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Rank Lowering Linear Maps and Multiple Dirichlet Series Associated to $GL(n, \mathbb{R})$

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Dedicated to John Coates

1. Introduction.

The Godement-Jacquet L-function associated to the discrete subgroup $SL(n, \mathbb{Z})$ (with $n \geq 2$) acting on

$$\mathfrak{h}^n := GL(n, \mathbb{R})/(O(n, \mathbb{R}) \cdot \mathbb{R}^\times)$$

was first introduced in [Godement-Jacquet, 1972] where its analytic properties: holomorphic continuation, functional equation, Euler product, etc. were obtained by a generalization of Tate's thesis [Tate, 1950]. Later, [Jacquet-Piatetski-Shapiro-Shalika, 1979] obtained a different derivation of the construction of the Godement-Jacquet L-function by the use of Whittaker models.

Following [Goldfeld, 2006], we review a classical construction of the Godement-Jacquet L-function. In order to do this, it is necessary to introduce the following notation. Let

$$z_n = x(n) \cdot y(n) \ \in \ \mathfrak{h}^n$$

with

$$x(n) = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix}, \quad y(n) = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

(where $x_{i,j} \in \mathbb{R}$ for $1 \leq i < j \leq n$ and $y_i > 0$ for $1 \leq i \leq n-1$) be in Iwasawa reduced form. Define $U_n(\mathbb{R})$ to be the subgroup of $SL(n, \mathbb{R})$ consisting of upper

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triangular unipotent matrices and let

$$u = \begin{pmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & 1 & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & u_{n-1,n} \\ & & & & 1 \end{pmatrix}$$

denote a generic element of $U_n(\mathbb{R})$, where the superdiagonal elements are relabeled as

$$u_1 = u_{n-1,n}, \quad u_2 = u_{n-2,n-1}, \qquad \ldots, \qquad u_{n-1} = u_{1,2}.$$

For $m = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}$, define ψ_m to be the character of $U_n(\mathbb{R})$ defined by

$$\psi_m(u) := e^{2\pi i \left[m_1 u_1 + m_2 u_2 + \cdots + m_{n-1} u_{n-1}\right]}.$$

For $\nu = (\nu_1, \nu_2, \cdots, \nu_{n-1}) \in \mathbb{C}^{n-1}$, and $z_n = x(n) \cdot y(n) \in \mathfrak{h}^n$, as above, define the function, $I_\nu : \mathfrak{h}^n \to \mathbb{C}$, by the condition:

$$I_\nu(z_n) := \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{i,j}\nu_j},$$

where

$$b_{i,j} = \begin{cases} ij & \text{if } i + j \leq n, \\ (n-i)(n-j) & \text{if } i + j \geq n. \end{cases}$$

Set

$$w_n = \begin{pmatrix} & & & (-1)^{\lfloor n/2 \rfloor} \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix} \in SL(n, \mathbb{Z}),$$

to be the long element of the Weyl group.

We may now define Jacquet's Whittaker function $W_{\text{Jacquet}}(z_n; \nu, \psi_m)$ by the integral formula

$$W_{\text{Jacquet}}(z_n; \nu, \psi_m) := \int_{U_n(\mathbb{R})} I_\nu(w_n \cdot u \cdot z_n)\, \overline{\psi_m(u)}\, d^*u,$$

when $\Re(v_i) \gg 1$ (for $1 \leq i \leq n-1$) and by meromorphic continuation to all $v \in \mathbb{C}^{n-1}$. The Whittaker function is characterized by the following properties. First, it satisfies the relation

$$W_{\text{Jacquet}}(uz_n; \nu, \psi_m) \;=\; \psi_m(u) \cdot W_{\text{Jacquet}}(z_n; \nu, \psi_m)$$

for all $u \in U_n(\mathbb{R})$. Second, it is a square integrable function (for the Haar measure) on a Siegel set Σ where

$$\Sigma = \left\{ z \in \mathfrak{h}^n \;\middle|\; \quad y_i > \frac{\sqrt{3}}{2}\,(1 \le i \le n-1), \qquad |x_{i,j}| \le \frac{1}{2}\,(1 \le i < j \le n) \right\}.$$

Finally, the Whittaker function is also an eigenfunction of all the $GL(n, \mathbb{R})$ invariant differential operators with the same eigenvalues as I_ν.

Definition 1.1. (Maass form) *Let $n \ge 2$. A Maass form for $SL(n, \mathbb{Z})$ of type $\nu \in \mathbb{C}^{n-1}$ is a smooth function $f : \mathfrak{h}^n \to \mathbb{C}$ which satisfies the following conditions:*

- $f(\gamma z_n) = f(z_n), \;\; \forall \gamma \in SL(n, \mathbb{Z}), \; z_n \in \mathfrak{h}^n,$

- $f \in \mathcal{L}^2\left(SL(n, \mathbb{Z})\backslash \mathfrak{h}^n\right),$

- *f is an eigenfunction of all the $GL(n, \mathbb{R})$ invariant differential operators (having the same eigenvalues as $I_\nu(z_n)$),*

- $\displaystyle\int\limits_{(SL(n,\mathbb{Z})\cap U)\backslash U} f(uz_n)\, du = 0,$

  for all upper triangular groups U of the form

$$U = \left\{ \begin{pmatrix} I_{r_1} & & & \\ & I_{r_2} & & * \\ & & \ddots & \\ & & & I_{r_b} \end{pmatrix} \right\},$$

with $r_1 + r_2 + \cdots r_b = n$. Here I_r denotes the $r \times r$ identity matrix, and $*$ denotes arbitrary real entries.

The Fourier expansion of a Maass form for $SL(n, \mathbb{Z})$ was first derived in [Piatetski-Shapiro, 1975], [Shalika, 1974]. The Fourier expansion takes the form (see [Goldfeld, 2006])

$$(1.2)\quad f(z_n) = \sum_{\gamma \in U_{n-1}(\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1}\ne 0} \frac{A(m_1, \ldots, m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}} \cdot W_{\text{Jacquet}}\left(M \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z_n,\; \nu,\; \psi_{1,\ldots,1,\frac{m_{n-1}}{|m_{n-1}|}} \right),$$

where

$$M = \begin{pmatrix} m_1 \cdots m_{n-2} \cdot |m_{n-1}| & & & & \\ & \ddots & & & \\ & & m_1 m_2 & & \\ & & & m_1 & \\ & & & & 1 \end{pmatrix}, \qquad A(m_1, \ldots, m_{n-1}) \in \mathbb{C}.$$

We may now give the definition of the Godement-Jacquet L-function.

Definition 1.3. (Godement-Jacquet L-function) *Let $s \in \mathbb{C}$ with $\Re(s) > \frac{n+1}{2}$, and let $f(z_n)$ be a Maass form for $SL(n, \mathbb{Z})$, with $n \geq 2$ as in definition 1.1, which is an eigenfunction of all the Hecke operators. We define the Godement-Jacquet L-function, $L_f(s)$, by the absolutely convergent series*

$$L_f(s) = \sum_{m=1}^{\infty} A(m, 1, \ldots, 1) m^{-s} = \prod_p \phi_p(s),$$

where

$$\phi_p(s) = \Big(1 - A(p, \ldots, 1)p^{-s} - A(1, p, \ldots, 1)p^{-2s} + \cdots + (-1)^{n-1} A(1, \ldots, p) p^{(-n+1)s} + (-1)^n p^{-ns}\Big)^{-1}. \tag{1.4}$$

It is natural to consider multiple Dirichlet series associated to f of the type

$$L_f(s_1, \ldots, s_{n-1}) := \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-1}=1}^{\infty} \frac{A(m_1, \ldots, m_{n-1})}{m_1^{s_1} \cdots m_{n-1}^{s_{n-1}}}, \tag{1.5}$$

where $s_1, s_2, \ldots, s_{n-1} \in \mathbb{C}$ with $\Re(s_i) \gg 1$ $(1 \leq i \leq n-1)$. Such a series was first considered by [Bump, 1984] in the case $n = 3$. Bump proved the identity

$$\sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{A(m_1, m_2)}{m_1^{s_1} m_2^{s_2}} = \frac{L_{\tilde{f}}(s_1) L_f(s_2)}{\zeta(s_1 + s_2)},$$

where $A(m_1, m_2)$ are the Fourier coefficients of a Maass form f (normalized Hecke eigenform) for $SL(3, \mathbb{Z})$ and $\tilde{f}$ denotes the dual of f. In [Bump-Friedberg, 1990], a generalization of this double Dirichlet series, and above identity, was obtained by considering the product of the standard and exterior square L-functions, i.e., a Rankin-Selberg construction involving two complex variables, one from the Eisenstein series, and one of "Hecke" type.

The multiple Dirichlet series of the type (1.5) are not, in general, expected to have meromorphic continuation to all $\mathbb{C}^{n-1}$ when $n \geq 4$ (see [Brubaker-Bump-Chinta-Friedberg-Hoffstein, 2005]). It is expected that they have natural boundaries.

Question 1.6. *One may ask if there exist multiple Dirichlet series associated to Maass forms for $SL(n,\mathbb{Z})$ with $n \geq 4$ beyond the series of Bump-Friedberg type [Bump-Friedberg, 1990]?*

A multiple Dirichlet series is said to be perfect if it satisfies a finite group of functional equations and has meromorphic continuation in all its complex variables. The main object of this paper is to construct, by the employment of a simple integral operator, classes of perfect multiple Dirichlet series associated to Maass forms on $SL(n,\mathbb{Z})$. Our construction makes use of a rank lowering map which maps automorphic forms for $SL(n,\mathbb{Z})$ (with $n \geq 3$) into automorphic forms for $SL(n-1,\mathbb{Z})$, i.e., the rank is lowered by one. The rank lowering map is given by a single Mellin transform and a restriction of variables. It has the very interesting property that the cuspidal image of a Maass form f for $SL(n,\mathbb{Z})$ is an an infinite sum of Maass forms for $SL(n-1,\mathbb{Z})$ weighted by twists of L_f by the Godement-Jacquet L-functions of the Maass forms on $SL(n-1,\mathbb{Z})$. In theorem 2.4, the cuspidal projection of the rank lowering map is precisely computed. The rank lowering linear map also satisfies natural functional equations so iterations of this map can be used to construct perfect multiple Dirichlet series.

2. Rank lowering linear maps acting on automorphic forms.

Fix $n \geq 2$ and $s \in \mathbb{C}$ with $\Re(s)$ sufficiently large. We shall construct a rank lowering linear map (denoted $\mathcal{P}_s^n$) acting on automorphic forms for the group $SL(n+1,\mathbb{Z})$ and mapping them to automorphic forms for $SL(n,\mathbb{Z})$ which is a discrete group of lower rank.

Definition 2.1. (rank lowering linear map) *Fix $n \geq 2$ and $s \in \mathbb{C}$ with $\Re(s)$ sufficiently large. For f an automorphic form for $SL(n+1,\mathbb{Z})$, define*

$$P_s^n(f)(z_n) = \int_0^\infty f\left(\begin{pmatrix} z_n & \\ & 1\end{pmatrix}\begin{pmatrix} y_0 I_n & \\ & 1\end{pmatrix}\right)\cdot\Big(y_0^n\cdot \mathrm{Det}(z_n)\Big)^s\,\frac{dy_0}{y_0},$$

where I_n is the $n\times n$ identity matrix. It is assumed that f has sufficient decay properties so that the above integral converges absolutely and uniformly on compact subsets with $\Re(s)$ sufficiently large. This will be the case, for example, if f is a Maass form as in definition 1.1.

Fix a constant $D \gg 1$. An automorphic form $f(z_n)$ for $SL(n,\mathbb{Z})$ is said to be strongly L^2 if for any fixed subset

$$Y = \{y_{\ell_1}, y_{\ell_2}, \cdots y_{\ell_r}\} \qquad \big(\text{with } 1 \leq \ell_1 < \cdots < \ell_r \leq n-1\big),$$

or $Y =$ the empty set, we have

$$|f(z_n)| \ll \prod_{y_i \notin Y} y_i^{-N} \tag{2.2}$$

for all $N = 1, 2, 3, \ldots$, and all $z_n \in \mathfrak{h}_n$ satisfying the condition that $y_i > D$ for $y_i \notin Y$. Here, the $\ll$–constant in (2.2) depends at most on f, N, D, and the values of the fixed variables $0 < y_{\ell_i} \in Y$. Note that this constant may blow up as a particular $y_{\ell_i} \to 0$.

Since a fundamental domain for $SL(n, \mathbb{Z})\backslash\mathfrak{h}^n$ is contained in a Siegel set (see [Goldfeld, 2006]), it easily follows that strongly L^2 implies $\mathcal{L}^2$ in the usual sense.

The key properties of the rank lowering map are given in the next proposition.

Proposition 2.3. *Fix $n \geq 2$ and $s \in \mathbb{C}$ with $\Re(s)$ sufficiently large. Then the rank lowering linear map P_s^n (as given in definition 2.1) maps strongly L^2 automorphic forms for $SL(n+1, \mathbb{Z})$ to strongly L^2 automorphic forms for $SL(n, \mathbb{Z})$.*

Proof: First we show that $P_s^n(f)$ is well defined on $\mathfrak{h}^n$, i.e., that

$$P_s^n\big(f\big)(z_n \cdot k \cdot rI_n) = P_s^n\big(f\big)(z_n)$$

for all $k \in O(n, \mathbb{R})$ and all $r \in \mathbb{R}^\times$. Since $\left(\begin{smallmatrix} k & \\ & 1 \end{smallmatrix}\right) \in O(n+1, \mathbb{R})$ and f is right invariant by $O(n+1, \mathbb{R})$ it immediately follows that

$$P_s^n\big(f\big)(z_n \cdot k) = P_s^n\big(f\big)(z_n).$$

It remains to show that $P_s^n\big(f\big)$ is right invariant by rI_n for $r \in \mathbb{R}^\times$. Clearly, we may assume $r > 0$. The invariance follows from the computation:

$$\begin{aligned} P_s^n\big(f\big)(r \cdot z_n) &= \int_0^\infty f\left(\begin{pmatrix} z_n & \\ & 1 \end{pmatrix}\begin{pmatrix} ry_0 I_n & \\ & 1 \end{pmatrix}\right) \cdot r^{ns} \cdot \Big(y_0^n \cdot \mathrm{Det}(z_n)\Big)^s \, \frac{dy_0}{y_0} \\ &= P_s^n\big(f\big)(z_n), \end{aligned}$$

after making the transformation $y_0 \mapsto \frac{y_0}{r}$.

If $\gamma \in SL(n, \mathbb{Z})$ then $\left(\begin{smallmatrix} \gamma & \\ & 1 \end{smallmatrix}\right) \in SL(n+1, \mathbb{Z})$. Since f is automorphic for $SL(n+1, \mathbb{Z})$ and $\mathrm{Det}\left(\left(\begin{smallmatrix} \gamma & \\ & 1 \end{smallmatrix}\right)\right) = 1$, it immediately follows from definition 2.1 that

$$P_s^n\big(f\big)(\gamma z_n) = P_s^n\big(f\big)(z_n)$$

for all $\gamma \in SL(n, \mathbb{Z})$. This shows that $P_s^n\big(f\big)$ is automorphic for $SL(n, \mathbb{Z})$.

Next, we establish the strongly L^2 property. Fix a set $Y = \{y_{\ell_1}, \ldots, y_{\ell_r}\}$. It follows from the definition 2.1 and the bounds (2.2) that for fixed s with $\Re(s)$

sufficiently large and $N \gg n\Re(s)$ that

$$|P_s^n(f)(z_n)| \le \int_0^D \left| f\left(\begin{pmatrix} z_n & \\ & 1 \end{pmatrix} \begin{pmatrix} y_0 I_n & \\ & 1 \end{pmatrix} \right) \right| \Big(y_0^n \cdot \mathrm{Det}(z_n) \Big)^{\Re(s)} \frac{dy_0}{y_0} + \int_D^\infty \left| f\left(\begin{pmatrix} z_n & \\ & 1 \end{pmatrix} \begin{pmatrix} y_0 I_n & \\ & 1 \end{pmatrix} \right) \right| \Big(y_0^n \cdot \mathrm{Det}(z_n) \Big)^{\Re(s)} \frac{dy_0}{y_0}$$

$$\ll \left(D^n \cdot \mathrm{Det}(z_n)^{\Re(s)} + \int_D^\infty y_0^{n\Re(s)-N} \cdot \mathrm{Det}(z_n)^{\Re(s)} \frac{dy_0}{y_0} \right) \cdot \prod_{y_i \notin Y} y_i^{-N}$$

$$\ll \prod_{y_i \notin Y} y_i^{-N'}.$$

for $N' = N - n\Re(s)$. In the estimation of the first integral above, we have used the fact that since f is automorphic and is bounded on a fundamental domain it is bounded everywhere.

□

Next, we will apply the rank lowering operator to Maass forms. It is necessary to show that Maass forms are strongly L^2. This is a long tedious computation which we omit but is based on the fact that the bound (2.2) holds for Jacquet's Whittaker function. This was first established in [Jacquet-Piatetski-Shapiro-Shalika, 1979]. It is easy to show that (2.2) holds on $GL(2)$ by classical estimates for the K-Bessel function. In theorem 2.1 [Stade, 1990], it is proved that Jacquet's Whittaker function on $GL(n)$ can be written as an integral of the classical K-Bessel function multiplied by another Jacquet Whittaker function for $GL(n-2)$. The bound (2.2) can then be obtained by induction. Alternatively, it is also possible to obtain sharp bounds by the methods in §7 in [Jacquet, 2004].

The following theorem gives the contribution of the discrete spectrum to the rank lowering map defined on Maass forms. It shows that this contribution is a Rankin-Selberg convolution.

Theorem 2.4. *Fix $n \ge 2$ and $s \in \mathbb{C}$, sufficiently large. Let f be an even Maass form of type $\nu \in \mathbb{C}^n$ for $SL(n+1,\mathbb{Z})$. For every $(m_1, m_2, \dots m_n) \in \mathbb{Z}^n$, let $A(m_1, \dots, m_n)$ be the corresponding Fourier coefficient of f. Let ϕ be a Maass form of type $\lambda \in \mathbb{C}^{n-1}$ for $SL(n,\mathbb{Z})$. For every $(r_1, \dots r_{n-1}) \in \mathbb{Z}^{n-1}$, let $B(r_1, \dots, r_{n-1})$ be the corresponding Fourier coefficient of ϕ. Let $\langle\ ,\ \rangle$ denote the*

Petersson inner product on $SL(n,\mathbb{Z})\backslash\mathfrak{h}^n$. *Then*

$$\left\langle P_s^n(f),\ \phi\right\rangle = \sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\frac{A(m_1,\ldots,m_n)\overline{B(m_2,\ldots,m_n)}}{\prod\limits_{k=1}^{n} m_k^{(n+1-k)(s+\frac{1}{2})}}\cdot G_{\nu,\lambda}(s)$$

where

$$G_{\nu,\lambda}(s) = \int_0^\infty\cdots\int_0^\infty W_{Jacquet}\left(\begin{pmatrix} y_0y_1y_2\cdots y_{n-1} & & & & \\ & y_0y_1\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_0 & \\ & & & & 1\end{pmatrix}, \nu, \psi_{1,\ldots,1}\right)$$
$$\cdot\,\overline{W_{Jacquet}}\left(\begin{pmatrix} y_1y_2\cdots y_{n-1} & & & & \\ & y_1\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1\end{pmatrix}, \lambda, \psi_{1,\ldots,1}\right)\prod_{k=0}^{n-1} y_k^{(n-k)(s-k)}\prod_{i=0}^{n-1}\frac{dy_i}{y_i}.$$

Note that a similar theorem can be obtained for odd Maass forms.

Proof: To compute the inner product, we will use the Rankin-Selberg unfolding technique. The first step will be to write the Fourier expansion of $P_s^n(f)$ as a sum over the coset representatives of $U_n(\mathbb{Z})\backslash SL(n,\mathbb{Z})$. With this expansion, we can unfold the integral over $SL(n,\mathbb{Z})\backslash\mathfrak{h}^n$ to obtain an integral over $U_n(\mathbb{Z})\backslash\mathfrak{h}^n$. The computation goes as follows.

$$P_s^n(f)(z_n) = \sum_{m_1,..,m_n=1}^{\infty}\ \sum_{\gamma\in U_n(\mathbb{Z})\backslash SL(n,\mathbb{Z})}\frac{A(m_1,..,m_n)}{\prod\limits_{k=1}^{n} m_k^{\frac{(n+1-k)k}{2}}}\cdot$$
$$\cdot\int_0^\infty W_{\text{Jacquet}}\left(M\begin{pmatrix}\gamma z_n & \\ & 1\end{pmatrix}\begin{pmatrix}y_0 I_n & \\ & 1\end{pmatrix}, \nu, \psi_{1,\ldots,1}\right)\left(y_0^n\cdot\text{Det}(z_n)\right)^s\frac{dy_0}{y_0},$$

where M is as in definition 1.2.

Now let $z_n(\gamma)$ be the Iwasawa form of γz_n. There exists $k(\gamma)\in O(n,\mathbb{R})$ and $d(\gamma)\in\text{diag}(n,\mathbb{R})$ such that $\gamma z_n = z_n(\gamma)k(\gamma)d(\gamma)$. Note that taking determinants implies

$$\text{Det}(z_n) = |d(\gamma)|^n\,\text{Det}\big(z_n(\gamma)\big).$$

Hence

$$P_s^n(f)(z_n) \;=\; \sum_{m_1,..,m_n=1}^{\infty} \;\sum_{\gamma\in U_n(\mathbb{Z})\backslash SL(n,\mathbb{Z})} \frac{A(m_1,..,m_n)}{\prod\limits_{k=1}^{n} m_k^{\frac{(n+1-k)k}{2}}} \cdot$$

$$\cdot \int_0^\infty W_{\text{Jacquet}}\left(M \begin{pmatrix} z_n(\gamma) & \\ & 1 \end{pmatrix} \begin{pmatrix} y_0|d(\gamma)|I_n & \\ & 1 \end{pmatrix}, \nu, \psi_{1,\ldots,1} \right) \left(y_0^n \cdot \mathrm{Det}(z_n) \right)^s \frac{dy_0}{y_0}.$$

As before, we make the change of variables $y_0|d(\gamma)| \to y_0$. Consequently

$$P_s^n(f)(z_n) \;=\; \sum_{m_1,..,m_n=1}^{\infty} \;\sum_{\gamma\in U_n(\mathbb{Z})\backslash SL(n,\mathbb{Z})} \frac{A(m_1,..,m_n)}{\prod\limits_{k=1}^{n} m_k^{\frac{(n+1-k)k}{2}}} \cdot$$

$$\cdot \int_0^\infty W_{\text{Jacquet}}\left(M \begin{pmatrix} z_n(\gamma) & \\ & 1 \end{pmatrix} \begin{pmatrix} y_0 I_n & \\ & 1 \end{pmatrix}, \nu, \psi_{1,\ldots,1} \right) \left(y_0^n \, \mathrm{Det} z_n(\gamma) \right)^s \frac{dy_0}{y_0}.$$

When f is a Maass form for $SL(3,\mathbb{Z})$, for example, the above calculation gives $P_s^2(f)$ as follows:

$$P_s^2(f)(x+iy) \;=\; \sum_{m_1,m_2=1}^{\infty} \;\sum_{\gamma\in U_2(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \frac{A(m_1,m_2)}{m_1 m_2}$$

$$\cdot \int_0^\infty W_{\text{Jacquet}}\left(\begin{pmatrix} \frac{m_1 m_2 y_0 y}{|cz+d|^2} & & \\ & m_1 y_0 & \\ & & 1 \end{pmatrix}, \nu, \psi_{1,1,1} \right) e(m_2 \mathrm{Re}(\gamma z)) \frac{y^s}{|cz+d|^{2s}} y_0^{2s} \frac{dy_0}{y_0}.$$

The above calculations give another proof of the invariance of $P_s^n(f)$ under $SL(n,\mathbb{Z})$.

The inner product can now be evaluated by unfolding. We have

$$\Big\langle P_s^n(f),\ \phi\Big\rangle = \int\limits_{SL(n,\mathbb{Z})\backslash\mathfrak{h}^n} P_s^n(f)(z_n)\,\overline{\phi}(z_n)\,d^*z_n$$

$$= \sum_{m_1,..,m_n=1}^{\infty}\ \sum_{\gamma\in U_n(\mathbb{Z})\backslash SL(n,\mathbb{Z})} \frac{A(m_1,..,m_n)}{\prod\limits_{k=1}^{n} m_k^{\frac{(n+1-k)k}{2}}} \int\limits_{SL(n,\mathbb{Z})\backslash\mathfrak{h}^n}\int_0^\infty \overline{\phi}(z_n)$$

$$\cdot\, W_{\text{Jacquet}}\left(M\begin{pmatrix} z_n(\gamma) & \\ & 1\end{pmatrix}\begin{pmatrix} y_0 I_n & \\ & 1\end{pmatrix}, \nu, \psi_{1,\dots,1}\right)\Big(y_0^n\cdot \text{Det}\big(z_n(\gamma)\big)\Big)^s\, d^*z_n\,\frac{dy_0}{y_0}$$

$$= \sum_{m_1,..,m_n=1}^{\infty} \frac{A(m_1,..,m_n)}{\prod\limits_{k=1}^{n} m_k^{\frac{(n+1-k)k}{2}}} \int\limits_{U_n(\mathbb{Z})\backslash\mathfrak{h}^n}\int_0^\infty \overline{\phi}(z_n)$$

$$\cdot\, W_{\text{Jacquet}}\left(M\begin{pmatrix} z_n & \\ & 1\end{pmatrix}\begin{pmatrix} y_0 I_n & \\ & 1\end{pmatrix}, \nu, \psi_{1,\dots,1}\right)\Big(y_0^n\cdot \text{Det}(z_n)\Big)^s\, d^*z_n\,\frac{dy_0}{y_0}.$$

Consequently

$$\Big\langle P_s^n(f),\ \phi\Big\rangle = \sum_{m_1,..,m_n=1}^{\infty} \frac{A(m_1,..,m_n)}{\prod\limits_{k=1}^{n} m_k^{\frac{(n+1-k)k}{2}}} \int_0^1\cdots\int_0^1\int_0^\infty\cdots\int_0^\infty$$

$$\cdot\, W_{\text{Jacquet}}\left(M\begin{pmatrix} y_0\cdot y(n) & \\ & 1\end{pmatrix}, \nu, \psi_{1,\dots,1}\right)$$

$$\cdot\,\Big(\prod_{k=0}^{n-1} y_k^{n-k}\Big)^s\overline{\phi}(z_n)\,e(m_2x_1+\cdots+m_nx_{n-1})\,d^*x(n)\,d^*y(n)\,\frac{dy_0}{y_0}$$

$$= \sum_{m_1,..,m_n=1}^{\infty} \frac{A(m_1,..,m_n)\,\overline{B(m_2,..,m_n)}}{\prod\limits_{k=1}^{n} m_k^{\frac{(n+1-k)}{2}(2k-1)}} \int_0^\infty\cdots\int_0^\infty$$

$$\cdot\, W_{\text{Jacquet}}\left(M\begin{pmatrix} y_0\cdot y(n) & \\ & 1\end{pmatrix}, \nu, \psi_{1,\dots,1}\right)\Big(\prod_{k=0}^{n-1} y_k^{n-k}\Big)^s$$

$$\cdot\,\overline{W_{\text{Jacquet}}}\left(\begin{pmatrix} m_2..m_n y_1..y_{n-1} & & & \\ & \ddots & & \\ & & m_2 y_1 & \\ & & & 1\end{pmatrix}, \nu, \psi_{1,\dots,1}\right)\prod_{k=1}^{n-1} y_k^{-k(n-k)}\prod_{i=0}^{n-1}\frac{dy_i}{y_i}.$$

Now, in the above integral, make the change of variables

$$m_{i+1}y_i \to y_i$$

for each i from 0 to $n-1$. The theorem follows.

We now show that the rank lowering map applied to a Maass form satisfies natural functional equations. Recall the long element

$$w = \begin{pmatrix} & & & \pm 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}$$

of the Weyl group of $SL(n+1,\mathbb{Z})$.

Theorem 2.5. (Functional Equations) *Fix $n \geq 2$ and let $s \in \mathbb{C}$ with $\Re(s)$ sufficiently large. Let f be a Maass form of type $\nu \in \mathbb{C}^n$ for $SL(n+1,\mathbb{Z})$. Define the contragredient Maass form $\tilde{f}$ by $\tilde{f}(z) = f(w\,{}^t z^{-1} w)$. Then $P_s^n\big(f\big)$ has analytic continuation to all $s \in \mathbb{C}$ and satisfies the following functional equations:*

(i) $P_s^n(f)(z_n) \;=\; P_{-s}^n(\tilde{f})({}^t z_n^{-1})$

(ii) $P_{s_1}^{n-1}(P_{s_2}^n(f)) \;=\; P_{v_1}^{n-1}(P_{v_2}^n(f))$,

where $v_1 = \frac{1}{n}s_1 + \frac{n+1}{n}s_2$ and $v_2 = -\frac{1}{n}s_2 + \frac{n-1}{n}s_1$. By iterating the rank lowering map, perfect multiple Dirichlet series may be created.

Proof: We compute, for $\Re(s) \geq 1$,

$$P_s^n\big(f\big)(z_n) \;=\; \int_0^\infty f\left(\begin{pmatrix} z_n & \\ & 1\end{pmatrix}\begin{pmatrix} y_0 I_n & \\ & 1\end{pmatrix}\right) \cdot \Big(y_0^n \cdot \mathrm{Det}(z_n)\Big)^s \, \frac{dy_0}{y_0}$$

$$= \int_0^1 f\left(\begin{pmatrix} z_n & \\ & 1\end{pmatrix}\begin{pmatrix} y_0 I_n & \\ & 1\end{pmatrix}\right) \cdot \Big(y_0^n \cdot \mathrm{Det}(z_n)\Big)^s \, \frac{dy_0}{y_0}$$

$$+ \int_1^\infty f\left(\begin{pmatrix} z_n & \\ & 1\end{pmatrix}\begin{pmatrix} y_0 I_n & \\ & 1\end{pmatrix}\right) \cdot \Big(y_0^n \cdot \mathrm{Det}(z_n)\Big)^s \, \frac{dy_0}{y_0}$$

$$= \int_0^1 f\left(w\begin{pmatrix} z_n & \\ & 1\end{pmatrix}\begin{pmatrix} y_0 I_n & \\ & 1\end{pmatrix}\right)\cdot\left(y_0^n\cdot\mathrm{Det}(z_n)\right)^s\,\frac{dy_0}{y_0}$$
$$+\int_1^\infty f\left(\begin{pmatrix} z_n & \\ & 1\end{pmatrix}\begin{pmatrix} y_0 I_n & \\ & 1\end{pmatrix}\right)\cdot\left(y_0^n\cdot\mathrm{Det}(z_n)\right)^s\,\frac{dy_0}{y_0}$$

$$= \int_0^1 \tilde f\left(\begin{pmatrix} {}^t z_n^{-1} & \\ & 1\end{pmatrix}\begin{pmatrix} y_0^{-1} I_n & \\ & 1\end{pmatrix}\right)\cdot\left(y_0^n\cdot\mathrm{Det}(z_n)\right)^s\,\frac{dy_0}{y_0}$$
$$+\int_1^\infty f\left(\begin{pmatrix} z_n & \\ & 1\end{pmatrix}\begin{pmatrix} y_0 I_n & \\ & 1\end{pmatrix}\right)\cdot\left(y_0^n\cdot\mathrm{Det}(z_n)\right)^s\,\frac{dy_0}{y_0}.$$

In the first integral above, make the change of variables $\frac{1}{y_0}\to y_0$. Note that the second integral converges absolutely for any $s\in\mathbb{C}$ since the Maass cusp form f has rapid decay as $y_0\to\infty$. It follows that

$$P_s^n(f)(z_n) = \int_1^\infty \tilde f\left(\begin{pmatrix} {}^t z_n^{-1} & \\ & 1\end{pmatrix}\begin{pmatrix} y_0 I_n & \\ & 1\end{pmatrix}\right)\cdot\left(y_0^n\cdot\mathrm{Det}\left({}^t z_n^{-1}\right)\right)^{-s}\,\frac{dy_0}{y_0}$$
$$+\int_1^\infty f\left(\begin{pmatrix} z_n & \\ & 1\end{pmatrix}\begin{pmatrix} y_0 I_n & \\ & 1\end{pmatrix}\right)\cdot\left(y_0^n\cdot\mathrm{Det}(z_n)\right)^s\,\frac{dy_0}{y_0}. \tag{2.7}$$

Let τ_n denote the Iwasawa form of ${}^t z_n^{-1}$, so that

$$\tau_n = {}^t z_n^{-1}\, k(z_n)\, d(z_n),$$

for some

$$k(z_n)\in O(n,\mathbb{R})$$

and

$$d(z_n)\in \mathrm{diag}(n,\mathbb{R}).$$

By the Iwasawa decomposition, we have the following equivalence:

$$\begin{pmatrix} {}^t z_n^{-1} & \\ & 1\end{pmatrix}\begin{pmatrix} y_0 I_n & \\ & 1\end{pmatrix} \equiv \begin{pmatrix} \tau_n & \\ & 1\end{pmatrix}\begin{pmatrix} y_0 d(z_n) I_n & \\ & 1\end{pmatrix} \quad \mathrm{mod}\left(O(n+1,\mathbb{R})\mathrm{diag}(n+1,\mathbb{R})\right).$$

It is then clear from this representation that as $y_0\to\infty$, the contragrediant Maass form $\tilde f$ will have rapid decay.

Consequently, both the first and second integrals in (2.7) converge absolutely for any $s\in\mathbb{C}$ and define holomorphic functions on all of $\mathbb{C}$. The function

$P_s^n\big(f\big)(z_n)$ can then be defined for every $s \in \mathbb{C}$ by analytic continuation. It follows that for any $s \in \mathbb{C}$,

$$
\begin{aligned}
P_{-s}^n\big(\tilde{f}\big)({}^t z_n^{-1}) &= \int_1^\infty f\left(\begin{pmatrix} z_n & \\ & 1 \end{pmatrix}\begin{pmatrix} y_0 I_n & \\ & 1 \end{pmatrix}\right)\cdot\Big(y_0^n\cdot \mathrm{Det}(z_n)\Big)^s \frac{dy_0}{y_0} \\
&\qquad + \int_1^\infty \tilde{f}\left(\begin{pmatrix} {}^t z_n^{-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} y_0 I_n & \\ & 1 \end{pmatrix}\right)\cdot\Big(y_0^n\cdot \mathrm{Det}\left({}^t z_n^{-1}\right)\Big)^{-s} \frac{dy_0}{y_0}, \\
&= P_s^n\big(f\big)(z_n).
\end{aligned}
$$

The last line gives the functional equation $P_s^n\big(f\big)(z_n) = P_{-s}^n\big(\tilde{f}\big)({}^t z_n^{-1})$.

The second functional equation in theorem 2.5 does not involve the contragredient. We shall deduce it by a double iteration of the rank lowering linear map.

Let v_1, v_2 be as in the statement of theorem 2.5. We compute, for $\Re(s_1), \Re(s_2)$, $\Re(v_1), \Re(v_2)$ all simultaneously sufficiently large:

$$
\begin{aligned}
&P_{s_1}^{n-1}(P_{s_2}^n(f))(z_{n-1}) = \\
&\quad = \int_0^\infty P_{s_2}^n(f)\left(\begin{pmatrix} z_{n-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} y_0 I_{n-1} & \\ & 1 \end{pmatrix}\right)\cdot\Big(y_0^{n-1}\cdot \mathrm{Det}(z_{n-1})\Big)^{s_1} \frac{dy_0}{y_0} \\
&\quad = \int_0^\infty\int_0^\infty f\left(\begin{pmatrix} z_{n-1} & & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} y_0 I_{n-1} & & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} y_{-1} I_n & \\ & 1 \end{pmatrix}\right) \\
&\qquad\qquad \cdot\Big(y_{-1}^n\, y_0^{n-1}\,\mathrm{Det}(z_{n-1})\Big)^{s_2}\cdot\Big(y_0^{n-1}\cdot \mathrm{Det}(z_{n-1})\Big)^{s_1} \frac{dy_{-1}}{y_{-1}}\,\frac{dy_0}{y_0} \\
&\quad = \int_0^\infty\int_0^\infty f\left(\begin{pmatrix} z_{n-1} & & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} y_{-1}y_0 I_{n-1} & & \\ & y_{-1} & \\ & & 1 \end{pmatrix}\right) \\
&\qquad\qquad \cdot y_{-1}^{ns_2}\, y_0^{(n-1)(s_1+s_2)}\Big(\mathrm{Det}(z_{n-1})\Big)^{s_1+s_2} \frac{dy_{-1}}{y_{-1}}\,\frac{dy_0}{y_0}
\end{aligned}
$$

$$= \int_0^\infty \int_0^\infty f\left(\begin{pmatrix} I_{n-1} & & \\ & & 1 \\ & 1 & \end{pmatrix} \begin{pmatrix} z_{n-1} & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} y_{-1}y_0 I_{n-1} & & \\ & y_{-1} & \\ & & 1 \end{pmatrix}\right) \cdot y_{-1}^{ns_2}\, y_0^{(n-1)(s_1+s_2)} \Big(\mathrm{Det}(z_{n-1})\Big)^{s_1+s_2} \frac{dy_{-1}}{y_{-1}} \frac{dy_0}{y_0}$$

$$= \int_0^\infty \int_0^\infty f\left(\begin{pmatrix} z_{n-1} & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} I_{n-1} & & \\ & & 1 \\ & 1 & \end{pmatrix} \begin{pmatrix} y_{-1}y_0 I_{n-1} & & \\ & y_{-1} & \\ & & 1 \end{pmatrix}\right) \cdot y_{-1}^{ns_2}\, y_0^{(n-1)(s_1+s_2)} \Big(\mathrm{Det}(z_{n-1})\Big)^{s_1+s_2} \frac{dy_{-1}}{y_{-1}} \frac{dy_0}{y_0}$$

$$= \int_0^\infty \int_0^\infty f\left(\begin{pmatrix} z_{n-1} & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} y_{-1}y_0 I_{n-1} & & \\ & & 1 \\ & y_{-1} & \end{pmatrix} \begin{pmatrix} I_{n-1} & & \\ & & 1 \\ & 1 & \end{pmatrix}\right) \cdot y_{-1}^{ns_2}\, y_0^{(n-1)(s_1+s_2)} \Big(\mathrm{Det}(z_{n-1})\Big)^{s_1+s_2} \frac{dy_{-1}}{y_{-1}} \frac{dy_0}{y_0}$$

$$= \int_0^\infty \int_0^\infty f\left(\begin{pmatrix} z_{n-1} & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} y_{-1}y_0 I_{n-1} & & \\ & 1 & \\ & & y_{-1} \end{pmatrix}\right) \cdot y_{-1}^{ns_2}\, y_0^{(n-1)(s_1+s_2)} \Big(\mathrm{Det}(z_{n-1})\Big)^{s_1+s_2} \frac{dy_{-1}}{y_{-1}} \frac{dy_0}{y_0}.$$

It follows that

$$P_{s_1}^{n-1}(P_{s_2}^{n}(f))(z_{n-1}) = \int_0^\infty \int_0^\infty f\left(\begin{pmatrix} z_{n-1} & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 I_{n-1} & & \\ & \frac{1}{y_{-1}} & \\ & & 1 \end{pmatrix}\right) \cdot \cdot y_{-1}^{ns_2}\, y_0^{(n-1)(s_1+s_2)} \Big(\mathrm{Det}(z_{n-1})\Big)^{s_1+s_2} \frac{dy_{-1}}{y_{-1}} \frac{dy_0}{y_0}.$$

In the above integral, let's make the successive transformations: $y_{-1} \to \frac{1}{y_{-1}}$ and then $y_0 \to y_{-1}y_0$. We obtain

$$P^{n-1}_{s_1}(P^n_{s_2}(f))(z_{n-1}) = \int_0^\infty\int_0^\infty f\left(\begin{pmatrix} z_{n-1} & & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} y_{-1}y_0 I_{n-1} & & \\ & y_{-1} & \\ & & 1\end{pmatrix}\right) \cdot y_{-1}^{-ns_2}\,(y_{-1}y_0)^{(n-1)(s_1+s_2)}\Big(\mathrm{Det}(z_{n-1})\Big)^{s_1+s_2}\,\frac{dy_{-1}}{y_{-1}}\,\frac{dy_0}{y_0}$$

$$= \int_0^\infty\int_0^\infty f\left(\begin{pmatrix} z_{n-1} & & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} y_0 I_{n-1} & & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} y_{-1} I_n & \\ & 1\end{pmatrix}\right) \cdot \Big(y_{-1}^n\,y_0^{n-1}\,\mathrm{Det}(z_{n-1})\Big)^{v_2}\cdot\Big(y_0^{n-1}\,\mathrm{Det}(z_{n-1})\Big)^{v_1}\,\frac{dy_{-1}}{y_{-1}}\,\frac{dy_0}{y_0},$$

where v_1 and v_2 are as given in the statement of the theorem.

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