it\right)K_\beta(t,w)\;dt,$$

where $K(s;v,w)$, $K_\beta(t,w)$ are given by (3.1) and (3.2), if f and g are holomorphic, and by (3.3) and (3.4), if f and g are both Maass cusp forms.

Proof. We evaluate

$$I(v,w) \;=\; \langle P(*;v,w),F\rangle \;=\; \int\int_{\Gamma\backslash\mathcal{H}} P(z;v,w)f(z)\overline{g(z)}\,y^\kappa\;\frac{dx\,dy}{y^2}$$

by the unfolding technique. We have

$$\begin{aligned}
I(v,w) &= \\
&= \int\limits_0^{\infty}\int\limits_{-\infty}^{\infty} f(z)\overline{g(z)}\,|z|^{-w}y^{v+w+\kappa-2}\;dx\;dy = \\
&= \int\limits_0^{\pi}\int\limits_0^{\infty} f\left(re^{i\theta}\right)\overline{g\left(re^{i\theta}\right)}\,r^{v+\kappa-1}\,\sin^{v+w+\kappa-2}(\theta)\;dr\;d\theta = \\
&= \sum_{m,\,n\neq 0}\frac{a_m b_n}{|mn|^{\frac{1-\kappa}{2}}}\int\limits_0^{\pi}\int\limits_0^{\infty} W_{\frac12+ir_1}\left(mre^{i\theta}\right)\overline{W_{\frac12+ir_2}\left(nre^{i\theta}\right)}\,r^{v+\kappa-1}\,\sin^{v+w+\kappa-2}(\theta)\;dr\,d\theta.
\end{aligned}$$

By Mellin transform theory, we may express

$$W_{\frac{1}{2}+ir_1}\left(mre^{i\theta}\right) \;=\; \frac{1}{2\pi i}\int\limits_{(\sigma)}\int\limits_{0}^{\infty} W_{\frac{1}{2}+ir_1}\left(mue^{i\theta}\right)\, u^s\,\frac{du}{u}\, r^{-s}\, ds.$$

Making the substitution $u \mapsto \frac{u}{|m|}$, we have

$$W_{\frac{1}{2}+ir_1}\left(re^{i\theta}\right) \;=\; \frac{1}{2\pi i}\int\limits_{(\sigma)}\int\limits_{0}^{\infty} W_{\frac{1}{2}+ir_1}\left(\frac{m}{|m|}\, ue^{i\theta}\right)\frac{u^s}{|m|^s}\,\frac{du}{u}\, r^{-s}\, ds.$$

Plugging this in the last expression of $\langle P(\cdot; v, w), F\rangle$, we obtain

$$\begin{aligned} I(v,w) \;=\;& \frac{1}{2\pi i}\int\limits_{(\sigma)}\sum_{m,\,n\neq 0}\frac{a_m b_n}{|m|^{s+\frac{1-\kappa}{2}}|n|^{\frac{1-\kappa}{2}}}\int\limits_{0}^{\pi}\int\limits_{0}^{\infty} W_{\frac{1}{2}+ir_1}\left(\frac{m}{|m|}\, ue^{i\theta}\right) u^s\,\frac{du}{u}\\ &\cdot\int\limits_{0}^{\infty}\overline{W_{\frac{1}{2}+ir_2}\left(nre^{i\theta}\right)}\, r^{v-s+\kappa}\,\frac{dr}{r}\cdot \sin^{v+w+\kappa-2}(\theta)\, d\theta\, ds. \end{aligned}$$

Recall that if f and g are Maass forms, then both are even. The proposition immediately follows by making the substitution $r \mapsto \frac{r}{|n|}$.

The second formula in Proposition 3.5. can be proved by a similar argument. □

4. The kernels $K(t,w)$ and $K_\beta(t,w)$

In this section, we shall study the behavior in the variable t of the kernels

$$(4.1)\qquad \begin{aligned} K(t,w) \;&:=\; K\Big(\frac{\kappa}{2}+it; 0, w\Big)\\ &=\; 2^{1-w-2\kappa}\,\pi^{-\kappa}\,\frac{\Gamma(w+\kappa-1)\,\Gamma\left(\frac{\kappa}{2}+it\right)\,\Gamma\left(\frac{\kappa}{2}-it\right)}{\Gamma\left(\frac{w}{2}+\frac{\kappa}{2}+it\right)\Gamma\left(\frac{w}{2}+\frac{\kappa}{2}-it\right)} \end{aligned}$$

and $K_\beta(t,w)$ given by (3.2). This will play an important role in the sequel. We begin by proving the following.

Proposition 4.2. *For $t \gg 0$, the kernels $K(t,w)$ and $K_\beta(t,w)$ are meromorphic functions of the variable w. Furthermore, for $-1 < \Re(w) < 2$, $|\Im(w)| \to \infty$, we have the asymptotic formulae*

$$(4.3)\qquad K(t,w) \;=\; \mathcal{A}(w)\, t^{-w}\cdot\left(1 \;+\; \mathcal{O}_\kappa\left(\frac{|\Im(w)|^4}{t^2}\right)\right),$$

$$(4.4)\qquad \begin{aligned} &K_\beta(t,w) \;=\\ &=\; 2^{1-\kappa}\,\pi^{-\kappa-1}\left|\Gamma\Big(\frac{\kappa}{2}+it\Big)\right|^2\int\limits_{0}^{\frac{\pi}{2}}\beta\big(e^{i\theta},w\big)\sin^{\kappa-2}(\theta)\,\cosh[t(2\theta-\pi)]\, d\theta\\ &=\; \mathcal{B}(w)\, t^{-w}\left(1+\mathcal{O}_\kappa\left(\frac{|\Im(w)|^3}{t^2}\right)\right), \end{aligned}$$

where

$$\mathcal{A}(w) = \frac{\Gamma(w+\kappa-1)}{2^{2\kappa+w-1}\,\pi^\kappa} \quad \textit{and} \quad \mathcal{B}(w) = \frac{2\pi^{w-\frac{1}{2}}\Gamma(w)\Gamma(w+\kappa-1)}{\Gamma(w+\frac{1}{2})(4\pi)^{\kappa+w-1}}.$$

Proof. Let s and a be complex numbers with $|a|$ large and $|a| < |s|^{\frac{1}{2}}$. Using the well-known asymptotic representation for large values of $|s|$:

$$\Gamma(s) = \sqrt{2\pi}\cdot s^{s-\frac{1}{2}}\, e^{-s}\left(1+\frac{1}{12\,s}+\frac{1}{288\,s^2}-\frac{139}{51840\,s^3}\;+\;\mathcal{O}\left(|s|^{-4}\right)\right),$$

which is valid provided $-\pi < \arg(s) < \pi$, we have

$$\frac{\Gamma(s)}{\Gamma(s+a)} \;=\; s^{-a}\left(1+\frac{a}{s}\right)^{-s-a+\frac{1}{2}} e^{a}\cdot\left(1-\frac{1}{12\,(s+a)}\;+\;\mathcal{O}\left(|s|^{-2}\right)\right)\left(1+\frac{1}{12\,s}\;+\;\mathcal{O}\left(|s|^{-2}\right)\right).$$

Since $|s| > |a|^2$, it easily follows that

$$\left(\tfrac{1}{2}-s-a\right)\log\left(1+\frac{a}{s}\right)+a \;=\; \frac{a\,(1-a)}{2\,s}\;+\;\frac{a^3}{6\,s^2}\;+\;\mathcal{O}\left(|a|^2|s|^{-2}\right).$$

Consequently,

$$\frac{\Gamma(s)}{\Gamma(s+a)} \;=\; s^{-a}e^{\frac{a\,(1-a)}{2\,s}+\frac{a^3}{6\,s^2}+\mathcal{O}\left(|a|^2|s|^{-2}\right)}\cdot\left(1-\frac{1}{12\,(s+a)}\;+\;\mathcal{O}\left(|s|^{-2}\right)\right)\cdot\left(1+\frac{1}{12\,s}\;+\;\mathcal{O}\left(|s|^{-2}\right)\right).$$

Now, we have by the Taylor expansion that

$$e^{\frac{a\,(1-a)}{2\,s}+\frac{a^3}{6\,s^2}} = 1+\frac{a(1-a)}{2s}+\mathcal{O}\left(\frac{|a|^4}{|s|^2}\right).$$

It follows that

$$\frac{\Gamma(s)}{\Gamma(s+a)} \;=\; s^{-a}\left(1+\frac{a(1-a)}{2s}+\mathcal{O}\left(\frac{|a|^4}{|s^2|}\right)\right). \tag{4.5}$$

Now

$$K(t,w) = 2^{1-w-2\kappa}\,\pi^{-\kappa}\,\Gamma(w+\kappa-1)\,\frac{\Gamma\left(\frac{\kappa}{2}+it\right)\,\Gamma\left(\frac{\kappa}{2}-it\right)}{\Gamma\left(\frac{w}{2}+\frac{\kappa}{2}+it\right)\Gamma\left(\frac{w}{2}+\frac{\kappa}{2}-it\right)}.$$

We may apply (4.5) (with $s=\frac{\kappa}{2}\pm it,\ a=\frac{w}{2}$) to obtain (for $t\to\infty$)

$$\begin{aligned} K(t,w) \;&=\; \frac{\Gamma(w+\kappa-1)}{2^{2\kappa+w-1}\,\pi^{\kappa}}\,\left|\tfrac{\kappa}{2}+it\right|^{-w}\cdot\left(1\;+\;\mathcal{O}\left(\frac{|w|^4}{\kappa^2+t^2}\right)\right)\\ &=\; \frac{\Gamma(w+\kappa-1)}{2^{2\kappa+w-1}\,\pi^{\kappa}}\,t^{-w}\cdot\left(1\;+\;\mathcal{O}\left(\frac{|w|^4}{t^2}\right)\right). \end{aligned}$$

This proves the asymptotic formula (4.3). □

We now continue on to the proof of (4.4). Recall that

$$K_\beta(t,w) \;=\; \frac{4\left|\Gamma\left(\frac{\kappa}{2}+it\right)\right|^2}{(2\pi)^{\kappa+1}}\int\limits_0^{\frac{\pi}{2}}\beta\left(e^{i\theta},w\right)\sin^{\kappa-2}(\theta)\,\cosh[t(2\theta-\pi)]\,d\theta.$$

We shall split the θ–integral into two parts. Accordingly, we write

$$K_\beta(t,w) \;=\; \frac{4\left|\Gamma\left(\frac{\kappa}{2}+it\right)\right|^2}{(2\pi)^{\kappa+1}}\left(\int\limits_0^{|\Im(w)|^{-\frac{1}{2}}}+\int\limits_{|\Im(w)|^{-\frac{1}{2}}}^{\frac{\pi}{2}}\right)\beta\left(e^{i\theta},w\right)\sin^{\kappa-2}(\theta)\,\cosh[t(2\theta-\pi)]\,d\theta.$$

First of all, we may assume $t \gg |\Im(w)|^{\frac{3}{2}+\epsilon}$. Otherwise, the asymptotic formula (4.4) is not valid.

$$\begin{aligned}\int_{|\Im(w)|^{-\frac{1}{2}}}^{\frac{\pi}{2}} & \beta\big(e^{i\theta}, w\big) \sin^{\kappa-2}(\theta) \cosh[t(2\theta-\pi)]\, d\theta \\ \ll \quad & e^{\pi t} e^{-\frac{2t}{\sqrt{|\Im(w)|}}} \cdot \max_{|\Im(w)|^{-\frac{1}{2}} \le \theta \le \frac{\pi}{2}} \big|\beta\big(e^{i\theta}, w\big)\big| \\ \ll \quad & e^{\pi t} e^{-|\Im(w)|^{1+\epsilon}},\end{aligned}$$

since $t \gg |\Im(w)|^{\frac{3}{2}+\epsilon}$ and $\beta\left(e^{i\theta}, w\right)$ is bounded. It follows that

$$\begin{aligned}K_\beta(t,w) \quad &= \quad \frac{4\left|\Gamma\left(\frac{\kappa}{2}+it\right)\right|^2}{(2\pi)^{\kappa+1}} \int_0^{|\Im(w)|^{-\frac{1}{2}}} \beta\big(e^{i\theta}, w\big) \sin^{\kappa-2}(\theta) \cosh[t(2\theta-\pi)]\, d\theta \\ &\qquad + \quad \mathcal{O}\left(e^{-|\Im(w)|^{1+\epsilon}}\right) \\ &= \quad \frac{2\left|\Gamma\left(\frac{\kappa}{2}+it\right)\right|^2 \cdot e^{\pi t}}{(2\pi)^{\kappa+1}} \int_0^{|\Im(w)|^{-\frac{1}{2}}} \beta\big(e^{i\theta}, w\big) \sin^{\kappa-2}(\theta)\, e^{-2\theta t}\, d\theta \\ &\qquad + \mathcal{O}\left(e^{-|\Im(w)|^{1+\epsilon}}\right).\end{aligned}$$

Now, for $\theta \ll |\Im(w)|^{-\frac{1}{2}}$, we have

$$\begin{aligned}&\beta\left(e^{i\theta}, w\right) \\ &= \int_{-\theta}^{\theta} \left(\frac{\sin\theta}{\cos u - \cos\theta}\right)^{1-w} du \\ &= 2(\sin\theta)^{1-w} \cdot \theta \int_0^1 \big(\cos(\theta u) - \cos(\theta)\big)^{w-1} du \\ &= 2(\sin\theta)^{1-w} \cdot \theta \int_0^1 \left(\theta^2\ \frac{(1-u^2)}{2!} - \theta^4\ \frac{(1-u^4)}{4!} + \theta^6\ \frac{(1-u^6)}{6!}\ -\ \cdots\right)^{w-1} du \\ &= \sqrt{\pi}\ 2^{1-w}(\sin\theta)^{1-w} \cdot \theta^{2w-1}\left[\frac{\Gamma(w)}{\Gamma\left(\frac{1}{2}+w\right)}\ +\ \frac{\theta^2(w-1)}{6}\left(-\frac{2\Gamma(w)}{\Gamma\left(\frac{1}{2}+w\right)} + \frac{\Gamma(1+w)}{\Gamma\left(\frac{3}{2}+w\right)}\right)\ +\ \cdots\right] \\ &= \sqrt{\pi}\ 2^{1-w}(\sin\theta)^{1-w} \cdot \theta^{2w-1}\left[\frac{\Gamma(w)}{\Gamma\left(\frac{1}{2}+w\right)}\left(1 + \theta^2 h_2(w) + \theta^4 h_4(w) + \theta^6 h_6(w)\ +\ \cdots\right)\right],\end{aligned}$$

where

$$h_2(w) = \frac{1-w^2}{6+12w}, \qquad h_4(w) = \frac{(w-1)(-21-5w+9w^2+5w^3)}{360(3+8w+4w^2)},$$

$$h_6(w) = \frac{(1-w)(3+w)(465-314w-80w^2+14w^3+35w^4)}{45360(1+2w)(3+2w)(5+2w)}, \qquad \cdots$$

and where $h_{2\ell}(w) = \mathcal{O}\left(|\Im(w)|^\ell\right)$ for $\ell = 1, 2, 3, \ldots$, and

$$\frac{\Gamma(w)}{\Gamma\left(\frac{1}{2}+w\right)}\left(1 + \theta^2 h_2(w) + \theta^4 h_4(w) + \theta^6 h_6(w)\ +\ \cdots\right)$$

converges absolutely for all $w \in \mathbb{C}$ and any fixed θ.

We may now substitute this expression for $\beta\left(e^{i\theta}, w\right)$ into the above integral for $K_\beta(t,w)$. We then obtain

$$K_\beta(t,w) =$$

$$= \frac{\left|\Gamma\left(\frac{\kappa}{2}+it\right)\right|^2 \cdot e^{\pi t}\,\Gamma(w)}{2^{\kappa+w-1}\pi^{\frac12+\kappa}\,\Gamma\left(\frac12+w\right)} \int\limits_0^{|\Im(w)|^{-\frac12}} (\sin\theta)^{\kappa-w-1}\theta^{2w-1}\, e^{-2\theta t}\left(1+\theta^2 h_2(w) + \cdots\right) d\theta + \mathcal{O}\left(e^{-|\Im(w)|^{1+\epsilon}}\right)$$

$$= \frac{\left|\Gamma\left(\frac{\kappa}{2}+it\right)\right|^2 \cdot e^{\pi t}\,\Gamma(w)}{2^{\kappa+w-1}\pi^{\frac12+\kappa}\,\Gamma\left(\frac12+w\right)} \int\limits_0^{|\Im(w)|^{-\frac12}} \theta^{\kappa+w-2}\, e^{-2\theta t}\left(1+\theta^2 \tilde{h}_2(w) + \theta^4\tilde{h}_4(w) + \cdots\right) d\theta + \mathcal{O}\left(e^{-|\Im(w)|^{1+\epsilon}}\right)$$

$$= \frac{\left|\Gamma\left(\frac{\kappa}{2}+it\right)\right|^2 \cdot e^{\pi t}\,\Gamma(w)}{2^{\kappa+w-1}\pi^{\frac12+\kappa}\,\Gamma\left(\frac12+w\right)} \int\limits_0^{\infty} \theta^{\kappa+w-2}\, e^{-2\theta t}\left(1+\theta^2 \tilde{h}_2(w) + \theta^4\tilde{h}_4(w) + \cdots\right) d\theta + \mathcal{O}\left(e^{-|\Im(w)|^{1+\epsilon}}\right)$$

$$= \frac{\left|\Gamma\left(\frac{\kappa}{2}+it\right)\right|^2 \cdot e^{\pi t}\,\Gamma(w)\Gamma(\kappa+w-1)}{t^{\kappa+w-1}\cdot 4^{\kappa+w-1}\pi^{\frac12+\kappa}\,\Gamma\left(\frac12+w\right)} \left(1 \;+\; \mathcal{O}\left(\frac{|\Im(w)|^3}{t^2}\right)\right),$$

where, in the above, $\tilde{h}_{2\ell}(w) = \mathcal{O}\left(|\Im(w)|^\ell\right)$ for $\ell = 1, 2, \ldots$.

If we now apply the identity

$$\begin{aligned} \left|\Gamma\left(\tfrac{\kappa}{2}+it\right)\right|^2 &= t\cdot|1+it|^2|2+it|^2|3+it|^2\cdots\left|\tfrac{\kappa}{2}-1+it\right|^2 \tfrac{\pi}{\sinh \pi t} \\ &= 2\pi t^{\kappa-1}e^{-\pi t}\left(1 \;+\; \mathcal{O}_\kappa\left(t^{-2}\right)\right) \end{aligned}$$

in the above expression, we obtain the second part of Proposition 4.2. □

For t smaller than $|\Im(w)|^{2+\epsilon}$, we have the following

Proposition 4.6. *Fix $\epsilon > 0$, $\kappa \geq 12$. For $-1 < \Re(w) < 2$ and $0 \leq t \ll |\Im(w)|^{2+\epsilon}$, with $\Im(w) \to \infty$, we have*

$$\left|\sin\left(\frac{\pi w}{2}\right) K_\beta(t, 1-w) - \cos\left(\frac{\pi w}{2}\right) K_\beta(t,w)\right| \;\ll_\kappa\; t^{\frac12}|\Im(w)|^{\kappa-\frac32}.$$

Proof. Let $g(w,\theta)$ denote the function defined by

$$g(w,\theta) \;=\; \Gamma(w)\, P_{-\frac12}^{\frac12-w}(\cos\theta).$$

We observe that

$$\begin{aligned} \sin\left(\frac{\pi w}{2}\right) g(1-w,\theta) \quad - \quad \cos\left(\tfrac{\pi w}{2}\right) g(w,\theta) \;=& \\ = \; - \; \frac{\cos \pi w}{2\cos\left(\frac{\pi w}{2}\right)}\left[g(w,\theta) + g(w,\pi-\theta)\right].& \end{aligned} \tag{4.7}$$

To see this, apply (2.10) and (2.11) with $\nu = -\frac12$ and $\mu = \frac12 - w$. We have:

$$g(1-w,\theta) \;=\; g(w,\theta)\sin\pi w - \frac{2}{\pi}\,\Gamma(w)\, Q_{-\frac12}^{\frac12-w}(\cos\theta)\cos\pi w;$$

$$g(w,\pi-\theta) \;=\; g(w,\theta)\cos\pi w + \frac{2}{\pi}\,\Gamma(w)\, Q_{-\frac12}^{\frac12-w}(\cos\theta)\sin\pi w.$$

Multiplying the first by $\sin \pi w$, the second by $\cos \pi w$, and then adding the resulting identities, we obtain

$$g(1-w,\theta)\sin \pi w + g(w,\pi-\theta)\cos \pi w \;=\; g(w,\theta),$$

from which (4.7) immediately follows by adding $g(w,\theta)\cos \pi w$ on both sides.

Now, if f and g are holomorphic, it follows from (2.7), (3.3), and (4.7) that

$$\begin{aligned}
&\sin\left(\tfrac{\pi w}{2}\right) K_\beta(t,1-w) - \cos\left(\tfrac{\pi w}{2}\right) K_\beta(t,w)\\
&= \; -2^{\frac12-\kappa}\,\pi^{-\kappa-\frac12}\left|\Gamma\Big(\frac{\kappa}{2}+it\Big)\right|^2 \frac{\cos \pi w}{\cos\left(\frac{\pi w}{2}\right)} \int_0^{\frac{\pi}{2}} [g(w,\theta)+g(w,\pi-\theta)]\\
&\qquad \sin^{\kappa-\frac32}(\theta)\,\cosh[t(2\theta-\pi)]\,d\theta \qquad (4.8)\\
&= \; -2^{\frac12-\kappa}\,\pi^{-\kappa-\frac12}\left|\Gamma\Big(\frac{\kappa}{2}+it\Big)\right|^2 \frac{\Gamma(w)\cos \pi w}{\cos\left(\frac{\pi w}{2}\right)} \int_0^{\pi} P^{\frac12-w}_{-\frac12}(\cos\theta)\\
&\qquad \sin^{\kappa-\frac32}(\theta)\,\cosh[t(2\theta-\pi)]\,d\theta.
\end{aligned}$$

By (2.9), we have

$$P^{\frac12-w}_{-\frac12}(\cos\theta) \;=\; \frac{1}{\Gamma(w+\frac12)}\,\cot^{\frac12-w}\left(\frac{\theta}{2}\right) F\left(\frac12,\frac12;w+\frac12;\sin^2\left(\frac{\theta}{2}\right)\right).$$

Invoking the well-known transformation formula

$$F(\alpha,\beta;\gamma;z) \;=\; (1-z)^{-\alpha}F\left(\alpha,\gamma-\beta;\gamma;\frac{z}{z-1}\right),$$

we can further write

$$P^{\frac12-w}_{-\frac12}(\cos\theta) \;=\; \frac{\cos^{-w-\frac12}\left(\frac{\theta}{2}\right)\sin^{w-\frac12}\left(\frac{\theta}{2}\right)}{\Gamma(w+\frac12)}\,F\left(\frac12,w;w+\frac12;-\tan^2\left(\frac{\theta}{2}\right)\right).$$

Now, represent the hypergeometric function on the right hand side by its inverse Mellin transform obtaining:

$$\begin{aligned}
P^{\frac12-w}_{-\frac12}(\cos\theta) \;&=\; \frac{1}{\Gamma(\frac12)\Gamma(w)}\cos^{-w-\frac12}\left(\frac{\theta}{2}\right)\sin^{w-\frac12}\left(\frac{\theta}{2}\right)\\
&\quad\cdot\; \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(\frac12+z)\Gamma(w+z)\Gamma(-z)}{\Gamma(z+w+\frac12)}\,\tan^{2z}\left(\frac{\theta}{2}\right)dz. \qquad (4.9)
\end{aligned}$$

Here, the path of integration is chosen such that the poles of $\Gamma(\frac12+z)$ and $\Gamma(w+z)$ lie to the left of the path, and the poles of the function $\Gamma(-z)$ lie to the right of it.

It follows that

$$\begin{aligned}
&\sin\left(\frac{\pi w}{2}\right) K_\beta(t,1-w) - \cos\left(\frac{\pi w}{2}\right) K_\beta(t,w)\\
&= \; -2^{\frac12-\kappa}\,\pi^{-\kappa-\frac12}\left|\Gamma\Big(\frac{\kappa}{2}+it\Big)\right|^2 \frac{\Gamma(w)\cos(\pi w)}{\cos\left(\frac{\pi w}{2}\right)}\int_0^{\pi}\frac{\cos^{-w-\frac12}\left(\frac{\theta}{2}\right)\sin^{w-\frac12}\left(\frac{\theta}{2}\right)}{\Gamma(\frac12)\Gamma(w)}\\
&\cdot\;\left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(\frac12+z)\Gamma(w+z)\Gamma(-z)}{\Gamma(z+w+\frac12)}\,\tan^{2z}\left(\frac{\theta}{2}\right)dz\right)\cdot\sin^{\kappa-\frac32}(\theta)\,\cosh[t(2\theta-\pi)]\,d\theta.
\end{aligned}$$

In the above, we apply the identity $\sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$; after exchanging integrals and simplifying, we obtain

$$\begin{aligned}(4.10)\qquad \sin\left(\frac{\pi w}{2}\right)K_\beta(t,1-w)-\cos\left(\frac{\pi w}{2}\right)K_\beta(t,w) &= \frac{\left|\Gamma\left(\frac{\kappa}{2}+it\right)\right|^2}{2\pi^{\kappa+1}}\frac{\cos(\pi w)}{\cos\left(\frac{\pi w}{2}\right)}\\ \cdot\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty}\frac{\Gamma(\frac12+z)\Gamma(w+z)\Gamma(-z)}{\Gamma(z+w+\frac12)}\\ \cdot\int\limits_0^\pi \cos^{\kappa-w-2z-2}\left(\frac{\theta}{2}\right)\sin^{2z+w+\kappa-2}\left(\frac{\theta}{2}\right)\cosh[t(2\theta-\pi)]\,d\theta\,dz.\end{aligned}$$

Note that $\sin\left(\frac{\pi w}{2}\right)K_\beta(t,1-w)-\cos\left(\frac{\pi w}{2}\right)K_\beta(t,w)$ satisfies a functional equation $w\mapsto 1-w$. We may, therefore, assume, without loss of generality, that $\Im(w)>0$. Fix $\epsilon>0$. We break the z–integral in (4.10) into three parts according as

$$-\infty<\Im(z)<-\left(\tfrac12+\epsilon\right)\Im(w),\qquad -\left(\tfrac12+\epsilon\right)\Im(w)\le\Im(z)\le\left(\tfrac12+\epsilon\right)\Im(w),$$

$$\left(\tfrac12+\epsilon\right)\Im(w)<\Im(z)<\infty.$$

Under the assumptions that $\Im(w)\to\infty$ and $0\le t\ll\Im(w)^{2+\epsilon}$, it follows easily from Stirling's estimate for the Gamma function that

$$\int\limits_{-i\infty}^{-i\left(\frac12+\epsilon\right)\Im(w)}\left|\frac{\Gamma(\frac12+z)\Gamma(w+z)\Gamma(-z)}{\Gamma(z+w+\frac12)}\right|dz \;=\; \mathcal{O}\left(e^{-\left(\frac{\pi}{2}+\epsilon\right)\Im(w)}\right),$$

$$\int\limits_{i\left(\frac12+\epsilon\right)\Im(w)}^{i\infty}\left|\frac{\Gamma(\frac12+z)\Gamma(w+z)\Gamma(-z)}{\Gamma(z+w+\frac12)}\right|dz \;=\; \mathcal{O}\left(e^{-\left(\frac{\pi}{2}+\epsilon\right)\Im(w)}\right),$$

and, therefore,

$$\begin{aligned}&\sin\left(\frac{\pi w}{2}\right)K_\beta(t,1-w)-\cos\left(\frac{\pi w}{2}\right)K_\beta(t,w)\\ (4.11)\qquad &=-\frac{\left|\Gamma\left(\frac{\kappa}{2}+it\right)\right|^2}{2\pi^{\kappa+1}}\frac{\cos\pi w}{\cos\left(\frac{\pi w}{2}\right)}\cdot\frac{1}{2\pi i}\int\limits_{-i\left(\frac12+\epsilon\right)\Im(w)}^{i\left(\frac12+\epsilon\right)\Im(w)}\frac{\Gamma(\frac12+z)\Gamma(w+z)\Gamma(-z)}{\Gamma(z+w+\frac12)}\\ &\cdot\int\limits_0^\pi\cos^{\kappa-w-2z-2}\left(\frac{\theta}{2}\right)\sin^{2z+w+\kappa-2}\left(\frac{\theta}{2}\right)\cosh[t(2\theta-\pi)]\,d\theta\,dz\\ &\qquad+\mathcal{O}\left(e^{-\epsilon\Im(w)}\right).\end{aligned}$$

Next, we evaluate the θ–integral on the right hand side of (4.11):

$$
\begin{aligned}
&\int_0^\pi \cos^{\kappa-w-2z-2}\left(\frac{\theta}{2}\right)\sin^{2z+w+\kappa-2}\left(\frac{\theta}{2}\right)\cosh[t(2\theta-\pi)]\,d\theta\\
&= \frac{e^{-\pi t}}{2}\int_0^\pi \cos^{\kappa-w-2z-2}\left(\frac{\theta}{2}\right)\sin^{2z+w+\kappa-2}\left(\frac{\theta}{2}\right)e^{2t\theta}\,d\theta\\
&\quad f+\frac{e^{\pi t}}{2}\int_0^\pi \cos^{\kappa-w-2z-2}\left(\frac{\theta}{2}\right)\sin^{2z+w+\kappa-2}\left(\frac{\theta}{2}\right)e^{-2t\theta}\,d\theta\\
&= e^{-\pi t}\int_0^{\pi/2}\cos^{\kappa-w-2z-2}(\theta)\,\sin^{2z+w+\kappa-2}(\theta)\,e^{4t\theta}\,d\theta\\
&\quad +e^{\pi t}\int_0^{\pi/2}\cos^{\kappa-w-2z-2}(\theta)\,\sin^{2z+w+\kappa-2}(\theta)\,e^{-4t\theta}\,d\theta,
\end{aligned}
\tag{4.12}
$$

where for the last equality we made the substitution

$$\theta \mapsto 2\theta.$$

Using the formula (see [**GR94**], page 511, 3.892-3),

$$
\begin{aligned}
&\int_0^{\pi/2} e^{2i\beta x}\sin^{2\mu}x\cos^{2\nu}x\,dx =\\
&= 2^{-2\mu-2\nu-1}\Big(e^{\pi i(\beta-\nu-\frac{1}{2})}\frac{\Gamma(\beta-\nu-\mu)\Gamma(2\nu+1)}{\Gamma(\beta-\mu+\nu+1)}F(-2\mu,\beta-\mu-\nu;1+\beta-\mu+\nu;-1)\\
&+e^{\pi i(\mu+\frac{1}{2})}\frac{\Gamma(\beta-\nu-\mu)\Gamma(2\mu+1)}{\Gamma(\beta-\nu+\mu+1)}F(-2\nu,\beta-\mu-\nu;1+\beta+\mu-\nu;-1)\Big),
\end{aligned}
$$

which is valid for $\Re(\mu)$, $\Re(\nu) > -\frac{1}{2}$, one can write the first integral in (4.12) as

$$
\begin{aligned}
2^{3-2\kappa}\sum_{\epsilon=\pm1} e^{-\epsilon\pi t}\cdot\Big(&e^{\pi i\frac{(1-\kappa+w+2z-4it\epsilon)}{2}}\frac{\Gamma(2-\kappa-2it\epsilon)\Gamma(-1+\kappa-w-2z)}{\Gamma(1-2it\epsilon-w-2z)}\\
&\cdot F(2-\kappa-w-2z,2-\kappa-2it\epsilon;1-w-2z-2it\epsilon;-1)\\
&+e^{\pi i\frac{(-1+\kappa+w+2z)}{2}}\frac{\Gamma(2-\kappa-2it\epsilon)\Gamma(-1+\kappa+w+2z)}{\Gamma(1-2it\epsilon+w+2z)}\\
&\cdot F(2-\kappa+w+2z,2-\kappa-2it\epsilon;1+w+2z-2it\epsilon;-1))\Big).
\end{aligned}
$$

If we replace the θ–integral on the right hand side of (4.11) by the above expression, it follows that

$$\begin{aligned}
& \sin\left(\frac{\pi w}{2}\right) K_\beta(t,1-w) - \cos\left(\frac{\pi w}{2}\right) K_\beta(t,w) \\
= \quad & -\frac{\left|\Gamma\left(\frac{\kappa}{2}+it\right)\right|^2}{2^{2\kappa-2}\pi^{\kappa+1}} \frac{\cos \pi w}{\cos\left(\frac{\pi w}{2}\right)} \cdot \sum_{\epsilon=\pm1} e^{-\epsilon\pi t}\Gamma(2-\kappa-2it\epsilon) \\
& \cdot \frac{1}{2\pi i} \int\limits_{-i\left(\frac{1}{2}+\epsilon\right)\Im(w)}^{i\left(\frac{1}{2}+\epsilon\right)\Im(w)} \frac{\Gamma(\frac{1}{2}+z)\Gamma(w+z)\Gamma(-z)}{\Gamma(z+w+\frac{1}{2})} \\
& \cdot \Bigg(e^{\pi i \frac{(1-\kappa+w+2z-4it\epsilon)}{2}} \frac{\Gamma(-1+\kappa-w-2z)}{\Gamma(1-2it\epsilon-w-2z)} \\
& \quad \cdot F(2-\kappa-w-2z, 2-\kappa-2it\epsilon; 1-w-2z-2it\epsilon; -1) \\
& \quad + e^{\pi i \frac{(-1+\kappa+w+2z)}{2}} \frac{\Gamma(-1+\kappa+w+2z)}{\Gamma(1-2it\epsilon+w+2z)} \\
& \quad \cdot F(2-\kappa+w+2z, 2-\kappa-2it\epsilon; 1+w+2z-2it\epsilon; -1)) \Bigg) dz \\
& \qquad\qquad + \mathcal{O}\left(e^{-\epsilon\Im(w)}\right).
\end{aligned} \tag{4.13}$$

To complete the proof of Proposition 4.6., we require the following Lemma.

LEMMA 4.14. *Fix* $\kappa \geq 12$. *Let* $-1 < \Re(w) < 2$, $0 \leq t \ll |\Im(w)|^{2+\epsilon}$, $\Re(z) = -\epsilon'$ *with* ϵ, ϵ' *small positive numbers, and* $|\Im(z)| < 2|\Im(w)|$. *Then, we have the following estimates:*

$$F(2-\kappa-w-2z, 2-\kappa-2it\epsilon; 1-w-2z-2it\epsilon; -1) \ll \sqrt{\min\{1, 2t, |\Im(w+2z)|\}},$$

$$F(2-\kappa+w+2z, 2-\kappa-2it\epsilon; 1+w+2z-2it\epsilon; -1) \ll \sqrt{\min\{1, 2t, |\Im(w+2z)|\}}.$$

PROOF. We shall make use of the following well-known identity of Kummer:

$$F(a,b,c;-1) = 2^{c-a-b}F(c-a, c-b, c; -1).$$

It follows that

$$\begin{aligned}
& F(2-\kappa-w-2z, 2-\kappa-2it\epsilon, 1-w-2z-2it\epsilon; -1) \\
& \qquad = 2^{2\kappa-3}F(\kappa-1-2it\epsilon, \kappa-1-w-2z, 1-w-2z-2it\epsilon; -1)
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
& F(2-\kappa+w+2z, 2-\kappa-2it\epsilon; 1+w+2z-2it\epsilon; -1) \\
& \qquad = 2^{2\kappa-3}F(\kappa-1-2it\epsilon, \kappa-1+w+2z, 1+w+2z-2it\epsilon, -1).
\end{aligned} \tag{4.16}$$

Now, we represent the hypergeometric function on the right hand side of (4.15) as

$$F(a,b,c;-1) \;=\; \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \cdot \frac{1}{2\pi i} \int\limits_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(a+\xi)\Gamma(b+\xi)\Gamma(-\xi)}{\Gamma(c+\xi)}\, d\xi, \tag{4.17}$$

with

$$\begin{aligned}
a &= \kappa-1-2it\epsilon \\
b &= \kappa-1-w-2z \\
c &= 1-w-2z-2it\epsilon.
\end{aligned}$$

This integral representation is valid, if, for instance, $-1 < \delta < 0$. We may also shift the line of integration to $0 < \delta < 1$ which crosses a simple pole with residue 1. Clearly, the main contribution comes from small values of the imaginary part of ξ.

If, for example, we use Stirling's formula

$$\Gamma(s) = \sqrt{2\pi} \cdot |t|^{\sigma - \frac{1}{2}} \; e^{-\frac{1}{2}\pi|t| + i\left(t\log|t| - t + \frac{\pi}{2}\cdot\frac{t}{|t|}\left(\sigma - \frac{1}{2}\right)\right)} \cdot \Big(1 + \mathcal{O}\left(|t|^{-1}\right)\Big),$$

where $s = \sigma + it,\ 0 \le \sigma \le 1,\ |t| \gg 0$, we have

$$\left|\frac{\Gamma(a+\xi)\Gamma(b+\xi)\Gamma(c)\Gamma(-\xi)}{\Gamma(a)\Gamma(b)\Gamma(c+\xi)}\right| \ll e^{\frac{\pi}{2}\left(-|W-\xi|+|2t+W-\xi|-|\xi|-|\xi-2t|\right)} \tag{4.18}$$
$$\cdot \frac{t^{\frac{3}{2}-\kappa}\, W^{\frac{3}{2}-\kappa}\, |W-\xi|^{-\frac{3}{2}+\kappa+\delta}\, |\xi-2t|^{-\frac{3}{2}+\kappa+\delta}\sqrt{2t+W}}{|\xi|^{\frac{1}{2}+\delta}\, |2t+W-\xi|^{\frac{1}{2}+\delta}},$$

where $W = \Im(w+2z) \ge 0$. This bound is valid provided

$$\min\Big(|W-\xi|, |2t+W-\xi|, |\xi|, |\xi-2t|\Big)$$

is sufficiently large. If this minimum is close to zero, we can eliminate this term and obtain a similar expression. There are 4 cases to consider.

Case 1: $|\xi| \le W,\ |\xi| \le 2t$. In this case, the exponential term in (4.18) becomes $e^0 = 1$ and we obtain

$$\left|\frac{\Gamma(a+\xi)\Gamma(b+\xi)\Gamma(c)\Gamma(-\xi)}{\Gamma(a)\Gamma(b)\Gamma(c+\xi)}\right| \ll |\xi|^{-\frac{1}{2}}.$$

Case 2: $|\xi| \le W,\ |\xi| > 2t$. In this case the exponential term in (4.18) becomes

$$+e^{\frac{\pi}{2}\left(-W+\xi+2t+W-\xi-|\xi|-|\xi|+2t\right)}$$

which has exponential decay in $(|\xi| - t)$.

Case 3: $|\xi| > W,\ |\xi| \le 2t$. Here, the exponential term in (4) takes the form

$$e^{\frac{\pi}{2}\left(-|\xi|+W+2t+W-\xi-|\xi|-2t+\xi\right)}$$

which has exponential decay in $(|\xi| - W)$.

Case 4: $|\xi| > W,\ |\xi| > 2t$. In this last case, we get

$$e^{\frac{\pi}{2}\left(-|\xi|-W+2t+W+|\xi|-2|\xi|-2t\right)}$$

if ξ is negative. Note that this has exponential decay in $|\xi|$. If ξ is positive, we get

$$e^{\frac{\pi}{2}\left(-|\xi|+W+|2t+W-\xi|-2|\xi|+2t\right)}.$$

This last expression has exponential decay in $(2|\xi| - W - 2t)$ if $2t + W - \xi > 0$. Otherwise it has exponential decay in $|\xi|$.

It is clear that the major contribution to the integral (4.17) for the hypergeometric function will come from case 1. This gives immediately the first estimate in Lemma 4.14. The second estimate in Lemma 4.14. can be established by a similar method. □

We remark that for $t = 0$, one can easily obtain the estimate in Proposition 4.6. by directly using the formula (see [**GR94**], page 819, 7.166),

$$\int_0^\pi P_\nu^{-\mu}(\cos\theta)\sin^{\alpha-1}(\theta)\,d\theta \;=\; 2^{-\mu}\pi\frac{\Gamma(\frac{\alpha+\mu}{2})\Gamma(\frac{\alpha-\mu}{2})}{\Gamma(\frac{1+\alpha+\nu}{2})\Gamma(\frac{\alpha-\nu}{2})\Gamma(\frac{\mu+\nu+2}{2})\Gamma(\frac{\mu-\nu+1}{2})},$$

which is valid for $\Re(\alpha\pm\mu) > 0$, and then by applying the Stirling's formula. It follows from this that

$$\sin\left(\frac{\pi w}{2}\right)K_\beta(0,1-w) - \cos\left(\frac{\pi w}{2}\right)K_\beta(0,w) \ll |\Im(w)|^{\kappa-2}.$$

Finally, we return to the estimation of $\sin\left(\frac{\pi w}{2}\right)K_\beta(0,1-w)-\cos\left(\frac{\pi w}{2}\right)K_\beta(0,w)$ using (4.13) and Lemma 4.14. If we apply Stirling's asymptotic expansion for the Gamma function, as we did before, it follows (after noting that $t, \Im(w) > 0$) that

$$\left|\sin\left(\frac{\pi w}{2}\right)K_\beta(0,1-w) - \cos\left(\frac{\pi w}{2}\right)K_\beta(0,w)\right|$$

$$\ll\; t^{\frac12}\int\limits_{-i\left(\frac12+\epsilon\right)\Im(w)}^{i\left(\frac12+\epsilon\right)\Im(w)} \frac{|\Im(w+2z)|^{\kappa-\frac32}}{\Im(w)^{\frac12}(1+|\Im(z)|)^{\frac12}|\Im(w+2z+2\epsilon t)|^{\frac12}}\;\sqrt{\min\{1,2t,|\Im(w+2z)|\}}\;dz$$

$$\ll\; t^{\frac12}\Im(w)^{\kappa-\frac32}.$$

This completes the proof of Proposition 4.6. □

5. The analytic continuation of $I(v,w)$

To obtain the analytic continuation of

$$I(v,w) \;=\; \langle P(*;v,w),F\rangle \;=\; \int\int_{\Gamma\backslash\mathcal{H}} P(z;v,w)f(z)\overline{g(z)}\,y^\kappa\,\frac{dx\,dy}{y^2},$$

we will compute the inner product $\langle P(*;v,w),F\rangle$ using Selberg's spectral theory. First, let us fix u_0, u_1, $u_2,\ldots$ an orthonormal basis of Maass cusp forms which are simultaneous eigenfunctions of all the Hecke operators T_n, $n = 1,\,2,\ldots$ and T_{-1}, where

$$(T_{-1}\,u)(z) \;=\; u(-\bar z).$$

We shall assume that u_0 is the constant function, and the eigenvalue of u_j, for $j = 1,\,2,\ldots,$ will be denoted by $\lambda_j = \frac14+\mu_j^2$. Since the Poincaré series $P_k(z;v,s)$ ($k\in Z$, $k\neq 0$) is square integrable, for $|\Re(s)|+\frac34 > \Re(v) > |\Re(s)|+\frac12$, we can spectrally decompose it as

$$\begin{aligned}P_k(z;v,s) \;&=\; \sum_{j=1}^\infty \langle P_k(*;v,s),u_j\rangle\,u_j(z)\\ &+\; \frac{1}{4\pi}\int\limits_{-\infty}^{\infty}\langle P_k(*;v,s),E(*,\tfrac12+i\mu)\rangle E(z,\tfrac12+i\mu)\,d\mu.\end{aligned} \tag{5.1}$$

Here we used the simple fact that $\langle P_k(*;v,s),u_0\rangle \;=\; 0.$

We shall need to write (5.1) explicitly. In order to do so, let u be a Maass cusp form in our basis with eigenvalue $\lambda = \frac{1}{4} + \mu^2$. Writing

$$u(z) \;=\; \rho(1) \sum_{\nu\neq 0} c_\nu \, |\nu|^{-\frac{1}{2}} \, W_{\frac{1}{2}+i\mu}(\nu z),$$

then by (2.3) and an unfolding process, we have

$$\begin{aligned}
\langle P_k(*; v, s), u\rangle \;&=\; |k|^{-\frac{1}{2}} \int_0^\infty \int_0^1 y^v \, W_{\frac{1}{2}+s}(kz) \, \overline{u(z)} \, \frac{dx\, dy}{y^2} \\
&=\; \overline{\rho(1)} \sum_{\nu\neq 0} \frac{c_\nu}{\sqrt{|k\nu|}} \int_0^\infty \int_0^1 y^{v-1} \, W_{\frac{1}{2}+s}(kz) \, W_{\frac{1}{2}+i\mu}(-\nu z) \, \frac{dx\, dy}{y} \\
&=\; \overline{\rho(1)} \, c_k \int_0^\infty y^v \, K_s(2\pi|k|y) \, K_{i\mu}(2\pi|k|y) \, \frac{dy}{y} \\
&=\; \pi^{-v} \, \frac{\overline{\rho(1)}}{8} \, \frac{c_k}{|k|^v} \, \frac{\Gamma\big(\frac{-s+v-i\mu}{2}\big)\Gamma\big(\frac{s+v-i\mu}{2}\big)\Gamma\big(\frac{-s+v+i\mu}{2}\big)\Gamma\big(\frac{s+v+i\mu}{2}\big)}{\Gamma(v)}.
\end{aligned}$$

Let $\mathcal{G}(s; v, w)$ denote the function defined by

$$\mathcal{G}(s; v, w) \;=\; \pi^{-v-\frac{w}{2}} \, \frac{\Gamma\big(\frac{-s+v+1}{2}\big)\Gamma\big(\frac{s+v}{2}\big)\Gamma\big(\frac{-s+v+w}{2}\big)\Gamma\big(\frac{s+v+w-1}{2}\big)}{\Gamma\big(v + \frac{w}{2}\big)}. \tag{5.2}$$

Then, replacing v by $v + \frac{w}{2}$ and s by $\frac{w-1}{2}$ in (5.2), we obtain

$$\Big\langle P_k\Big(*; v + \frac{w}{2}, \frac{w-1}{2}\Big), u\Big\rangle \;=\; \frac{\overline{\rho(1)}}{8} \, \frac{c_k}{|k|^{v+\frac{w}{2}}} \, \mathcal{G}\big(\tfrac{1}{2} + i\mu; v, w\big). \tag{5.3}$$

Next, we compute the inner product between $P_k\Big(z; v+\frac{w}{2}, \frac{w-1}{2}\Big)$ and the Eisenstein series $E(z, \bar{s})$. This is well-known to be the Mellin transform of the constant term of $P_k\Big(z; v + \frac{w}{2}, \frac{w-1}{2}\Big)$. More precisely, if we write

$$P_k\Big(z; v{+}\frac{w}{2}, \frac{w-1}{2}\Big) \;=\; y^{v+\frac{w}{2}+\frac{1}{2}} K_{\frac{w-1}{2}}(2\pi|k|y) e(kx) + \sum_{n=-\infty}^{\infty} a_n\Big(y; v{+}\frac{w}{2}, \frac{w-1}{2}\Big) e(nx),$$

where we denoted $e^{2\pi i x}$ by $e(x)$, then for $\Re(s) > 1$,

$$\Big\langle P_k\Big(\cdot; v + \frac{w}{2}, \frac{w-1}{2}\Big), \, E(\cdot, \bar{s})\Big\rangle \;=\; \int_0^\infty a_0\Big(y; v + \frac{w}{2}, \frac{w-1}{2}\Big) y^{s-2} \, dy.$$

Now, by a standard computation, we have

$$\begin{aligned}
a_0\Big(y; v + \frac{w}{2}, \frac{w-1}{2}\Big) \;&=\; \sum_{c=1}^\infty \sum_{\substack{r=1 \\ (r,c)=1}}^{c} e\left(\frac{kr}{c}\right) \int_{-\infty}^{\infty} \left(\frac{y}{c^2x^2 + c^2y^2}\right)^{v+\frac{w+1}{2}} \\
&\qquad \cdot K_{\frac{w-1}{2}}\left(\frac{2\pi|k|y}{c^2x^2 + c^2y^2}\right) e\left(\frac{-kx}{c^2x^2 + c^2y^2}\right) dx.
\end{aligned}$$

Making the substitution $x \mapsto \frac{x}{c^2}$ and $y \mapsto \frac{y}{c^2}$, we obtain

$$\left\langle P_k\Big(*; v + \frac{w}{2}, \frac{w-1}{2}\Big), \, E(*, \bar{s}) \right\rangle \;=\; \sum_{c=1}^{\infty} \tau_c(k)\, c^{-2s} \cdot \int_0^{\infty} \int_{-\infty}^{\infty} \frac{y^{s+v+\frac{w-3}{2}}}{(x^2+y^2)^{v+\frac{w+1}{2}}} \cdot K_{\frac{w-1}{2}}\left(\frac{2\pi |k| y}{x^2+y^2}\right) \cdot e\left(\frac{-kx}{x^2+y^2}\right) dx\, dy.$$

Here, $\tau_c(k)$ is the Ramanujan sum given by

$$\tau_c(k) \;=\; \sum_{\substack{r=1 \\ (r,c)=1}}^{c} e\left(\frac{kr}{c}\right).$$

Recalling that

$$\sum_{c=1}^{\infty} \tau_c(k)\, c^{-2s} \;=\; \frac{\sigma_{1-2s}(|k|)}{\zeta(2s)},$$

where for a positive integer n, $\sigma_s(n) = \sum_{d|n} d^s$, it follows after making the substitution $x \mapsto |k|x$, $y \mapsto |k|y$ that

$$\left\langle P_k\Big(*; v + \frac{w}{2}, \frac{w-1}{2}\Big), \, E(\cdot, \bar{s}) \right\rangle = |k|^{s-v-\frac{w}{2}-\frac{1}{2}} \cdot \frac{\sigma_{1-2s}(|k|)}{\zeta(2s)} \int_0^{\infty} \int_{-\infty}^{\infty} \frac{y^{s+v+\frac{w-3}{2}}}{(x^2+y^2)^{v+\frac{w+1}{2}}} \cdot K_{\frac{w-1}{2}}\left(\frac{2\pi y}{x^2+y^2}\right) e\left(-\frac{k}{|k|}\,\frac{x}{x^2+y^2}\right) dx\, dy. \tag{5.4}$$

The double integral on the right hand side can be computed in closed form by making the substitution $z \mapsto -\frac{1}{z}$. For $\Re(s) > 0$ and for $\Re(v-s) > -1$, we successively have:

$$\begin{aligned} \int_0^{\infty} \int_{-\infty}^{\infty} & \frac{y^{s+v+\frac{w-3}{2}}}{(x^2+y^2)^{v+\frac{w+1}{2}}} \cdot K_{\frac{w-1}{2}}\left(\frac{2\pi y}{x^2+y^2}\right) e\left(-\frac{k}{|k|}\,\frac{x}{x^2+y^2}\right) dx\, dy \\ &= \int_0^{\infty} \int_{-\infty}^{\infty} y^{s+v+\frac{w-3}{2}}\, (x^2+y^2)^{-s} \cdot K_{\frac{w-1}{2}}(2\pi y)\, e\left(\frac{k}{|k|}\, x\right) dx\, dy \\ &= \int_0^{\infty} y^{s+v+\frac{w-3}{2}}\, K_{\frac{w-1}{2}}(2\pi y) \cdot \int_{-\infty}^{\infty} (x^2+y^2)^{-s}\, e\left(\frac{k}{|k|}\, x\right) dx\, dy \\ &= \frac{2^{-v-\frac{w}{2}+1}\, \pi^{s-v-\frac{w}{2}}}{\Gamma(s)} \int_0^{\infty} y^{v+\frac{w}{2}-1}\, K_{\frac{w-1}{2}}(y)\, K_{s-\frac{1}{2}}(y)\; dy \\ &= \frac{\mathcal{G}(s; v, w)}{4\, \pi^{-s}\, \Gamma(s)}. \end{aligned} \tag{5.5}$$

Combining (5.4) and (5.5), we obtain

$$\left\langle P_k\Big(*; v + \frac{w}{2}, \frac{w-1}{2}\Big), \, E(\cdot, \bar{s}) \right\rangle = |k|^{s-v-\frac{w}{2}-\frac{1}{2}} \cdot \frac{\sigma_{1-2s}(|k|)}{4\, \pi^{-s}\, \Gamma(s)\, \zeta(2s)}\; \mathcal{G}(s; v, w) \tag{5.6}$$

Using (5.1), (5.3) and (5.6), one can decompose $P_k\big(\cdot; v+\frac{w}{2}, \frac{w-1}{2}\big)$ as

(5.7)

$$P_k\Big(z; v+\frac{w}{2}, \frac{w-1}{2}\Big) = \sum_{j=1}^{\infty} \frac{\overline{\rho_j(1)}}{8} \frac{c_k^{(j)}}{|k|^{v+\frac{w}{2}}} \mathcal{G}(\tfrac{1}{2}+i\mu_j; v, w)\, u_j(z) + \frac{1}{16\pi} \int_{-\infty}^{\infty} \frac{1}{\pi^{-\frac{1}{2}+i\mu}\,\Gamma(\frac{1}{2}-i\mu)\,\zeta(1-2i\mu)} \frac{\sigma_{2i\mu}(|k|)}{|k|^{v+\frac{w}{2}+i\mu}}\, \mathcal{G}(\tfrac{1}{2}-i\mu; v, w) E(z, \tfrac{1}{2}+i\mu)\, d\mu.$$

Now from (2.2) and (5.7), we deduce that

$$\pi^{-\frac{w}{2}}\Gamma\Big(\frac{w}{2}\Big) P(z; v, w) = \pi^{\frac{1-w}{2}}\,\Gamma\left(\frac{w-1}{2}\right) E(z, v+1) + \frac{1}{2} \sum_{u_j-\text{even}} \overline{\rho_j(1)}\, L_{u_j}(v+\tfrac{1}{2})\, \mathcal{G}(\tfrac{1}{2}+i\mu_j; v, w)\; u_j(z) + \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\zeta(v+\frac{1}{2}+i\mu)\,\zeta(v+\frac{1}{2}-i\mu)}{\pi^{-\frac{1}{2}+i\mu}\,\Gamma(\frac{1}{2}-i\mu)\,\zeta(1-2i\mu)}\, \mathcal{G}(\tfrac{1}{2}-i\mu; v, w) E(z, \tfrac{1}{2}+i\mu)\, d\mu. \tag{5.8}$$

The series corresponding to the discrete spectrum converges absolutely for $(v, w) \in C^2$, apart from the poles of $\mathcal{G}(\frac{1}{2}+i\mu_j; v, w)$. To handle the continuous part of the spectrum, we write the above integral as

$$\frac{1}{4\pi i} \int_{(\frac{1}{2})} \frac{\zeta(v+s)\zeta(v+1-s)}{\pi^{s-1}\Gamma(1-s)\zeta(2-2s)}\, \mathcal{G}(1-s; v, w) E(z, s)\, ds.$$

As a function of v and w, this integral can be meromorphically continued by shifting the line $\Re(s) = \frac{1}{2}$. For instance, to obtain continuation to a region containing $v = 0$, take v with $\Re(v) = \frac{1}{2}+\epsilon$, $\epsilon > 0$ sufficiently small, and take $\Re(w)$ large. By shifting the line of integration $\Re(s) = \frac{1}{2}$ to $\Re(s) = \frac{1}{2} - 2\epsilon$, we are allowed to take $\frac{1}{2} - \epsilon \le \Re(v) \le \frac{1}{2} + \epsilon$. We now assume $\Re(v) = \frac{1}{2} - \epsilon$, and shift back the line of integration to $\Re(s) = \frac{1}{2}$. It is not hard to see that in this process we encounter simple poles at $s = 1 - v$ and $s = v$ with residues

$$\pi^{\frac{1-w}{2}}\, \frac{\Gamma\big(\frac{w}{2}\big)\Gamma\big(\frac{2v+w-1}{2}\big)}{\Gamma\big(v+\frac{w}{2}\big)} E(z, 1-v),$$

and

$$\pi^{\frac{3}{2}-2v-\frac{w}{2}}\, \frac{\Gamma(v)\Gamma\big(\frac{2v+w-1}{2}\big)\Gamma\big(\frac{w}{2}\big)}{\Gamma(1-v)\Gamma\big(v+\frac{w}{2}\big)} \frac{\zeta(2v)}{\zeta(2-2v)} E(z, v) = \pi^{\frac{1-w}{2}}\, \frac{\Gamma\big(\frac{2v+w-1}{2}\big)\Gamma\big(\frac{w}{2}\big)}{\Gamma\big(v+\frac{w}{2}\big)} E(z, 1-v),$$

respectively, where for the last identity we applied the functional equation of the Eisenstein series $E(z, v)$. In this way, we obtained the meromorphic continuation of the above integral to a region containing $v = 0$. Continuing this procedure, one can prove the meromorphic continuation of the Poincaré series $P(z; v, w)$ to $\mathbb{C}^2$.

Using Parseval's formula, we obtain

$$\pi^{-\frac{w}{2}}\Gamma\Big(\frac{w}{2}\Big)\, I(v,w) \;=\; \pi^{\frac{1-w}{2}}\,\Gamma\left(\frac{w-1}{2}\right)\langle E(\cdot, v+1), F\rangle \tag{5.9}$$
$$+\;\frac{1}{2}\sum_{u_j-\text{even}} \overline{\rho_j(1)}\, L_{u_j}(v+\tfrac12)\,\mathcal{G}(\tfrac12+i\mu_j; v,w)\,\langle u_j, F\rangle$$
$$+\;\frac{1}{4\pi}\int\limits_{-\infty}^{\infty} \frac{\zeta(v+\frac12+i\mu)\,\zeta(v+\frac12-i\mu)}{\pi^{-\frac12+i\mu}\,\Gamma(\frac12-i\mu)\,\zeta(1-2i\mu)}\;\mathcal{G}(\tfrac12-i\mu; v,w)\,\langle E(\cdot,\tfrac12+i\mu), F\rangle\; d\mu,$$

which gives the meromorphic continuation of $I(v,w)$. We record this fact in the following

PROPOSITION 5.10. *The function* $I(v,w)$*, originally defined for* $\Re(v)$ *and* $\Re(w)$ *sufficiently large, has meromorphic continuation to* $\mathbb{C}^2$.

We conclude this section by remarking that from (5.9), one can also obtain information about the polar divisor of the function $I(v,w)$. When $v=0$, this issue is further discussed in the next section.

6. Proof of Theorem 1.3

To prove the first part of Theorem 1.3, assume for the moment that $f=g$. By Proposition 5.10, we know that the function $I(v,w)$ admits meromorphic continuation to $\mathbb{C}^2$. Furthermore, if we specialize $v=0$, the function $I(0,w)$ has its first pole at $w=1$. Using the asymptotic formula (4), one can write

$$I(0,w) \;=\; \int\limits_{-\infty}^{\infty} \left|L_f(\tfrac12+it)\right|^2 K(t,w)\; dt \;=\; 2\int\limits_{0}^{\infty} \left|L_f(\tfrac12+it)\right|^2 K(t,w)\; dt, \tag{6.1}$$

for at least $\Re(w)$ sufficiently large. Here the kernel $K(t,w)$ is given by (4.1). As the first pole of $I(0,w)$ occurs at $w=1$, it follows from (4.3) and Landau's Lemma that

$$Z(w) \;=\; \int\limits_{1}^{\infty} \left|L_f(\tfrac12+it)\right|^2 t^{-w}\; dt$$

converges absolutely for $\Re(w)>1$. If $f\neq g$, the same is true for the integral defining $Z(w)$ by Cauchy's inequality. The meromorphic continuation of $Z(w)$ to the region $\Re(w)>-1$ follows now from (4.3). This proves the first part of the theorem.

To obtain the polynomial growth in $|\Im(w)|$, for $\Re(w)>0$, we invoke the functional equation (see [**Goo86**])

$$\cos\left(\frac{\pi w}{2}\right) I_\beta(w) - \sin\left(\frac{\pi w}{2}\right) I_\beta(1-w) \tag{6.2}$$
$$= \frac{2\pi\,\zeta(w)\,\zeta(1-w)}{(2w-1)\,\pi^{-w}\,\Gamma(w)\,\zeta(2w)}\langle E(\cdot, 1-w), F\rangle.$$

It is well-known that $\langle E(\cdot,1-w),F\rangle$ is (essentially) the Rankin-Selberg convolution of f and g. Precisely, we have:

$$\langle E(\cdot, 1-w), F\rangle \;=\; (4\pi)^{w-\kappa}\Gamma(\kappa-w)\, L(1-w, f\times g). \tag{6.3}$$

It can be observed that the expression on the right hand side of (6.2) has polynomial growth in $|\Im(w)|$, away from the poles for $-1 < \Re(w) < 2$.

On the other hand, from the asymptotic formula (4), the integral

$$I_\beta(w) := \int_0^\infty L_f\left(\frac{1}{2}+it\right) L_g\left(\frac{1}{2}-it\right) K_\beta(t,w)\, dt$$

is absolutely convergent for $\Re(w) > 1$. We break $I_\beta(w)$ into two integrals:

$$\begin{aligned} I_\beta(w) &= \int_0^\infty L_f\left(\frac{1}{2}+it\right) L_g\left(\frac{1}{2}-it\right) K_\beta(t,w)\, dt \\ &= \int_0^{T_w} + \int_{T_w}^\infty := I_\beta^{(1)}(w) + I_\beta^{(2)}(w), \end{aligned} \tag{6.4}$$

where $T_w \ll |\Im(w)|^{2+\epsilon}$ (for small fixed $\epsilon > 0$), and T_w will be chosen optimally later.

Now, take w such that $-\epsilon < \Re(w) < -\frac{\epsilon}{2}$, and write the functional equation (6.2) as

$$\begin{aligned} \cos\left(\frac{\pi w}{2}\right) I_\beta^{(2)}(w) &= \left(\sin\left(\frac{\pi w}{2}\right) I_\beta^{(1)}(1-w) - \cos\left(\frac{\pi w}{2}\right) I_\beta^{(1)}(w)\right) \\ &\quad + \sin\left(\frac{\pi w}{2}\right) I_\beta^{(2)}(1-w) \\ &\quad + \frac{2\pi\,\zeta(w)\,\zeta(1-w)}{(2w-1)\,\pi^{-w}\,\Gamma(w)\,\zeta(2w)}\,\langle E(\cdot,1-w),F\rangle. \end{aligned} \tag{6.5}$$

Next, by Proposition 4.2,

$$\begin{aligned} \frac{I_\beta^{(2)}(w)}{\mathcal{B}(w)} &= \int_{T_w}^\infty L_f\left(\frac{1}{2}+it\right) L_g\left(\frac{1}{2}-it\right) t^{-w}\left(1+\mathcal{O}\left(\frac{|\Im(w)|^3}{t^2}\right)\right)\, dt \\ &= Z(w) - \int_1^{T_w} L_f\left(\frac{1}{2}+it\right) L_g\left(\frac{1}{2}-it\right) t^{-w}\, dt + \mathcal{O}\left(\frac{|\Im(w)|^3}{T_w^{1-\epsilon}}\right) \\ &= Z(w) + \mathcal{O}\left(T_w^{1+\epsilon} + \frac{|\Im(w)|^3}{T_w^{1-\epsilon}}\right). \end{aligned}$$

It follows that

$$Z(w) = \frac{I_\beta^{(2)}(w)}{\mathcal{B}(w)} + \mathcal{O}\left(T_w^{1+\epsilon} + \frac{|\Im(w)|^3}{T_w^{1-\epsilon}}\right). \tag{6.6}$$

We may estimate $\frac{I_\beta^{(2)}(w)}{\mathcal{B}(w)}$ using (6.5). Consequently,

$$\begin{aligned}\frac{I_\beta^{(2)}(w)}{\mathcal{B}(w)} &= \frac{1}{\mathcal{B}(w)}\Bigg[\left(\tan\left(\frac{\pi w}{2}\right) I_\beta^{(1)}(1-w) \;-\; I_\beta^{(1)}(w)\right) + \tan\left(\frac{\pi w}{2}\right) I_\beta^{(2)}(1-w) \\ &\qquad + \frac{2\pi\,\zeta(w)\,\zeta(1-w)}{\cos\left(\frac{\pi w}{2}\right)(2w-1)\,\pi^{-w}\,\Gamma(w)\,\zeta(2w)}\;\langle E(\cdot,1-w),F\rangle\Bigg].\end{aligned} \tag{6.7}$$

We estimate each term on the right hand side of (6.7) using Proposition 4.2 and Proposition 4.6. First of all

$$\begin{aligned}\frac{\tan\left(\frac{\pi w}{2}\right) I_\beta^{(1)}(1-w) \;-\; I_\beta^{(1)}(w)}{\mathcal{B}(w)} &= \frac{\sin\left(\frac{\pi w}{2}\right) I_\beta^{(1)}(1-w) \;-\; \cos\left(\frac{\pi w}{2}\right) I_\beta^{(1)}(w)}{\cos\left(\frac{\pi w}{2}\right)\mathcal{B}(w)} \\ &= \int_0^{T_w} L_f\left(\frac{1}{2}+it\right) L_g\left(\frac{1}{2}-it\right)\cdot\frac{t^{\frac{1}{2}}|\Im(w)|^{\kappa-\frac{3}{2}}}{|\Im(w)|^{\kappa-2-\epsilon}}\,dt \\ &\ll T_w^{\frac{3}{2}+\epsilon}|\Im(w)|^{\frac{1}{2}+\epsilon}.\end{aligned} \tag{6.8}$$

Next, using Stirling's formula to bound the Gamma function,

$$\begin{aligned}\frac{\tan\left(\frac{\pi w}{2}\right) I_\beta^{(2)}(1-w)}{\mathcal{B}(w)} &= \int_{T_w}^{\infty} L_f(\cdot)L_g(\cdot)\frac{\mathcal{B}(1-w)}{\mathcal{B}(w)}t^{-1-\frac{\epsilon}{2}}\left(1+\mathcal{O}\left(\frac{|\Im(w)|^3}{t^2}\right)\right)\,dt \\ &= \mathcal{O}\left(\frac{\mathcal{B}(1-w)}{\mathcal{B}(w)}\cdot\left(1+\frac{|\Im(w)|^3}{T_w^2}\right)\right) \\ &\ll \left|\frac{\Gamma(1-w)\Gamma(1-w+\kappa-1)\Gamma\left(\frac{1}{2}+w\right)}{\Gamma(w)\Gamma(w+\kappa-1)\Gamma\left(\frac{3}{2}-w\right)}\right|\cdot\left(1+\frac{|\Im(w)|^3}{T_w^2}\right) \\ &\ll |\Im(w)|^{1+2\epsilon}+\frac{|\Im(w)|^{4+2\epsilon}}{T_w^2}.\end{aligned} \tag{6.9}$$

Using the functional equation of the Riemann zeta-function (6.3), and Stirling's asymptotic formula, we have

$$\left|\frac{2\pi\,\zeta(w)\,\zeta(1-w)}{\mathcal{B}(w)\cos\left(\frac{\pi w}{2}\right)(2w-1)\,\pi^{-w}\,\Gamma(w)\,\zeta(2w)}\;\langle E(\cdot,1-w),F\rangle\right| \ll_\epsilon |\Im(w)|^{1+\epsilon}. \tag{6.10}$$

Now, we can optimize T_w by letting

$$T_w^{\frac{3}{2}+\epsilon}|\Im(w)|^{\frac{1}{2}+\epsilon} = \frac{|\Im(w)|^3}{T_w^{1-\epsilon}} \implies T_w = |\Im(w)|.$$

Thus, we get

$$Z(w) = \mathcal{O}\left(|\Im(w)|^{2+2\epsilon}\right).$$

One cannot immediately apply Phragmen-Lindelöf principle as the above function may have simple poles at $w = \frac{1}{2} \pm i\mu_j$, $j \geq 1$. To surmount this difficulty, let

$$\mathcal{G}_0(s,w) \;=\; \frac{\Gamma\big(w-\frac{1}{2}\big)}{\Gamma\big(\frac{w}{2}\big)} \left[\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{w-s}{2}\right) \;+\; \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{w+s-1}{2}\right)\right], \tag{6.11}$$

and define $\mathcal{J}(w) \;=\; \mathcal{J}_{\rm discr}(w) \;+\; \mathcal{J}_{\rm cont}(w)$, where

$$\mathcal{J}_{\rm discr}(w) \;=\; \frac{1}{2} \sum_{u_j-\text{even}} \overline{\rho_j(1)}\, L_{u_j}(\tfrac{1}{2})\, \mathcal{G}_0(\tfrac{1}{2}+i\mu_j, w)\, \langle u_j, F\rangle \tag{6.12}$$

and

$$\begin{aligned} &\mathcal{J}_{\rm cont}(w) \\ &= \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} \frac{\zeta(\frac{1}{2}+i\mu)\,\zeta(\frac{1}{2}-i\mu)}{\pi^{-\frac{1}{2}+i\mu}\,\Gamma(\frac{1}{2}-i\mu)\,\zeta(1-2i\mu)}\; \mathcal{G}_0(\tfrac{1}{2}-i\mu, w)\langle E(\cdot, \tfrac{1}{2}+i\mu), F\rangle\; d\mu. \end{aligned} \tag{6.13}$$

In (6.13), the contour of integration must be slightly modified when $\Re(w) = \frac{1}{2}$ to avoid passage through the point $s = w$.

From the upper bounds of Hoffstein-Lockhart [**HL94**] and Sarnak [**Sar94**], we have that

$$\left|\overline{\rho_j(1)}\,\langle u_j, F\rangle\right| \;\ll_\epsilon\; |\mu_j|^{N+\epsilon},$$

for a suitable N. It follows immediately that the series defining $\mathcal{J}_{\rm discr}(w)$ converges absolutely everywhere in C, except for points where $\mathcal{G}_0(\frac{1}{2}+i\mu_j, w)$, $j \geq 1$, have poles. The meromorphic continuation of $\mathcal{J}_{\rm cont}(w)$ follows easily by shifting the line of integration to the left. The key point for introducing the auxiliary function $\mathcal{J}(w)$ is that

$$I(0,w) \;-\; \mathcal{J}(w) \qquad (\Re(w) > -\epsilon)$$

(may) have poles only at $w = 0$, $\frac{1}{2}$, 1, and moreover,

$$\cos\left(\frac{\pi w}{2}\right)\mathcal{J}(w)$$

has polynomial growth in $|\Im(w)|$, away from the poles, for $-\epsilon < \Re(w) < 2$. To obtain a good polynomial bound in $|\Im(w)|$ for this function, it can be observed using Stirling's formula that the main contribution to $\mathcal{J}_{\rm discr}(w)$ comes from terms corresponding to $|\mu_j|$ close to $|\Im(w)|$. Applying Cauchy's inequality, we have that

$$\begin{aligned} \left|\frac{\mathcal{J}_{\rm discr}(w)}{2A(w)}\right| \ll \frac{1}{|A(w)|} &\cdot \Bigg(\sum_{\substack{u_j \\ |\mu_j|<2|\Im(w)|}} |\rho_j(1)\,\langle u_j, F\rangle|^2\Bigg)^{\frac{1}{2}} \\ &\cdot \Bigg(\sum_{\substack{u_j \\ |\mu_j|<2|\Im(w)|}} L^2_{u_j}(\tfrac{1}{2})\,|\mathcal{G}_0(\tfrac{1}{2}+i\mu_j, w)|^2\Bigg)^{\frac{1}{2}}. \end{aligned}$$

Using Stirling's asymptotic formula, we have the estimates

$$\begin{aligned} &\frac{1}{|A(w)|} \ll |\Im(w)|^{-\Re(w)-\kappa+\frac{3}{2}}\, e^{\frac{\pi}{2}|\Im(w)|} \\ &|\mathcal{G}_0(\tfrac{1}{2}+i\mu_j, w)| \ll_\epsilon |\Im(w)|^{\frac{\Re(w)}{2}-\frac{3}{4}+\epsilon} e^{-\frac{\pi}{2}|\Im(w)|} \qquad (\Re(w) < 1+\epsilon). \end{aligned}$$

Also, Hoffstein-Lockhart estimate [**HL94**] gives

$$|\rho_j(1)|^2 \ll_\epsilon |\Im(w)|^\epsilon e^{\pi|\mu_j|},$$

for $\mu_j \ll |\Im(w)|$. It follows that

$$\left|\frac{\mathcal{J}_{\rm discr}(w)}{2A(w)}\right| \ll |\Im(w)|^{-\frac{\Re(w)}{2}-\kappa+\frac{3}{4}+2\epsilon}\cdot\Bigg(\sum_{\substack{u_j\\ |\mu_j|<2|\Im(w)|}} e^{\pi|\mu_j|}\cdot|\langle u_j, F\rangle|^2\Bigg)^{\frac{1}{2}} \cdot\Bigg(\sum_{\substack{u_j\\ |\mu_j|<2|\Im(w)|}} L_{u_j}^2(\tfrac{1}{2})\Bigg)^{\frac{1}{2}}.$$

A very sharp bound for the first sum on the right hand side was recently obtained by Bernstein and Reznikov (see [**BR99**]). It gives an upper bound on the order of $|\Im(w)|^{\kappa+\epsilon}$. Finally, Kuznetsov's bound (see [**Mot97**]) gives an estimate on the order of $|\Im(w)|^{1+\epsilon}$ for the second sum. We obtain the final estimate

$$\left|\frac{\mathcal{J}_{\rm discr}(w)}{2A(w)}\right| \ll_\epsilon |\Im(w)|^{-\frac{\Re(w)}{2}+\frac{7}{4}+4\epsilon} \qquad (\Re(w)<1+\epsilon). \tag{6.14}$$

It is not hard to see that the same estimate holds for $\frac{\mathcal{J}_{\rm cont}(w)}{2A(w)}$. To see this, we apply in (6.3) the convexity bound for the Rankin-Selberg L–function together with Stirling's formula. It follows that

$$|\langle E(\cdot, \tfrac{1}{2}+i\mu), F\rangle| \ll_\epsilon |\mu|^{\kappa+\epsilon}\, e^{-\frac{\pi}{2}|\mu|}.$$

Then,

$$\left|\frac{\mathcal{J}_{\rm cont}(w)}{2A(w)}\right| \ll_\epsilon |\Im(w)|^{-\frac{\Re(w)}{2}+\frac{3}{4}+2\epsilon}\int\limits_{-2|\Im(w)|}^{2|\Im(w)|}\frac{|\zeta(\frac{1}{2}+i\mu)|^2}{|\zeta(1-2i\mu)|}\,d\mu \qquad (\Re(w)<1+\epsilon).$$

By the well-known bounds

$$|\zeta(1+it)|^{-1}\ll 1, \qquad \int\limits_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt \ll_\epsilon T^{1+\epsilon},$$

we obtain

$$\left|\frac{\mathcal{J}_{\rm cont}(w)}{2A(w)}\right| \ll_\epsilon |\Im(w)|^{-\frac{\Re(w)}{2}+\frac{7}{4}+3\epsilon} \qquad (\Re(w)<1+\epsilon). \tag{6.15}$$

It can be easily seen that the function

$$Z(w) - \frac{\mathcal{J}(w)}{2A(w)} \qquad (\Re(w)>-\epsilon)$$

(may) have poles only at $w = 0,\ \frac{1}{2},\ 1$. We can now apply Phragmen-Lindelöf principle, and Theorem 1.3 follows. □

Finally, we remark that the choice of the function $\mathcal{G}_0(s,w)$ defined by (6.11) is not necessarily the optimal one. We were rather concerned with making the method as transparent as possible, and in fact, the exponent $2-2\delta$ instead of $2-\frac{3}{4}\delta$ should be obtainable.

References

[BR99] J. Bernstein & A. Reznikov – "Analytic continuation of representations and estimates of automorphic forms", *Ann. of Math. (2)* **150** (1999), no. 1, p. 329–352.

[CF00] J. B. Conrey & D. W. Farmer – "Mean values of L-functions and symmetry", *Internat. Math. Res. Notices* (2000), no. 17, p. 883–908.

[CFK$^+$] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubenstein & N. C. Snaith – "Moments of zeta and L-functions".

[CG84] J. B. Conrey & A. Ghosh – "On mean values of the zeta-function", *Mathematika* **31** (1984), no. 1, p. 159–161.

[CG01] J. B. Conrey & S. M. Gonek – "High moments of the Riemann zeta-function", *Duke Math. J.* **107** (2001), no. 3, p. 577–604.

[Dav00] H. Davenport – *Multiplicative number theory*, third ed., Graduate Texts in Mathematics, vol. 74, Springer-Verlag, New York, 2000, Revised and with a preface by Hugh L. Montgomery.

[DG] A. Diaconu & D. Goldfeld – "Second moments of quadratic Hecke L–series and multiple Dirichlet series I", in *Multiple Dirichlet Series, Automorphic Forms and Analytic Number Theory, Proceedings of the Workshop on Multiple Dirichlet Series, July 11-14, 2005,* Proceedings of Symposia in Pure Mathematics, AMS, (to appear).

[DGH03] A. Diaconu, D. Goldfeld & J. Hoffstein – "Multiple Dirichlet series and moments of zeta and L-functions", *Compositio Math.* **139** (2003), no. 3, p. 297–360.

[Gau01] C. F. Gauss – "Disquisitiones Arithmeticae", 1801.

[Goo82] A. Good – "The square mean of Dirichlet series associated with cusp forms", *Mathematika* **29** (1982), no. 2, p. 278–295 (1983).

[Goo86] A. Good – "The convolution method for Dirichlet series", in *The Selberg trace formula and related topics (Brunswick, Maine, 1984)*, Contemp. Math., vol. 53, Amer. Math. Soc., Providence, RI, 1986, p. 207–214.

[GR94] I. S. Gradshteyn & I. M. Ryzhik – *Table of integrals, series, and products*, fifth ed., Academic Press Inc., Boston, MA, 1994.

[HB81] D. R. Heath-Brown – "An asymptotic series for the mean value of Dirichlet L-functions", *Comment. Math. Helv.* **56** (1981), no. 1, p. 148–161.

[HL18] G. H. Hardy & J. E. Littlewood – "Contributions to the theory of the Riemann zeta-function and the theory of the distributions of primes", *Acta Mathematica* **41** (1918), p. 119–196.

[HL94] J. Hoffstein & P. Lockhart – "Coefficients of Maass forms and the Siegel zero", *Ann. of Math. (2)* **140** (1994), no. 1, p. 161–181, With an appendix by Dorian Goldfeld, Hoffstein and Daniel Lieman.

[IJM00] A. Ivić, M. Jutila & Y. Motohashi – "The Mellin transform of powers of the zeta-function", *Acta Arith.* **95** (2000), no. 4, p. 305–342.

[Ing26] A. E. Ingham – "Mean-value theorems in the theory of the Riemann zeta-function", *Proceedings of the London Mathematical Society* **27** (1926), p. 273–300.

[Jut97] M. Jutila – "Mean values of Dirichlet series via Laplace transforms", in *Analytic number theory (Kyoto, 1996)*, London Math. Soc. Lecture Note Ser., vol. 247, Cambridge Univ. Press, Cambridge, 1997, p. 169–207.

[Jut05] ———, "The Mellin transform of the fourth power of Riemann's zeta-function", in *Number theory*, Ramanujan Math. Soc. Lect. Notes Ser., vol. 1, Ramanujan Math. Soc., Mysore, 2005, p. 15–29.

[KS99] N. M. Katz & P. Sarnak – *Random matrices, Frobenius eigenvalues, and monodromy*, American Mathematical Society Colloquium Publications, vol. 45, American Mathematical Society, Providence, RI, 1999.

[KS00] J. P. Keating & N. C. Snaith – "Random matrix theory and $\zeta(1/2 + it)$", *Comm. Math. Phys.* **214** (2000), no. 1, p. 57–89.

[Mot92] Y. Motohashi – "Spectral mean values of Maass waveform L-functions", *J. Number Theory* **42** (1992), no. 3, p. 258–284.

[Mot93] ———, "An explicit formula for the fourth power mean of the Riemann zeta-function", *Acta Math.* **170** (1993), no. 2, p. 181–220.

[Mot95] ———, "A relation between the Riemann zeta-function and the hyperbolic Laplacian", *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **22** (1995), no. 2, p. 299–313.

[Mot97] Y. Motohashi – *Spectral theory of the Riemann zeta-function*, Cambridge Tracts in Mathematics, vol. 127, Cambridge University Press, Cambridge, 1997.

[Sar94] P. Sarnak – "Integrals of products of eigenfunctions", *Internat. Math. Res. Notices* (1994), no. 6, p. 251 ff., approx. 10 pp. (electronic).

[Vin18] I. M. Vinogradov – "On the mean value of the number of classes of proper primitive forms of negative discriminant", *Soobchshneyia Khar'kovskogo mathematicheskogo obshestva (Transactions of the Kharkov Math. Soc.)* **16** (1918), p. 10–38.

[Zav89] N. I. Zavorotny – "Automorphic functions and number theory, part I, II (Russian)", *Akad. Nauk SSSR, Dal'nevostochn. Otdel., Vladivostok* **254** (1989), p. 69–124a.

[Zha] W. Zhang – "Integral mean values of modular L–functions", preprint.

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