AUTOMORPHIC REPRESENTATIONS AND L-FUNCTIONS FOR GL(n)

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Contents

1. Introduction

Two of the main achievements of Hecke are the investigation of the L–function attached to a Grössencharacter and the L–function attached to a modular form. The modern view is that these are instances of the general notion of the L–function $L(s,\pi)$ attached to an automorphic representation π of the group $GL(n)$ over a number field F. The simplest method to obtain the analytic properties of this function is to imitate the construction of Tate in his thesis [34]. But we would like to stress that Hecke's method based on the Fourier expansion of modular

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forms gives the same result. Moreover Hecke's method generalizes to $GL(n)$.

Our goal in this note is to briefly review this method as explained in [16, 23, 24]. We refer to the book of Moeglin and Waldspurger [29] as a convenient reference for the general theory of automorphic forms. There is a huge literature on the subject (see the references in the above works). Here, in addition to the early work of Godement ([13],[14]) and Tamagawa [33] we quote the work of Maloletkin [28] who, like Godement, saw that in the Poisson formula, one can ignore the singular matrices when dealing with cusp-forms.

Langlands was aware of the possibility of defining an L–function this way, even before the full theory was available and alludes to it in his famous letter to Weil.

We cannot in this elementary paper get into the Langlands' program. But we can at least state one conjecture which is alluded to in the letter to Weil. Let E/F be an extension of number fields of degree n and let χ be an idele class character for E. Then the L–function $L(s,\chi)$ attached to χ is equal to the L–function $L(s,\pi)$ attached to an automorphic representation π for the group $GL(n,F)$. This representation π needs not be cuspidal but the L–function $L(s,\pi)$ may be written as a product of L–functions $L(s,\pi_i)$ where the π_i are cuspidal automorphic representations for various groups $GL(n_i,F)$. In particular, we can take χ to be the trivial character. Then $L(s,\chi)$ is the Dedekind zeta function of E/F.

2. Local non-Archimedean theory

2.1. **Smooth representations.** In this section F is a non Archimedean local field. We denote by ψ a non-trivial additive character of F. We let $\mathcal{O}_F$ be the ring of integers of F, and by q_F or simply q, we mean the cardinality of the residual field.

We let G be the group $GL(n)$ regarded as an algebraic group. For $g \in G(F)$ we define its norm

$$||g|| = \sup_{\substack{1\le i\le n\\ 1\le j\le n}} \left(\sup\ \left(|g_{ij}|,\ |(g^{-1}_{ij})|\right)\right).$$

We first describe the smooth representations of $G(F)$ (or more generally of $G(F)$ here G is a product of GL groups). A representation π of $G(F)$ on a complex vector V is said to be smooth if the stabilizer of every vector $v \neq 0$ in V is an open subgroup of $G(F)$. A smooth representation π of $G(F)$ on V is admissible if, conversely, the space

$V^{K'}$ of vectors fixed by a compact open subgroup K' of $G(F)$ is finite dimensional. If (π, V) is a smooth representation we denote by V^* the algebraic dual and by $\widetilde{V}$ the subspace of those vectors (linear forms) fixed by some compact open subgroup. We denote by $\widetilde{\pi}$ the representation of $G(F)$ on $\widetilde{V}$. We say that $(\widetilde{\pi}, \widetilde{V})$ is the representation contragredient to (π, V). A smooth representation is said to be irreducible if it is algebraically irreducible. Any irreducible smooth representation is admissible. More precisely, given an open compact subgroup K' there is a constant c such that for any irreducible representation (π, V) the dimension of $V^{K'}$ is bounded by c [3]. If π is an irreducible representation of $GL(n, F)$ then there is a character ω_π of $F^\times$ such that

$$\pi(zI_n) = \omega_\pi(z)$$

for all $z \in F^\times$. We call ω_π the central character of π. A function of the form

$$f(g) = \langle \pi(g)v, \widetilde{v} \rangle \, , \, v \in V \, , \, \widetilde{v} \in \widetilde{V},$$

is a matrix coefficient of π. Then the function $\check{f}$ defined by

$$\check{f}(g) = f(g^{-1})$$

is a matrix coefficient of $\widetilde{\pi}$.

If π is a unitary (topologically) irreducible representation of $G(F)$ on a Hilbert space H with scalar product $(\bullet, \bullet)$ then the space V of smooth vectors (i.e. fixed by some compact open subgroup of $G(F)$) is invariant under $G(F)$ and the representation (also noted π) of $G(F)$ on V is algebraically irreducible and admissible. The representation $\widetilde{\pi}$ is then the imaginary conjugate of π. In particular, for v_1, v_2 in V the function

$$f(g) = (\pi(g)v_1, v_2)$$

is a matrix coefficient of the admissible representation π. We say that an irreducible admissible representation is unitarizable if it is the space of smooth vectors in a topologically irreducible unitary representation of $G(F)$ on a Hilbert space.

We first review the definition of an induced representation. We let $P = P^{(m_1, m_2, \ldots, m_n)}$ be the upper parabolic subgroup of type $(m_1, m_2, \ldots, m_r)$

with $\sum\limits_{1\le i\le r} m_i = n$. This is the group of matrices of the form

$$p = \begin{pmatrix} g_1 & u_{12} & u_{13} & \cdots & u_{1j} & \cdots & u_{1r} \\ 0 & g_2 & u_{23} & \cdots & u_{2j} & \cdots & u_{2r} \\ 0 & 0 & g_3 & \cdots & u_{3j} & \cdots & u_{3r} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & g_{r-1} & u_{(r-1)r} \\ 0 & 0 & 0 & \cdots & \cdots & 0 & g_r \end{pmatrix},$$

where $g_i \in GL(m_i)$ and $u_{i,j}$ is a matrix with m_i rows and m_j columns. The unipotent radical $U = U^{(m_1,m,\ldots m_n)}$ is the group of matrices with $g_i = 1$ for all i. We let $M = M_{(m_1,m_2,\ldots m_n)}$ be the subgroup of matrices for which $u_{ij} = 0$ for all (i,j). So we have the Levi decomposition

$$P = MU\,,$$

and

$$G(F) = P(F)K = U(F)M(F)K\,,$$

where K is the standard maximal compact subgroup

$$K := GL(n, \mathcal{O}_F)\,.$$

Let π_i, $1 \le i \le r$, be an irreducible (or simply admissible) representation of $GL(r_i, F)$ on a complex vector space V_i. We set

$$V = \bigotimes_{1\le i\le r} V_i$$

and denote by $\sigma = \bigotimes \pi_i$ the tensor product representation of $M(F)$ on V.

We denote by δ_P the topological module of the locally compact group $P(F)$. Recall δ_P is trivial on $U(F)$ and is given on $M(F)$ by the formula

$$d(mum^{-1}) = \delta_P(m)du$$

where du denotes a Haar measure on $U(F)$. In general, we denote by $\rho(g)$ the right translation of a function ϕ by g:

$$\rho(g)\phi(h) = \phi(hg)\,.$$

The space of the corresponding induced representation

$$\mathrm{Ind}(G, P; \pi_1, \pi_2, \ldots, \pi_r)$$

is the space of functions

$$\phi : G(F) \to V$$

such that

$$\phi(mug) = \delta_P^{1/2}\sigma(m)\phi(g)$$

for all $g \in G(F), m \in M(F), u \in U(F)$ and there is a compact open subgroup $K' \subset K$ such that, for all $k \in K'$,

$$\rho(k')\phi = \phi\,.$$

The representation π of $G(F)$ on the induced representation is by right-shifts.
We can consider the representation

$$\mathrm{Ind}(G, P; \widetilde{\pi}_1, \widetilde{\pi}_2, \ldots, \widetilde{\pi}_r)$$

with $\widetilde{V} = \bigotimes \widetilde{V}_i$, $\widetilde{\sigma} = \bigotimes \widetilde{\pi}_i$. We have on $V \times \widetilde{V}$ the invariant scalar product

$$\left\langle \otimes\, v_i,\, \otimes \widetilde{v_i} \right\rangle = \prod_i \langle v_i, \widetilde{v_i} \rangle$$

so that $(\widetilde{V}, \widetilde{\sigma})$ is contragredient to (V, σ). It follows that for $\phi \in \mathrm{Ind}\big(G, P; \pi_1, \pi_2, \ldots, \pi_r\big)$ and $\widetilde{\phi} \in \mathrm{Ind}\big(G, P; \widetilde{\pi}_1, \widetilde{\pi}_2, \ldots, \widetilde{\pi}_r\big)$, we have

$$\left\langle \phi(pg), \widetilde{\phi}(pg) \right\rangle = \delta_P(p) \left\langle \phi(g), \widetilde{\phi(g)} \right\rangle.$$

Hence if we set

$$\left\langle \phi, \widetilde{\phi} \right\rangle := \int_K \left\langle \phi(k), \widetilde{\phi}(k) \right\rangle dk$$

we obtain an invariant non-degenerate scalar product and $\widetilde{\pi}$ is indeed contragredient to π.

An irreducible representation of $GL(n, F)$ is said to be supercuspidal if it is not a component of an induced representation. A character of $GL(1, F)$ is by definition a supercuspidal representation. For $n > 1$ a matrix coefficient f of a supercuspidal representation transforms under the central character ω_π of π, that is,

$$f(zg) = \omega_\pi(z) f(g)$$

for all $z \in Z(F)$ and all g. The function f is compactly supported modulo the center. Moreover, if U is the unipotent radical of a proper parabolic subgroup of G, then

$$\int_{U(F)} f(g_1 u g_2)\, du = 0\,.$$

Any irreducible representation π is a sub-representation of an induced representation

$$\mathrm{Ind}(G, P^{n_1, n_2, \ldots, n_r}; \sigma_1, \sigma_2, \ldots, \sigma_r)$$

where each σ_i is a supercuspidal representation of $GL(n_i, F)$.

2.2. **The Main Theorem.** Let π be an irreducible smooth representation of $GL(n,F)$. Let Φ be a Schwartz-Bruhat function on $M(n\times n,F)$ and f a matrix coefficient of π. We consider the integral

$$Z(\Phi,f,s) := \int_{GL(n,F)} \Phi(g)f(g)\,|\det g|^{s+\frac{n-1}{2}}\,dg\,.$$

We define the Fourier transform $\widehat{\Phi}$ of a Schwartz-Bruhat function Φ on $M(n\times n,F)$ by

$$\widehat{\Phi}(X) = \int_{M(n\times n,F)} \Phi(y)\psi(-\mathrm{tr}XY)dY\,.$$

The Haar measure dY is self-dual, that is, for all Φ,

$$\int \widehat{\Phi}(X)dX = \Phi(0)\,.$$

Recall the notation $\check{f}(g) := f(g^{-1})\,.$

Theorem 2.1. *Let the notations be as above.*

(i) *The integral defining $Z(\Phi,f,s)$ converges absolutely for* $\mathrm{Re}(s)$ *sufficiently large* ($\mathrm{Re}(s)>0$ *if π is tempered and* $\mathrm{Re}(s)>\frac{n-1}{2}$ *if π is unitary).*

(ii) *$Z(\Phi,f,s)$ is a rational function of q^{-s},q^{s}. More precisely the space spanned by these integrals is a fractional ideal of $\mathbb{C}[q^{-s},q^{s}]$ with a unique generator of the form*

$$L(s,\pi) = \frac{1}{P(q^{-s})}, \qquad \big(P\in\mathbb{C}[q^{-s}],\ P(0)=1\big).$$

(iii) *There is a functional equation*

$$Z(1-s,\widehat{\Phi},\check{f}) = \gamma\,(s,\pi,\psi)\,Z(f,\Phi,s)$$

where $\gamma\,(s,\pi,\psi)$ is rational. Furthermore,

$$\gamma\,(s,\pi,\psi) = \frac{\epsilon(s,\pi,\psi)L(1-s,\widetilde{\pi})}{L(s,\pi)}$$

where $\epsilon(s,\pi,\psi)$ has the form cq^{-ms}.

The factors $\epsilon(s,\pi,\psi)$ and $\gamma(s,\pi,\psi)$ depend on ψ. It is easily seen that if $\psi_a(x) := \psi(x)$ where $a\in F^{\times}$ then

$$\epsilon(s,\pi,\psi_a) = \omega_\pi(a)|s|^{n(s-\frac{1}{2})}\epsilon(s,\pi,\psi)\,,$$

where ω_π is the central character of π.

From its definition it is clear that the factor $\epsilon(s,\pi,\psi)$ is a monomial in q^{-s}.

If we apply the functional equation twice we find

$$\gamma(1-s,\widetilde{\pi},\overline{\psi})\cdot\gamma(s,\pi,\psi)=1,$$

or equivalently

$$\epsilon(1-s,\widetilde{\pi},\overline{\psi})\cdot\epsilon(s,\pi,\psi)=1\,.$$

In particular for $s=\frac{1}{2}$ we find

$$\epsilon\left(\frac{1}{2},\widetilde{\pi},\overline{\psi}\right)\epsilon\left(\frac{1}{2},\pi,\psi\right)=1\,.$$

If π is unitary we have

$$\epsilon\left(\frac{1}{2},\widetilde{\pi},\overline{\psi}\right)=\overline{\epsilon\left(\frac{1}{2},\pi,\psi\right)},$$

and so

$$\left|\epsilon\left(\frac{1}{2},\pi,\psi\right)\right|=1\,.$$

2.3. **Convergence.** We first prove (i) for tempered representations. By definition an irreducible representation π is tempered if its central character is unitary and if any matrix coefficient of π is bounded by a constant multiple of the function Ξ defined as follows. Let $B=AN$ be the group of upper triangular matrices and δ_B its module function of teh group B. Extend δ_B to be invariant under K on the right. Then

$$\Xi(g)=\int_K\delta_B^{1/2}(kg)dk\,.$$

Thus we only need to prove that an integral

$$\int_{G(F)}\Phi(g)\,\Xi(g)\,|\det g|^{s+\frac{n-1}{2}}dg$$

is absolutely convergent for $\mathrm{Re}(s)>0$. Now Φ is bounded in absolute value by a function of the form

$$X\mapsto c\,\Phi_0(\varpi^m X)\,,$$

where Φ_0 is the characteristic function of $M(n\times n,\mathcal{O}_F)$. So it suffices to prove that the integral

$$\int_{G(F)}\Phi_0(g)\,\Xi(g)\,|\det g|^{s+\frac{n-1}{2}}dg$$

is finite for $s>0$. Since Φ_0 is K–invariant on the left, this integral, finite or infinite, is equal to

$$\int_{G(F)}\Phi_0(g)\,\delta_B^{1/2}(g)\,|\det g|^{s+\frac{n-1}{2}}dg\,,$$

This integral is computed below in the subsection devoted to unramified representations and equal to

$$(1-q^{-s})^{-n}$$

which is finite for $s>0$.

If π is unitary then its matrix coefficients are uniformly bounded. As in the previous case we are reduced to show that the integral

$$\int_{G(F)} \Phi_0(g)\,|\det g|^{s+\frac{n-1}{2}}\,dg$$

is finite for $s>\frac{n-1}{2}$. This integral can be computed as in the subsection devoted to unramified representations and is equal to

$$\prod_{k=1}^{k=n} \frac{1}{1-q^{-(s-\frac{n}{2}+\frac{k}{2})}}\,.$$

Our assertion follows.

For a general representation π one can first prove that a matrix coefficient is majorized by a power of the norm and prove that an integral of the form

$$\int \Phi_0(g)\;||g||^m\;|\det g|^s dg$$

is finite for $s>>0$ or one can use the reduction step below.

For an arbitrary π, by taking suitable functions Φ with compact support contained in $G(F)$ we see we can choose Φ and f so that, for all s,

$$Z(\Phi,f,s)=1\,.$$

Also we have, for $\mathrm{Re}(s)>>0$ and $h\in G(F)$,

$$\int \Phi(gh)f(gh)\,|\det g|^{s+\frac{n-1}{2}}\,dg = |\det h|^{-s-\frac{n-1}{2}}\,Z(\Phi,f,s)\,.$$

This shows that if we prove that the integrals are rational functions of q^{-s} then the complete assertion (ii) follows.

Consider now the case where π is a supercuspidal representation of $GL(n,F)$, If $n=1$ this means that π is a one dimensional character and the result follows from Tate's thesis. If $n>1$ then the matrix coefficients of π are compactly supported modulo the center and this can be used to prove the convergence of the integral for $\mathrm{Re}(s)>>0$ ($\mathrm{Re}(s)>0$ if the central character is unitary) and also that the integrals are polynomials in q^{-s},q^s, in other words that $L(s,\pi)=1$. To prove the functional equation one can imitate Tate's argument.

One then uses a reduction step.

Lemma 2.2 (Reduction step). *Let $P = MU$ be a parabolic subgroup of type $(n_1, n_2, \dots, n_r)$. For each i let π_i be an irreducible admissible representation of $GL(n_i, F)$. Let π be the induced representation*

$$\pi = \mathrm{Ind}(G, P; \pi_1, \pi_2, \dots, \pi_r)\,.$$

Suppose the assertions of the theorem are true for each π_i.

(i) *Then they are true for any irreducible component σ of π.*

(ii) *Furthermore $\gamma\,(s, \sigma, \psi) = \prod_{1\le i\le r} \gamma\,(s, \pi_i, \psi)$*

(iii) $L(s, \sigma) = R_\sigma(q^{-s}) \prod_{1\le i\le r} L(s, \pi_i)$. *where R_σ is a polynomial and*

$$L(s, \widetilde{\sigma}) = \widetilde{R}_\sigma(q^{-s}) \prod_{1\le i\le r} L(s, \widetilde{\pi}_i),$$

where $\widetilde{R}_\sigma$ is the polynomial determined by

$$\widetilde{R}_\sigma(q^{-s}) = R_\sigma(q^{-1+s})\,.$$

(iv) If the induced representation is irreducible, so that σ is the induced representation, then $R_\sigma = 1$.

Since every irreducible admissible representation of $GL(n, F)$ is induced by supercuspidal representations the lemma follows.

The above lemma gives the factor $\gamma(s, \pi, \psi)$ for any irreducible representation π. If π is tempered then $L(s, \pi)$ and $L(s, \widetilde{\pi})$ are given by convergent integraks for $\mathrm{Re}(s) > 0$ thus are holomorphic for $\mathrm{Re}(s) > 0$. It follows that the fraction

$$\frac{L(1-s, \widetilde{\pi})}{L(s, \pi)}$$

is an irreducible fraction of the ring $\mathbb{C}[q^{-s}, q^s]$. This observation determines completely the factors $L(s, \pi)$ and $L(s, \widetilde{\pi})$.

For a complete computation of the L–factors see [24].

2.4. **Unramified representations.** Because of its importance, we discuss the case of representations which have a vector fixed under $K := GL(n, \mathcal{O}_F)$. We first observe that a supercuspidal representation (π, V) of $GL(n, F)$ (for $n > 1$) cannot have a non-zero vector fixed under K. Indeed, assume that π has such a vector v. Then the contragredient representation $(\widetilde{\pi}, \widetilde{V})$ has also a vector $\widetilde{v} \neq 0$ fixed under K. The matrix coefficient

$$f(g) = \langle \pi(g)v, \widetilde{v} \rangle$$

is bi-invariant under K, transforms under a character of $Z(F) = F^\times$ and is compactly supported modulo $Z(F)$. Moreover, because of the cuspidality, for all g

$$\int_{N(F)} f(ug)du = 0\,,$$

where we recall n is the group o upper triangular matrices with unit diagonal. By Satake lemma [30] this implies $f = 0$, a contradiction. This result extends to a Levi subgroup M (which is a product of linear groups). A supercuspidal representation σ of $M(F)$ can have a non-zero vector fixed under $K \cap M(F)$ only if M is a product of groups $GL(1)$, that is $M = A$, the group of diagonal matrices and σ is a product of characters of $GL(1)$.

Now consider a general unramified representation π. It is a subrepresetnation of an induced representation

$$\mathrm{Ind}(G, MU; \sigma)$$

where σ is a supercuspidal representation of $M(F)$. The representation σ must have a vector fixed under $K \cap M(F)$. Thus it must be that $M = A$ (group of diagonal matrices) and σ is a product of one dimensional unramified characters of $F^\times$. Hence π is an irreducible component of

$$\mathrm{Ind}(G, AN; \chi_1, \chi_2, \ldots, \chi_n)\,,$$

where each χ_i is an unramified character of $F^\times$. This representation may not be irreducible but it has a finite composition series. Since

$$G(F) = N(F)A(F)K$$

this representation has a unique irreducible component having a non-zero vector fixed under K. We denote it by $\pi(\chi_1, \chi_2, \ldots, \chi_n)$. We stress that it appears only once in the irreducible quotients of a composition series. If we permute the χ_i the character of the induced representation does not change so the irreducible components do not change and in particular the representation $\pi(\chi_1, \chi_2, \ldots, \chi_n)$ does not change. An unramified character like χ_i is determined by its value $z_i = \chi_i(\varpi)$ where ϖ is a uniformizer. So we see that $\pi(\chi_1, \chi_2, \ldots, \chi_n)$ is determined by the conjugacy class in $GL(n, \mathbb{C})$ of the matrix

$$A = \mathrm{diag}(z_1, z_2, \ldots, z_n)\,.$$

This is the **Langlands conjugacy class** of the representation π.

Lemma 2.3. *Let π be an irreducible representation with a fixed vector under K. Then*

$$L(s, \pi) = \det\left(1_n - Aq^{-s}\right)^{-1},$$

where A is its Langlands conjugacy class of π. If moreover the conductor of ψ is $\mathcal{O}_F$ then $\epsilon(s,\pi,\psi)=1$.

Proof. We have $\pi=\pi(\chi_1,\chi_2,\dots,\chi_n)$ for unramfied characters χ_i. Let ϕ be the element of

$$\mathrm{Ind}(G,AN;\chi_1,\chi_2,\dots,\chi_n)$$

taking the value 1 on K. Define similarly $\widetilde{\phi}$ for the representation

$$\mathrm{Ind}(G,AN;\chi_1^{-1},\chi_2^{-1},\dots,\chi_n^{-1})\,.$$

Then the function

$$f(g):=\int_K \phi(kg)\widetilde{\phi}(k)dk=\int_K \phi(kg)dk$$

is a matrix coefficient of $\pi=\pi(\chi_1,\chi_2,\dots,\chi_n)$. Let Φ be the characteristic function of $M(n\times n,\mathcal{O}_F)$. Then

$$\begin{aligned} Z(\Phi,f,s) &= \int_G \Phi(g)\,f(g)\,|\det g|^{s+\frac{n-1}{2}}dg \\ &= \int_{G(F)\times K} \Phi(g)\,\phi(kg)\,|\det g|^{s+\frac{n-1}{2}}dgdk. \end{aligned}$$

Since Φ is K–invariant, this reduces to

$$\int_{G(F)} \Phi(g)\,\phi(g)\,|\det g|^{s+\frac{n-1}{2}}dg\,.$$

Using the Iwasawa decomposition $G(F)=A(F)N(F)K$ this reduces at once to

$$\prod_{1\le i\le n}\int_{F^\times}\Phi_0(a_i)\chi_i(a_i)|a_i|^s\,d^\times a_i$$

where Φ_0 is the characteristic function of $\mathcal{O}_F$ in F. This is equal to

$$\prod_{1\le i\le n} L(s,\chi_i)=\det\left(1_n-Aq^s\right)^{-1}\;,$$

which is the first assertion.

For the second assertion we remark that under the assumption on ψ we have $\widehat{\Phi}=\Phi$. Hence

$$Z(\widehat{\Phi},\check{f},1-s)=\int \Phi(g)\check{f}(g)\,|\det g|^{1-s+\frac{n-1}{2}}dg\,.$$

Replacing f by $\check{f}$ amounts to exchange ϕ and $\widetilde{\phi}$. So this integral is equal to

$$\prod_{1\le i\le n} L(1-s,\chi^{-1})\,.$$

Hence $\epsilon(s,\pi,\psi)=1$. □

3. Local theory for $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$

In this section G denotes a product of groups $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$ regarded as a real lie group. We define the norm of an element of G. If $g \in GL(n,\mathbb{R})$ we set

$$||g|| = \sqrt{\sum_{i,j}\left(g_{ij}^2 + (g^{-1})_{ij}^2\right)}$$

We could also use the supremum of the absolute values of the entries of g and g^{-1}.
If $g \in GL(n,\mathbb{C})$ we set

$$||g|| = \sum_{i,j}\left(g_{ij}\overline{g_{i,j}} + (g^{-1})_{ij}\overline{(g^{-1})_{ij}}\right).$$

We could also use the supremum of the $g_{ij}\overline{g_{i,j}}$ and $(g^{-1})_{ij}\overline{(g^{-1})_{ij}}$.

The norm of an element of G is then the product of the norms of its components.

We denote by $\mathfrak{g}$ the Lie algebra of G and by $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. We also denote by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. The standard maximal compact subgroup of $GL(n,\mathbb{R})$ is the orthogonal group $\mathbf{O}(n)$ and the standard maximal subgroup of $GL(n,\mathbb{C})$ is the unitary group $\mathbf{U}(n)$. The standard maximal compact subgroup of G is the product of the standard maximal subgroups of the factors. It is noted K. We note that because G is contained in a product of groups $GL(n,\mathbb{C})$ the center $Z(\mathfrak{g})$ is equal to $Z_G(\mathfrak{g})$, the set of elements of $U(\mathfrak{g})$ fixed by the operators $\mathrm{Ad}g$, $g \in G$.

We assume the reader is familiar with the notion of $(\mathfrak{g},K)$ module. A $(\mathfrak{g},K)$ is said to be admissible if any irreducible representation of K appears with finite multiplicity. We denote by $\mathcal{H}$ the category of admissible, finitely generated $(\mathfrak{g},K)$ modules.

Lemma 3.1. (Harish Chandra) *Consider a $(\mathfrak{g},K)$-module V which is finitely generated. If V is annihilated by an ideal of finite codimension in $Z(\mathfrak{g})$ then V is admissible.*

Lemma 3.2. *Let σ be an irreducible representation of K. Then the multiplicity of σ in an irreducible admissible $(\mathfrak{g},K)$ module is bounded by the dimension of σ.*

Let (π_0,V_0) be an admissible finitely generated $(\mathfrak{g},K)$ module. Then there exists a locally convex complete topological vector space V and a continuous representation π of G on V such that V_0 can be identified with the space of K–finite vectors in V and π_0 is the corresponding representation of $(\mathfrak{g}, K)$. There are many choices for the topological vector space V. If (π, V_0) is an admissible algebraically irreducible $(\mathfrak{g}, K)$ module then for any choice of V the representation of G on V is topologically irreducible and the center of G operates by a scalar. So if $G = GL(n, \mathbb{R})$ or $Gl(n, \mathbb{C})$ we can define the central character ω_π. It depends only on V_0 and not on the choice of V.

Let $\mathcal{H}(G, K)$ be the convolution algebra of bi–K–finite smooth functions of compact support on G. Then the operators $\pi(f)$, $f \in \mathcal{H}(G, K)$ leave V_0 invariant. We have thus a representation of $\mathcal{H}(G, K)$ on V_0. This representation does not depend on V but only V_0. The algebra $\mathcal{H}(G, K)$ does not have a unity but it has an approximation of unity. In particular, given vectors $v_0, v_1, \dots v_n$ in V_0 there is $f \in \mathcal{H}(G, K)$ such that $\pi_0(f)v_i = v_i$ for all i. The following lemma (Harish-Chandra [20]) follows from the above considerations.

Lemma 3.3. *Let G be as above. Given $\mathbb{C}^\infty$ functions*

$$f_1, f_2, \dots, f_r : G \to \mathbb{C},$$

which are K-finite and annihilated by an ideal of finite codimension in $Z(\mathfrak{g})$, then there exists $h \in \mathbb{C}_c^\infty(G)$ such that

$$f_i = \rho(h) f_i, \qquad (\textit{for } i = 1, 2, \dots, r).$$

We recall a lemma of Dixmier Malliavin [9] which similarly can be used to show that some functions can be written as convolutions. Let again G be a Lie group, say a product of $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$. Let π be a unitary representation of G on a Hilbert space H. Then let V be the subspace of C^∞ vectors in H. The space V is equipped with the topology defined by the semi-norms $v \mapsto ||\pi(X)v||$ where X is in the enveloping algebra of the Lie algebra of G. It is complete for this topology. The group G operates on V.

Lemma 3.4. *Any vector $v \in V$ can be written as a finite sum*

$$v = \sum_{1 \le j \le r} \pi(f_j) v_j$$

where the vectors v_j are in V and the functions f_j are C^∞ functions of compact support on G.

Let again (π, V_0) be a finitely generated admissible $(\mathfrak{g}, K)$ module. There is a completion V of v_0 with the following properties (Casselman-Wallach, see [35],[2]). The space V is a Frechet space and the representation π of G be C^∞. This means that for each vector v the map

$v \mapsto \pi(g)v$ is C^∞. Finally, we demand for any continuous semi-norm λ on V there is another continuous semi-norm ν_λ and $m > 0$ such that

$$\lambda(\pi(g)v) \leq ||g||^m \nu_\lambda(v)$$

for all v and g. The representation (π, V) is uniquely determined by these conditions. We call it the canonical completion of (π_0, V_0).

Moreover, let $\widetilde{V}_0$ be the contragredient module: this is the vector space of K–finite linear forms on V_0. Let $\widetilde{V}$ be the canonical completion of $\widetilde{V}_0$. Then the natural bilinear form on $V_0 \times \widetilde{V}_0$ extends to a continuous, invariant bilinear form on $V \times \widetilde{V}$. Usually, this bilinear form is noted $\langle v, \widetilde{v}\rangle$. The functions

$$g \mapsto \langle \pi(g)v, \widetilde{v}\rangle$$

are the matrix coefficients of π.

Finally, let (π, H) be a unitary (topologically) irreducible representation of G on a Hilbert space H with Hermitian scalar product (v_1, v_2) and norm $||v|| = (v, v)^{\frac{1}{2}}$. Let H_K be the space of K–finite vectors. Every vector v in H_K is C^∞ so that $\mathfrak{g}$ operates on H_K and H_k is a $(\mathfrak{g}, K)$ module admissible and irreducible. Let V be the space of C^∞ vectors in H. The space V equipped with the topology defined by the semi-norms

$$v \to ||\pi(X)v||, \quad \big(X \in U(\mathfrak{g})\big),$$

is the canonical completion of the $(\mathfrak{g}, K)$ module H_K. The space $\widetilde{V}$ is simply the space imaginary conjugate of V, that is the same space, with the same addition and the same topology and scalar multiplication defined

$$\lambda._{\widetilde{V}} v = \overline{\lambda}._V v\,.$$

Thus a matrix coefficient of V have the form

$$g \mapsto (\pi(g)v_1, v_2), \qquad \big(v_1, v_2 \in H_K\big).$$

If the ground field is $\mathbb{R}$ we let ψ be a non-trivial additive character. We write ψ in the form $\psi(x) = \exp(2i\pi ax)$, $a \in \mathbb{R}^\times$. We denote by $\mathcal{S}_0(M(n \times n, \mathbb{R}))$ the subspace of $\mathcal{S}(M(n \times n, \mathbb{R}))$ spanned by the functions of the form

$$\Phi(X) = \exp\big(-\pi \mathrm{tr}({}^tXX)\big)\, P(X),$$

where P is a polynomial.

If the ground field is $\mathbb{C}$, we let ψ be a non-trivial additive character. We write ψ in the form $\psi(z) = \exp\left(2i\pi(az + \overline{az})\right)$, $a \in \mathbb{C}^\times$. We denote by $\mathcal{S}_0(M(n \times n, \mathbb{C}))$ the subspace of $\mathcal{S}(M(n \times n, \mathbb{C}))$ spanned by the functions of the form

$$\Phi(X) = \exp\big(-2\pi \mathrm{tr}({}^t\overline{X}X)\big)\, P(X, \overline{X}),$$

where P is a polynomial. Often, we write $\mathcal{S}$ and $\mathcal{S}_0$ for these spaces.

We define the Fourier transform

$$\widehat{\Phi}(X) = \int \Phi(Y)\psi(-\mathrm{Tr}XY)\,dY$$

of a function Φ. The Haar measure dX is self dual, that is, for all Φ,

$$\int \widehat{\Phi}(X)\,dX = \Phi(0)\,.$$

The space $\mathcal{S}$ (resp. $\mathcal{S}_0$) is invariant under the Fourier transform (resp. if $a = \pm 1$).

From now on we do not distinguish between a $(\mathfrak{g}, K)$ module and its canonical completion.

Theorem 3.5. *Let π be an irreducible $(\mathfrak{g}, K)$ and (π, V) its canonical completion. Let f be a smooth matrix coefficient of π and let $\Phi \in \mathcal{S}(M(n \times n, \mathbb{R}))$.*

(i) *The integral*

$$Z(\Phi, f, s) := \int_{GL(n,\mathbb{R})} \Phi(g)f(g)\,|\det g|^{s+\frac{n-1}{2}}\,dg$$

converges absolutely for $\mathrm{Re}(s) >> 0$ ($\mathrm{Re}(s) > 0$ if π is tempered and $\mathrm{Re}(s) > \frac{n-1}{2}$ if π is unitary).

(ii) *If $P(s)$ is any polynomial, then*

$$P(s)Z(\Phi, f, s) = \sum_{1\le i\le r} Z(\Phi_i, f_i, s)$$

for suitable Φ_i and f_i. If Φ is in $\mathcal{S}_0$ and f is bi–K–finite then one can take $\Phi_i \in \mathcal{S}_0$ and f_i bi–K–finite.

(iii) *The integrals $Z(\Phi, f, s)$ extend to meromorphic function of s. There is a meromorphic function $L(s, \pi)$ which never vanishes with the following properties. The integrals $Z(\Phi, f, s)$ are entire multiple of $L(s, \pi)$. If Φ is in $\mathcal{S}_0$ and f is K–finite then $Z(\Phi, f, s)$ is a polynomial multiple of $L(s, \pi)$. Conversely, if P is any polynomial then one can find $\Phi_i \in \mathcal{S}_0$ and K–finite coefficients f_i such that*

$$P(s)L(s, \pi) = \sum_i Z(\Phi_i, f_i, s)\,.$$

(iv) *As a meromorphic function of s, the integral $Z(\Phi, f, s)$ satisfies the functional equation:*

$$Z(\widehat{\Phi}, \check{f}, 1 - s) = \gamma\,(s, \sigma, \psi)\,Z(\Phi, f, s)\,,$$

where $\gamma(s,\sigma,\psi)$ is a suitable meromorphic function.
The factor γ has the form

$$\gamma(s,\sigma,\psi)=\frac{\epsilon(\pi,s,\psi)L(1-s,\widetilde{s})}{L(s,\pi)}$$

where $\epsilon(\pi,s,\psi)$ is an exponential function of s.

These conditions determine the factor $L(s,\pi)$ up to a scalar factor. It will turn out to be a product of Γ factors. In a vertical strip it has only finitely many poles.

We pass to the assertion (ii). We have a representation of $G(F)\times G(F)$ on $\mathcal{S}$, the action of (g_1,g_2) being given by

$$\lambda(g_1)\rho(g_2)\Phi[X]:=\Phi(g_1^{-1}Xg_2)\,.$$

This action is C^∞ so we have a corresponding action of $\mathfrak{g}\times\mathfrak{g}$. For instance, let $X\in\mathfrak{g}$. Then

$$\rho(X)\Phi(X)=\frac{d}{dt}\Phi[Xe^{tX}]\bigg|_{t=0}\,.$$

The space $\mathcal{S}_0$ is invariant under the action of $K\times K$ and its elements are $K\times K$ finite. Furthermore the space $\mathcal{S}_0$ is invariant under $\mathfrak{g}\times\mathfrak{g}$. Finally, if Φ is in $\mathcal{S}_0$ and f is a matrix coefficient, then

$$Z(\Phi,f,s)=Z(\Phi,f_0,s)$$

where f_0 is a bi–K–finite coefficient.

Now let $X\in\mathfrak{g}$. Then

$$Z(\Phi,\rho(X)f,s)$$

$$\begin{aligned}
&=\frac{d}{dt}\int\Phi(g)f(ge^{tX})\,|\det g|^{s+\frac{n-1}{2}}dg\bigg|_{t=0}\\
&=\frac{d}{dt}\int\Phi(ge^{-tX})f(g)\,\left|\det ge^{-tX}\right|^{s+\frac{n-1}{2}}dg\bigg|_{t=0}\\
&=-\int\rho(X)\,\Phi(g)f(g)\,|\det g|^{s+\frac{n-1}{2}}dg\;-\;\left(s+\frac{n-1}{2}\right)Z(\Phi,f,s)\,.
\end{aligned}$$

Assertion (ii) follows.

For $n=1$ our assertions are essentially contained in Tates' thesis. In the context of $(\mathfrak{g},K)$ modules or their canonical completion, we have a notion of induced representations which we take for granted. Then we have again a reduction step.

Lemma 3.6. *Let π_i, $1 \leq i \leq r$ be irreducible representations of $GL(n_i, F)$. Suppose the assertions of the theorem are true for each representation π_i. Let σ be an irreducible component of the induced representation*

$$\mathrm{Ind}(G, P; \pi_1, \pi_2, \ldots, \pi_r)\,.$$

(i) *The assertions of the theorem are true for the representation σ.*

(ii) *We have*

$$\gamma(s, \sigma, \psi) = \prod_{1 \leq i \leq r} \gamma(s, \pi_i, \psi)\,,$$

$$L(s, \sigma) = P(s) \prod_{1 \leq i \leq r} L(s, \pi_i)\,,$$

$$L(s, \widetilde{\sigma}) = \widetilde{P}(s) \prod_{1 \leq i \leq r} L(s, \widetilde{\pi}_i)\,,$$

where P and $\widetilde{P}$ are polynomials and

$$\widetilde{P}(s) = P(1-s)\,.$$

(iii) *If the induced representation is irreducible (and equal to σ) then $P = 1$.*

We use the reduction step in the following way. Let π be an irreducible admissible $(\mathfrak{g}, K)$ module (0r its canonical completion). Then there are n characters $\pi_i : F^\times \to \mathbb{C}$ with the following property. Consider the induced representation

$$\mathrm{Ind}(G, P; \pi_1, \pi_2, \ldots, \pi_n)\,,$$

where P is the group of upper triangular matrices. The space of this induced representation is the space of C^∞ functions

$$f : GL(n, F) \to \mathbb{C}$$

such that

$$f(nag) = \delta_P(a)^{1/2} \mu(a) f(g)\,,$$

where

$$\mu(a) = \mu_1(a_{1,1})\mu_2(a_{2,2}) \cdots \mu_n(a_{n,n})\,.$$

The canonical completion of π is a subquotient of this induced representation and the $(\mathfrak{g}, K)$ module π is then a subquotient of the $(\mathfrak{g}, K)$ module of K–finite functions in the induced representation.

One then proves that the integrals $Z(\Phi, f, s)$ for f a matrix coefficient of π extend to meromorphic functions which are entire multiples of

$$\prod_{i=1}^{n} L(s, \pi_i)$$

bounded at infinity in vertical strips. This space of meromorphic functions has a natural topology defined as follows. Consider a strip $A \leq \mathrm{Re}(s) \leq B$ and a polynomial $P(s)$ which cancel the poles of $\prod L(s, \mu_i)$ in the strip. We define then a semi-norm

$$\sup_{A \leq \mathrm{Re}(s) \leq B} |P(s) f(s)| \,.$$

The topology is then the one defined by these semi-norms. The map

$$\Phi \mapsto Z(\Phi, f, s)$$

is then continuous for this topology. If we write $f(g) = \langle \pi(g)v, \widetilde{v} \rangle$ the bilinear form

$$(v, \widetilde{v}) \mapsto Z(\Phi, f, s)$$

is also continuous. We have also the functional equation:

$$Z(\widehat{\Phi}, \check{f}, 1-s) = \prod_{i=1}^{n} \gamma(s, \mu_i, \psi) Z(\Phi, f, s) \,.$$

If we take f to be K–finite and $\Phi \in S_0$ then the integrals are polynomial multiples of $\prod_{i=1}^n L(s, \mu_i)$. The vector space spanned by these polynomials is an ideal with a generator P_π, well defined up to a scalar multiple. We set $L(s, \pi) = P_\pi(s) \prod_{i=1}^n L(s, \mu_i)$. So $L(s, \pi)$ is defined up to multiplication by a constant. We define similarly $L(s, \widetilde{\pi})$. We have also a functional equation. A density argument implies that the integrals $Z(\Phi, f, s)$ for Φ arbitrary are again holomorphic multiple of $L(s, \pi)$. Since given f and s_0 one can choose Φ of compact support on $GL(n, F)$ such that $Z(\Phi, f, s_0) \neq 0$ one concludes that $L(s_0, \pi) \neq 0$. In other words the zeroes of P_π must cancel poles of $\prod_{i=1}^n L(s, \mu_i)$. We have a similar polynomial $P_{\widetilde{\pi}}$ and the factor $L(s, \widetilde{\pi})$. Moreover, in the functional equation

$$\frac{Z(\widehat{\Phi}, \check{f}, 1-s)}{\prod L(1-s, \mu_i^{-1})} = \prod \epsilon(s, \mu_i, \psi) \frac{Z(\Phi, f, s)}{\prod L(s, \mu_i)}$$

the product of the epsilon factors is in fact a constant. If f is a K –finite matrix coefficient of π and Φ in $\mathcal{S}_0$ then the right hand side is a polynomial multiple of $P_\pi(s)$ and the left-hand side a polynomial multiple of $P_{\widetilde{\pi}}(1-s)$. We conclude that $P_{\widetilde{\pi}}(1-s) = c P_\pi(s)$ for a suitable constant c. Finally, we can write the functional equation in the form

$$Z(\widehat{\Phi}, \check{f}, 1-s) = \gamma(s, \pi, \psi) Z(\Phi, s, f)$$

where

$$\gamma(s, \pi, \psi) = \frac{\epsilon(s, \pi, \psi) L(1-s, \widetilde{\pi})}{L(s, \pi)} \,,$$

and $\epsilon(s,\pi,\psi)$ is an exponential function of s (in fact a constant with our choice of ψ).

In principle the above considerations determine the factor $\gamma(s,\pi,\psi)$. It remains to compute the factor $L(s,\pi)$. Thus we need to consider the case of a representation π of $GL(n,F)$, square integrable (modulo the center) other than a character of $GL(1)$ of module 1. Such a representation exists only if $F=\mathbb{R}$ and $n=2$. There exists 2 characters μ_1,μ_2 of $\mathbb{R}^\times$ such that π is a subrepresentation of the induced representation

$$\mathrm{Ind}(G,P;\mu_1,\mu_2)\,.$$

Since the representation π is tempered, the integrals $Z(\Phi,f,s)$ and $Z(\Phi,\check{k},s)$ (where f is a matrix coefficient of π) converge for $\mathrm{Re}(s)>0$. Thus the products

$$P_\pi(s)L(s,\mu_1)L(s,\mu_2)\,,\qquad P_{\widetilde{\pi}}(s)L(s,\mu_1^{-1})L(s,\mu_2^{-1})\,,$$

are holomorphic for $\mathrm{Re}(s)>0$. This added condition determines the polynomials and the factors $L(s,\pi)$, $L(s,\widetilde{\pi})$. See [24] for a computation of the L factor in all cases.

4. Tensor product of representations

Let F be number field and let G be the group $GL(n)$ regarded as an algebraic group over F. Let $\mathbb{A}$ be the ring of adeles of F. Let π be a (topologically) irreducible unitary representation of $G(\mathbb{A})$ on a Hilbert space H.

If v is a finite place we let $\mathcal{O}_v$ be the ring of integers of F_v and we set

$$K_v := GL(n,\mathcal{O}_v)\,.$$

If v is a real place we set $K_v:=\mathbf{O}(n)$. If v is a complex place we set $K_v:=\mathbf{U}(n)$. We set

$$K_\infty := \prod_{v\in\infty} K_v\,,\; K_f := \prod_{v\notin\infty} K_v\,,\; K := K_\infty\cdot K_f\,,$$

We can restrict this representation to the maximal compact subgroup K. This representation decomposes into a discrete sum of unitary irreducible representations of K. In particular, the space of K–finite vectors is dense in H. Consider an irreducible representation σ of K. Then there is a finite set of places S containing all the Archimedean places, for each $v\in S$ an irreducible representation σ_v of G_v such that σ is the tensor product of the σ_v, $v\in S$ and the trivial representation of $K^S:=\prod_{v\notin S}K_v$. It follows that the union of the closed subvector spaces H^{K^S} is dense in H. Consider one of them H^{K^S} say. Then the product group $G_S:=\prod_{v\in S}G_v$ leaves that space invariant and so does the Hecke algebra $\mathcal{H}^S$. Fix a vector $v_0 \neq 0 \in H^{K^D}$. If v is any other vector in the same space then v can be approached by vectors of the form

$$\sum_i c_i \pi_i(g_i)\pi(h_i)v_0$$

with c_i in $\mathbb{C}$, $g_i \in G_S$, $h_i \in G^s$. Since V and v_0 are in H^{K^S} we see that v can be approached by vectors the form

$$\sum_i c_i \pi_i(g_i) \int_{K^S} \pi(k)dk\pi(h_i) \int_{K^S} \pi(k)dkv_0$$

and $\int_{K^S} \pi(k)dk\pi(h_i) \int_{K^S} \pi(k)dk = \pi(\phi)$ for some ϕ in $\mathcal{H}^S$. So H^{K^S} must be irreducible under the action of G_S and $\mathcal{H}^S$.

But $\mathcal{H}^S$ is commutative and the operators $\pi(\phi)$, $\phi \in \mathcal{H}^S$ commute to the operators $\pi(g)$, $g \in G_S$. So the operators $\pi(\phi)$, $\phi \in \mathcal{H}^S$ must be scalars. It follows that the representation of G_S on H^{K^S} is topologically irreducible. Concretely because the local groups G_v are of type I the representation must be the tensor product $\bigotimes_{v\in S} \pi_v$ where the π_v are irreducible unitary representations. It $T \supset S$ then we get unitary irreducible representations π'_t, $t \in T$. For $s \in S$ we have $\pi_s \simeq \pi'_s$. For $t \in T-S$ the representation π_t; contains a unit vector e_t invariant under K_t and $\bigotimes_{t\in T} H'_t \simeq \bigotimes_{s\in S} H_s \bigotimes_{t\in T-S} e_t$. Finally, we have obtained for every place v a unitary irreducible representation (π_v, H_v). For almost all v. the space H_v contains a unit vector fixed under K_v (unique up to a scalar factor of module 1). If S is sufficiently large and $T \supset S$

$$\bigotimes_{t\in T} H_t \supset \bigotimes_{v\in S} H_v \bigotimes_{v\in T-S} e_t \simeq \bigotimes_{v\in S} H_v\,.$$

We can define the algebraic inductive limit of the spaces $\bigotimes_{v\in S} H_v$ and H is the completed space of the algebraic limit. In a more concrete way, choose for almost all places V a unit vector invariant under K_v. We have the pure tensor vectors

$$\bigotimes_{\text{all}\, v} u_v$$

with $u_v = e_v$ for almost all v. The linear span of the pure tensors is dense in H. The scalar product of two pure tensors is given

$$\left(\bigotimes_{\text{all}\, v} u_v, \bigotimes_{\text{all}\, v} u;_v \right) = \prod_{\text{all}\, v} (u_v, u'_v)$$

Concretely, the matrix coefficients $(\pi(g)u, u')$ for u and u' pure tensors are given by the infinite product:

$$(\pi(g)u, u') = \prod_{\text{all } v} (\pi_v(g_v)u_v, u'_v)\,.$$

For a given g almost all factors are equal to 1. This description applies to the space V of K–finite vectors. (see [15] for a discussion in the case of $GL(2)$). The space V is invariant and irreducible under the action of $(\mathfrak{g}, K_\infty)$ and $G(\mathbb{A}_f)$. It is also admissible in the sense that any irreducible representation of K appears with finite multiplicity.

More generally, consider a $(\mathfrak{g}_\infty, K_\infty) \times GL(n, \mathbb{A}_f)$ module (π, V). This means that V is a $(\mathfrak{g}_\infty, K_\infty)$ module and a $GL(n, \mathbb{A}_f)$ module and the actions commute. We assume that each vector in V is fixed under some compact open subgroup of $GL(n, \mathbb{A}_f)$. We say that V is admissible if each irreducible representation of $K_\infty \times K_f$ appears with finite multiplicity. We say that (π, V) is irreducible if there no non trivial invariant subspaces. Then π is isomorphic to a restricted infinite product

$$\bigotimes_v (\pi_v, V_v)\,.$$

For v infinite (π_v, V_v) is an irreducible $(\mathfrak{g}_v, K_v)$. For v finite (π_v, V_v) is an irreducible (admissible) representation of $GL(n, F_v)$. For almost all finite v the vector space contains a non-zero vector e_v fixed by $K_v := GL(n, \mathcal{O}_v)$. We have a similar description of the contragredient representation $(\widetilde{\pi}, \widetilde{V})$ as the infinite tensor product $\bigotimes \widetilde{\pi}_v, \bigotimes V_v)$. One can choose $\widetilde{e}_v$ to be such that $\langle e_v, \widetilde{e}_v \rangle = 1$. See [22] and [18] for a detailed discussion in the case of $GL(2)$ and [10] for the general case.

5. Reduction theory for $GL(n)$

Let F be a number field and $\mathbb{A}_F$ or simply $\mathbb{A}$ its ring of adeles. We denote by $\mathbb{A}^\times_{>0}$ the group of ideles whose finite components are 1 and whose infinite components are all equal to the same positive number. We can identify this group with $\mathbb{R}_{>0}$. Now let

$$x = (t, t, \dots, t, 1, 1, \dots) \in \mathbb{A}^\times_{>0}, \qquad (t \in \mathbb{R}_{>0}).$$

We must remember that if v is a real place then $|x_v|_v = t$ and if v is a complex place then $|x_v|_v = t^2$. In particular,

$$|x| = t^{r+2c},$$

where r is the number of real places, c the number of complex places. Define $\mathbb{A}^1$ to be the group of ideles of norm one, and let $|x|$ be the usual absolute value on $\mathbb{A}$. The we have the decomposition

$$\mathbb{A}^\times = \mathbb{A}^\times_{>0} \cdot \mathbb{A}^1.$$

Now $F^\times \subset \mathbb{A}^1$ and $\mathbb{A}^1/F^\times$ is compact (reduction theory for $GL(1)$).

Let $G = GL(n)$ regarded as an algebraic group over F. We let G^1 be the set of $g \in G(\mathbb{A})$ such that $|\det g| = 1$. We have, of course, $G(F) \subset G^1$. We let Z be the center of G. We define $Z_{>0} \subset Z(\mathbb{A})$ as the subgroup of elements whose entries are in $\mathbb{A}^\times_{>0}$. We have (direct product)

$$G(\mathbb{A}) = Z_{>0} \cdot G^1 .$$

Let A be the group of diagonal matrices, regarded as an algebraic group. Let A^1 be the subgroup of elements in $A(\mathbb{A})$ whose entries have absolute value 1 and let $A_{>0}$ be the subgroup of elements whose entries are in $\mathbb{A}^\times_{>0}$. We have

$$A(\mathbb{A}) = A_{>0} \cdot A^1 .$$

Finally we let A_1 be the subgroup of elements of $A_{>0}$ whose determinant is 1. Then we have

$$A_{>0} = Z_{>0} \cdot A_1$$

and

$$A(\mathbb{A}) = Z_{>0} \cdot A_1 \cdot A^1$$

as well as

$$A(\mathbb{A}) \cap G^1 = A_1 \cdot A^1 .$$

We let N be the group of upper triangular matrices with unit diagonal.

We have the Iwasawa decomposition

$$G(\mathbb{A}) = N(\mathbb{A}) \cdot A(\mathbb{A}) \cdot K$$

and

$$G^1 = N(\mathbb{A}) \cdot A_1 \cdot A^1 \cdot K .$$

Recall that $N(F)\backslash N(\mathbb{A})$ and $A(F)\backslash A^1$ are compact. We also recall the simple roots:

$$\alpha_i : A \to GL(1)$$

defined by

$$\alpha_i(a) := a_{i,i}/a_{i+1,i+1}.$$

If $t > 0$ we denote by $A(t)$ the subset of elements a of A_1 satisfying

$$|\alpha_i(a)| \geq t, \qquad 1 \leq i \leq n-1 .$$

Let Ω_N be a compact subset of $N(\mathbb{A})$ and let Ω_A be a compact subset of A^1 and let $t > 0$. We denote by $\mathfrak{S}_{t,\Omega_N,\Omega_A}$ the set of $g \in G^1$ of the form

$$g = \omega_N \cdot a \cdot \omega_A \cdot k$$

with $a \in A(t)$, $\omega_N \in \Omega_N$, $\omega_A \in \Omega_A$, $k \in K$. Such a set is called a Siegel set. It is elementary that a Siegel set has finite volume for the Haar measure of G^1. Moreover, if g is as above then $a^{-1} \cdot \omega_N \cdot a$ remain in a compact set of $N(\mathbb{A})$ so that

$$g = a \cdot \omega_G \,,$$

where ω_G remain in a compact set Ω_G of G^1.

The basic result of reduction theory is as follows (see [12]):

Theorem 5.1. *For any Siegel set $\mathfrak{S}$ the set*

$$X_{\mathfrak{S}} := \{\gamma \in G(F) \ \mid \gamma\mathfrak{S} \cap \mathfrak{S} \neq \emptyset\}$$

is finite.
There is a Siegel set $\mathfrak{S}$ such that

$$G^1 = G(F)\,\mathfrak{S} \,.$$

As a consequence we see the volume of $G(F)\backslash G^1$ is finite. More precisely we have the following result.

Theorem 5.2. *There is a Siegel set $\mathfrak{S}$ such that*

$$\mathrm{Vol}(G(F)\backslash G^1) \leq \mathrm{Vol}(\mathfrak{S}) \,.$$

For any Siegel set $\mathfrak{S}$, there is a constant c such that

$$\mathrm{Vol}(\mathfrak{S}) \leq c\mathrm{Vol}(G(F)\backslash G^1) \,.$$

Proof. Since we may always replace a Siegel set by a larger one, it suffices to consider a Siegel set $\mathfrak{S}$ such that $G^1 = G(F)\mathfrak{S}$. Let $\mathfrak{S}'$ be a measurable section of $G(F)\backslash G^1$ contained in $\mathfrak{S}$. We have then (disjoint union):

$$G^1 = \bigsqcup_{\gamma \in G(F)} \gamma\mathfrak{S}'$$

and (finite disjoint union)

$$\mathfrak{S} \subset \bigsqcup_{\gamma \in X_{\mathfrak{S}}} \gamma\mathfrak{S}' \,.$$

Let c be the cardinality of X_σ. Then

$$\mathrm{Vol}(G(F)\backslash G^1) = \mathrm{Vol}(\mathfrak{S}') \leq \mathrm{Vol}(\mathfrak{S}) \leq c\mathrm{Vol}(\mathfrak{S}') = c\mathrm{Vol}(G(F)\backslash G^1) \,.$$

□

We have also the weak approximation theorem.

Theorem 5.3. *Let K' be an open compact subgroup of K_f. Then there are finitely many elements c_i, $1 \leq i \leq r$, of $G(\mathbb{A}_f)$ such that we have a disjoint union*

$$G(\mathbb{A}) = \bigcup_{1\leq i\leq r} G(F)G_\infty c_i K' .$$

Finally, for $g \in G(\mathbb{A})$, we let $||g|| = \prod_v |||g_v||_v$ denote the norm of the element g.

Lemma 5.4. *Given a Siegel set $\mathfrak{S}$ then for every $g \in \mathfrak{S}$ we have*

$$||g|| \asymp \inf_{\gamma\in G(F)} ||\gamma g|| .$$

Proof. Of course we have, for all $g \in G(\mathbb{A})$,

$$\inf_{\gamma\in G(F)} ||\gamma g|| \leq ||g|| .$$

We prove an inequality in the reverse direction for g in a Siegel set. If g is in a Siegel set then it has the form

$$g = a\omega$$

where ω is in a compact set and $a \in A(t)$. Thus it suffices to prove that there is a constant c such that

$$||a|| \leq c \cdot ||\gamma a||$$

for all $\gamma \in G(F)$ and $a \in A_{>0}$. At this point we may use the supremum norm at each infinite place, and for any place v, we adopt the convention that for an adele $a \in \mathbb{A}$

$$|a|_v := |a_v|_v.$$

Similarly, for $g \in G(\mathbb{A})$ and any place v, we set

$$||g||_v := ||g_v||_v .$$

We have now for any index j

$$a_{j,j} = (t_j, t_j, \ldots, t_j, 1, 1, \ldots, 1 \ldots) \;\; (t_j > 0)$$

and

$$||a|| = \left(\sup_j \sup(t_j, t_j^{-1})\right)^r \cdot \left(\sup_j \sup(t_j^2, t_j^{-2})\right)^c$$

Let j be an index. There is an index i such that $\gamma_{i,j} \neq 0$ (i-th row and j-th column) For a real place v

$$t_j|\gamma_{i,j}|_v = |\gamma_{i,j}a_{j,j}| \leq ||\gamma a||_v .$$

For v a complex place, we have

$$t_j^2|\gamma_{i,j}|_v = |\gamma_{i,j}a_{j,j}|_v \leq ||\gamma a||_v .$$

For v finite, we have

$$|\gamma_{i,j}|_v = |\gamma_{i,j}a_{j,j}|_v \leq ||\gamma a||_v \,.$$

Taking the product of these inequalities we get

$$t_j^{r+2c} \leq ||\gamma a|| \,.$$

Similarly,

$$t_j^{-r-2c} \leq ||a^{-1}\gamma^{-1}|| = ||\gamma a|| \,.$$

So we get

$$||a|| \leq ||\gamma a|| \,.$$

We are done. □

Thus to check that a function ϕ on G^1 invariant on the left under $G(F)$ is of moderate growth, that is bounded by a constant multiple of the power of the norm, it suffices to check it is of moderate growth on a Siegel set.

For $a \in A(\mathbb{A})$ define

$$\beta(a) := \prod_{1\leq i\leq n-1} \big|\alpha_i(a)\big|.$$

Lemma 5.5. *Given $t > 0$,there exist $c_1, c_2, m_1, m_2 > 0$ such that*

$$c_1\beta(a)^{m_1} \leq ||a|| \leq c_2\beta(a)^{m_2}$$

for all $a \in A(t)$.

Proof. This follows from the fact that

$$a_{1,1}^n = \prod_{i=1}^{n-1} \alpha_i(a)^{n_i}$$

with $n_i, (i = 1, \ldots, n-1)$ positive integers and for $j \geq 2$

$$a_{j,j} = a_{1,1} \prod_{i=1}^{n-1} \alpha_i(a)^{-u_i},$$

where $u_i \geq 0$ are integers.

□

Now consider a Siegel set $\mathfrak{S}_{t,\Omega_N,\Omega_A}$. Since Ω_N, Ω_A, K are compact sets, it follows that for $\omega_N \in \Omega_N$, $\omega_A \in \Omega_A$, $a \in A(t)$, and $k \in K$,

$$||\omega_N a\omega_A k|| \asymp ||a||.$$

It immediately follows from Lemma 5.5 that on the Siegel set $\mathfrak{S}_{t,\Omega_N,\Omega_A}$ we have

$$\beta(a)^{m_1} \ll ||\omega_N a\omega_A k|| \ll \beta(a)^{m_2} \,.$$

Finally, we see that a function ϕ on G^1 invariant on the left under $G(F)$ is of moderate growth if and only if for every $t > 0$, there is a constant m such that for every compact set Ω, there is a constant c with

$$|\phi(a\omega)| \leq c\beta(a)^m$$

for $a \in A(t)$ and $\omega \in \Omega$.

Another application is the following Lemma.

Lemma 5.6. *Let C be a compact subset of $G(\mathbb{A})$. Then there is $c > 0$ and $m > 0$ such that, for all x in $G(\mathbb{A})$ the cardinality of the set*

$$G(F) \cap xCx^{-1}$$

is bounded by $c||x||^m$.

Proof. Let Ω be a compact subset of $G(\mathbb{A}_f)$ and $t > 0$. Let $G_{t,\Omega}$ be the set of $g \in G_\infty\Omega$ such that $||g|| \leq t$. It easy to see that the volume of $G_{t,\Omega}$ for a Haar measure of $G(\mathbb{A})$ is bounded by ct^m for suitable $c > 0$ and $m > 0$.

Now, as a function of x the cardinality of the set $G(F) \cap xCx^{-1}$ is invariant under $G(F)$ on the left. Thus, to prove our contention we have assume that x is in a Siegel set, and a fortiori, in the set $G_\infty C'$ where C' is a compact set of $G(\mathbb{A}_f)$. Replacing the set C by the set $C'CC'^{-1}$ we see we may assume that x is in G_∞. For $\gamma \in G(F) \cap xCx^{-1}$ we have

$$||\gamma|| \leq c||x|| \cdot ||x^{-1}|| = c||x||^2 .$$

Now let V be a compact neighborhood of 1 in $G(\mathbb{A})$ such that

$$G(F) \cap (V \cdot V^{-1}) = \{1\} .$$

For $v \in V$ and $\gamma \in G(F) \cap xCx^{-1}$ we have

$$||v\gamma|| \leq c_1||\gamma|| \leq c_2||x||^2 .$$

On the other hand,

$$v\gamma \in xCx^{-1}V .$$

Since C and V are contained in the product of a compact set of G_∞ and a compact set of $G(\mathbb{A}_f)$ the set $xCx^{-1}V$ is contained in a product $G_\infty\Omega$ where Ω is a compact set of $G(\mathbb{A}_f)$. We see that if V is as above and γ in the intersection then

$$\gamma V \subset G_{c_2||x||^2,\Omega} .$$

The disjoint union

$$\cup_{\gamma \in G(F) \cap xCx^{-1}} \gamma V$$

is contained in

$$G_{c_2||x||^2,\Omega}$$

Thus

$$\mathrm{Vol}\,(\cup\gamma V) \leq \mathrm{Vol}(G_{c_2||x||^2,\Omega}) \leq c||x||^{2m}\,.$$

But the volume on the left is Vol(V) times the cardinality we are trying to bound. □

Lemma 5.7. *Let $x, y \in G(\mathbb{A})$ and C a compact set of $G(\mathbb{A})$. Then the cardinality of the set*

$$G(F) \cap (xCy)$$

is bounded by $c||x||^m$ for suitable $c > 0$ and $m > 0$.

Proof. Fix an element δ in the set in question. Then for any other element γ we have

$$\delta^{-1}\gamma \in xCC^{-1}x^{-1}\,.$$

Since CC^{-1} is a compact set, it suffices to apply the previous lemma. □

On the group $G(\mathbb{A})$ (and in general for any group) we denote by $\rho(x)$ the right translation by x:

$$\rho(x)\phi(h) = \phi(hx)\,.$$

Moreover, if f is a function on $G(\mathbb{A})$, we set

$$\rho(f)\phi(h) = \int_{G(\mathbb{A})} \phi(hg)dh\,,$$

where dx is a Haar measure on $G(\mathbb{A})$.

Lemma 5.8. *Suppose f is a continuous function of compact support on $G(\mathbb{A})$. Then there are constants $c > 0$ and $m > 0$ such that, for any $\phi \in L^2(Z_{>0}G(F)\backslash G(\mathbb{A}))$ and every $x \in G(\mathbb{A})$,*

$$|\rho(f)\phi(x)| \;\leq\; c||x||^m||\,\phi||_2\,.$$

Proof. Set

$$f_1(g) := \int_{Z_{>0}} f(zg)dz\,.$$

Then f_1 is a continuous function of compact support on $Z_{>0}\backslash G(\mathbb{A})$. We have

$$\begin{aligned}\rho(f)\phi(g) &= \int_{Z_{>0}\backslash G(\mathbb{A})} \phi(hg)f_1(h)dh \\ &= \int_{Z_{>0}\backslash G(\mathbb{A})} \phi(h)f_1(g^{-1}h)dh \\ &= \int_{Z_{>0}G(F)\backslash G(\mathbb{A})} \phi(h) \sum_{\gamma\in G(F)} f_1(g^{-1}\gamma h)dh\,.\end{aligned}$$

But the inner integral is bounded in absolute value by $\sup|f_1|$ times the cardinality of the set

$$\left\{\gamma \in G(F) \,\middle|\, g^{-1}\gamma h \in \Omega\right\}$$

where Ω is the support of f_1. Since this is also the set

$$G(F) \cap g\Omega h^{-1}$$

we can apply the previous lemma. We find

$$|\rho(f)\phi(g)| \leq c||x||^m \int |\phi(h)|dh \leq c||x||^m ||\phi||_2 \mathrm{Vol}(Z_{>0}G(F)\backslash G(\mathbb{A}))\,.$$

□

6. Definition of automorphic forms

In the adelic setting of $G(\mathbb{A})$ an automorphic form is a function

$$\phi : G(\mathbb{A}) \to \mathbb{C}$$

invariant under $G(F)$ on the left and K–finite on the right. Further we demand that ϕ be C^∞ and $Z(\mathfrak{g})$-finite. Finally we demand that ϕ be of moderate growth, that is,

$$|\phi(g)| \leq c||g||^M$$

for some $c > 0$ and some $M > 0$.

Since

$$0 \leq ||g_1 g_2|| \leq ||g_1|| \cdot ||g_2||$$

for all $g_1, g_2 \in G(\mathbb{A})$, the right translates of ϕ or the convolution of ϕ on the right with a smooth function of compact support are of moderate growth with the same exponent m.

An automorphic form ϕ is thus annihilated by an ideal $\mathfrak{i}$ of $Z(\mathfrak{g})$ of finite codimension and the space V_0 of its right translates by K is finite dimesional. The K–type of ϕ is the set θ of irreducible representations of K which appears in V_0. The pair $(\mathfrak{i}, \theta)$ is the type of ϕ.

Lemma 6.1. *Suppose ϕ is an automorphic form. Then there is a smooth function of compact support f on $G(\mathbb{A})$ such that*

$$\rho(f)\phi = \phi\,.$$

Proof. Indeed, the theorem of weak approximation asserts that we have a finite disjoint union

$$G(\mathbb{A}) = \bigcup_{1 \leq i \leq r} G(F) \cdot G_\infty \cdot g_i \cdot K'$$

with $g_i \in G(\mathbb{A}_f)$. We apply lemma 3.3 to the functions

$$g_\infty \mapsto \phi(g_\infty c_i)\,.$$

Thus there is a C^∞ function of compact support f_∞ on G_∞ such that

$$\int_{G^\infty} \phi(g_\infty h c_i) f_\infty(h) dh = \phi(g_\infty c_i)$$

for all i. Now define a function f on $G(\mathbb{A})$ by

$$f(g_\infty g_f) = \begin{cases} \frac{1}{\mathrm{Vol}(K')} f_\infty(g_\infty) & \text{if} \quad g_f \in K', \\ 0 & \text{if} \quad g_f \notin K'. \end{cases}$$

We claim that

$$\int_{G((A)} \phi(gh) f(h) dh = \phi(g)$$

for all $g \in G(\mathbb{A})$. Since the functions of g on the left hand side and the right hand side are invariant under K' on the right it suffices to check this relation for $g = g_\infty c_i$ for some i. But then it reduces to

$$\int_{G_\infty} \phi(g_\infty h_\infty c_i) f_\infty(h_\infty) dh \infty = \phi(g_\infty c_i)$$

which is true by the choice of f_∞. □

By Lemma 6.1 there is a smooth function of compact support f on $G(\mathbb{A})$ such that $\phi = \rho(f)\phi$. Then for every $X \in U(\mathfrak{g})$ we have

$$\rho(X)\phi = \rho(\rho(X)f)\phi$$

and so $\rho(X)\phi$ is still of moderate growth with the same exponent M.

For $n = 1$ the condition of moderate growth is superfluous. An automorphic form on $GL(1, \mathbb{A}) = \mathbb{A}^\times$ is a finite sum

$$\phi(x) = \sum_j \chi_j(x) P_j(\log|x|)$$

where each χ_j is an idele class character and each P_j is a polynomial.

For $n > 1$ an automorphic form ϕ is $Z(\mathbb{A})$ finite. In particular, it is $Z_{>0}$ finite . We write $|z| = |z_{i,i}|$ (recall all the $z_{i,i}$ are equal and in $\mathbb{A}^\times_{>0}$.). Then, for $z \in Z_{>0}$,

$$\phi(zg) = \sum_{1 \le j \le r} |z|^{s_j} \left(\sum_{1 \le i \le M_{i,j}} (\log|z|)^{m_{i,j}} \phi_{i,j}(g) \right),$$

where the functions $\phi_{i,j}$ are automorphic forms. We will be mostly concerned in the case where, for $z \in Z_{>0}$,

$$\phi(zg) = |z|^s \phi(g).$$

In fact, multiplying by a power of $|\det g|$ we may reduce ourselves to the case where

$$\phi(zg) = \phi(g)$$

for all $z \in Z_{>0}$ and this will be the case of interest.

We can also consider square integrable automorphic forms. Those are elements of $L^2(Z_{>0}G(F)\backslash G(\mathbb{A}))$, K_∞ finite on the right and annihilated by an ideal of finite codimension of $Z(\mathfrak{g})$. A priori, this last condition must be taken in the distribution sense. But the two conditions together imply that such a function is real analytic so the differential equation can be taken in the ordinary sense. The conclusion of lemma 6.1 applies. Thus there is a smooth function of compact support f such that

$$\rho(f)\phi = \phi\,.$$

For any $X \in U(\mathfrak{g})$ we have

$$\rho(X)f = \rho\left((\rho(X)f)\right)\phi\,.$$

This implies that for all $X \in U(\mathfrak{g})$ the function $\rho(X)\phi$ is still square integrable. Moreover the function ϕ is of moderate growth. Indeed by Lemma 5.8 we have

$$|\phi(x)| = |\rho(f)\phi(x)| \leq c||x||^m||f||_2\,.$$

Thus ϕ is a slowly increasing automorphic form.

We could also consider more generally functions transforming on the left under a unitary character of $Z_{>0}$.

We have also the following result.

Lemma 6.2. *Let V be the space of C^∞ vectors in $L^2(Z_{>0}G(F)\backslash G(\mathbb{A}))$. Every $v \in V$ can be written as a fintie sum*

$$v = \sum_{1\leq i\leq r} \rho(f_i)v_i\,,$$

where $v_i \in V$ and the f_i are smooth functions of compact support on $G(\mathbb{A})$.

Proof. One argue as in lemma 6.1 using lemma 3.4 instead of lemma 3.3. □

We comment briefly on the relation with Harish-Chandra's notion of automorphic forms [20]. Let ϕ be an adelic automorphic form. In particular, it is invariant under a compact open subgroup K' of $G(\mathbb{A}_f)$. Consider the intersection subgroup

$$\Gamma := K' \cap G(F)$$

where $G(F)$ is embedded diagonally into $G(\mathbb{A}_f)$. If we embed $G(F)$ into $G\infty$ we can regard Γ as discrete subgroup of G_∞. It is an arithmetic subgroup. Let ϕ_o be the restriction of ϕ to G_∞. Then we have

$$\phi_0(\gamma g) = \phi_0(g)$$

for all $\gamma \in \Gamma$ and then ϕ_0 is an automorphic form in the sense of Harish Chandra for the group Γ. Suppose we translate ϕ on the right by an element $C \in G(\mathbb{A}_f)$. Then the function

$$\phi^c(g) := \phi(gc)$$

is invariant under the open compact subgroup $c^{-1}K'c$ and its restriction ϕ_0^c to G_∞ is invariant under another discrete subgroup Γ^c of G_∞. By the weak approximation theorem we have (disjoint union)

$$G(\mathbb{A}) = \bigcup G(F)G_\infty c_i K'$$

with $c_i \in G(\mathbb{A}_f)$. Thus we see that the adelic form ϕ is completely determined by the Harish-Chadra automorphic forms $\phi_0^{c_i}$ for different arithmetic groups Γ^{c_i}. In favorable circumstances ϕ is determined by one single Harish-Chandra automorphic form.

7. Two lemmas of functional analysis

We recall two lemmas of functional analysis

Lemma 7.1. *Let X be a locally compact space, μ a Borel measure on X such that $\mu(X) < +\infty$. Suppose V is a closed subspace of $L^2(X, \mu)$ such that any element $f \in V$ is a uniformly bounded continuous function. Then V is finite dimensional.*

This lemma is due to Godement. A proof due to Hörmander can be found in [7] Lemma 8.3 or [20] pp. 17,18.

Lemma 7.2. *Let X be a locally compact space, countable at infinity, with a countable dense subset. Let μ a Borel measure on X such that $\mu(X) < +\infty$. Suppose T is a continuous operator*

$$T : L^2(X, \mu) \to C_b(X)$$

where C_b is the space of bounded continuous functions with sup norm. Then T viewed as an operator $T : L^2(X, \mu) \to L^2(X, \mu)$ is a Hilbert-Schmidt operator and, in particular, a compact operator.

A complete proof can be found in [25], XII §3, Theorem 6.

8. Cusp forms and square integrable forms

A continuous function ϕ on $G(F)\backslash G(\mathbb{A})$ is said to be cuspidal if

$$\int_{U(F)\backslash U(\mathbb{A})} \phi(ug)du = 0$$

each time U is the unipotent radical of a proper parabolic subgroup of G and $g \in G(\mathbb{A})$. Here du is a Haar masure on $U(\mathbb{A})$.

If ϕ is invariant under $Z_{>0}$ and square integrable on $Z_{>0}G(F)\backslash G(\mathbb{A})$ we say it it is cuspidal if for every smooth function of compact support f the continuous function

$$g \mapsto \int \phi(gh)f(h)dh$$

is cuspidal. Thus the space $L^2_{\text{cusp}}(Z_{>0}G(F)\backslash G(\mathbb{A}))$ of cuspidal elements of $L^2(Z_{>0}G(F)\backslash G(\mathbb{A}))$ is a closed subspace.

We are going to see that a cusp form which is invariant under $Z_{>0}$ is in fact square integrable.

Lemma 8.1. *Suppose ϕ is a cusp form invariant under $Z_{>0}$. Then ϕ is bounded and in particular square integrable.*

We review the elegant proof of Godement in [11] (which is somewhat incomplete).

Proof. We only need to prove that ϕ is bounded on any Siegel set. In fact we prove that it is rapidly decreasing in an appropriate sense. Since a Siegel set is contained in a set of the form $A(t)\Omega$ where Ω is a compact set, it will suffice to prove that for any $m \geq 1$ there is a constant c such that

$$|\phi(a\omega)| \leq c\beta(a)^{-m}\,,$$

for $a \in A(t)$ and $\omega \in \Omega$. We recall the definition

$$\beta(a) := \prod_{1\leq i\leq n-1} |\alpha_i(a)|\,.$$

Indeed, we can write

$$\phi(g) = \int_{G^1} \phi(gh)f(h)dh = \int_{G^1} \phi(h)f(g^{-1}h)dh$$

where f is a smooth function of compact support on G^1. Using the Iwasawa decomposition for G^1, we get

$$\phi(a\omega) = \int \phi(ubk)f(\omega^{-1}a^{-1}ubk)du\delta(b)^{-1}dbdk$$

with $u \in N(\mathbb{A})$, $b \in A(\mathbb{A}) \cap G^1$, $k \in K$. Since f has compact support, we see that, in the above integral, the element

$$a^{-1}ubk = (a^{-1}ua)(a^{-1}b)k$$

remains in a fixed compact set. This implies that $a^{-1}b$ remains in a compact set Ω_A of $A(\mathbb{A})$.

Since ϕ is invariant on the left under $N(F)$ we can write this as

$$\int \left(\int_{N(F)\backslash N(\mathbb{A})} \sum_{\gamma \in N(F)} f(\omega^{-1}a^{-1}\gamma ubk)\phi(ubk)du \right) \delta(b)^{-1} dbdk\,.$$

We now use the cuspidality of ϕ.

If α and β are sums of positive roots we write $\alpha \succeq \beta$ if $\alpha - \beta$ is a sum (possibly empty) of positive roots. Let $\Delta = \{\alpha_i, 1 \le i \le n\}$ be the set of simple positive roots. For every subset $\theta \subseteq \Delta$ we denote by V^θ the subgroup of N defined by the following condition.: for each positive root α, the one dimensional subgroup N_α is contained in V^θ if and only there is a simple root $\alpha_i \in \theta$ such that $\alpha \succeq \alpha_i$. Thus V^θ is the unipotent radical of a parabolic subgroup $P^\theta = M^\theta V^\theta$. We set $M^\theta \cap N = N^\theta$. We have a semi-direct product where V^θ is normal:

$$N = N^\theta V^\theta\,.$$

Thus if $\theta = \emptyset$ then $P^\emptyset = G$, $N^\emptyset = N$, $V^\emptyset = \{e\}$. For $\theta = \Delta$ then P^Δ is the minimal parabolic subgroup, i.e. the group of triangular matrices and $V^\Delta = N$, $N^\Delta = \{e\}$. For instance, for $n = 3$, we have $\Delta = \{\alpha_1, \alpha_2\}$ and

$$V^{\{\alpha_1\}} = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, N^{\alpha_1} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

$$V^{\{\alpha_2\}} = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}, N^{\{\alpha_2\}} = \left\{ \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Now we consider the following alternating sum

$$\sum_{\theta \subseteq \Delta} (-1)^{|\theta|} \int_{V^\theta(\mathbb{A})} \sum_{\gamma \in N^\theta(F)} f(\omega^{-1}a^{-1}\gamma vubk)dv\,,$$

as a function of $u \in N(\mathbb{A})$. Here $|\theta|$ denotes the cardinality of θ. The term corresponding to $\theta = \emptyset$ is just our original expression, namely

$$\sum_{\gamma \in N(F)} f(\omega^{-1}a^{-1}\gamma ubk)\,.$$

In this sum, for a given $\theta \neq \emptyset$ the corresponding term

$$\int_{V^\theta(\mathbb{A})} \sum_{\gamma \in N^\theta(F)} f(\omega^{-1}a^{-1}\gamma vubk)dv\,,$$

as a function of u, is invariant on the let under $V^\theta(\mathbb{A})$ and $N^\theta(F)$ so is also invariant under $N(F)$. If now we integrate against ϕ on $N(F)\backslash N(\mathbb{A})$, we get

$$\int_{N(F)\backslash N(\mathbb{A})}\left(\int_{V^\theta(\mathbb{A})}\sum_{\gamma\in N^\theta(F)} f(\omega^{-1}a^{-1}\gamma vubk)dv\right)\phi(ubk)du\,.$$

But the integral over $N(F)\backslash N(\mathbb{A})$ can be decomposed as an integral over $V^\theta(F)\backslash V^\theta(\mathbb{A})$ followed by an integral over $N^\theta(F)\backslash N^\theta(\mathbb{A})$ (because V^θ is a normal subgroup). Since ϕ is cuspidal this integral is 0.

Thus our expression for $\phi(a\omega)$ can be replaced by

$$\int_{N(F)\backslash N(\mathbb{A})\times A^1\times K}\mathrm{Alt}\cdot\delta^{-1}(b)\cdot\phi(ubk)\cdot dudbdk\,,$$

where

$$\mathrm{Alt} := \sum_\theta(-1)^\theta\int_{V^\theta(F)\backslash V^\theta(\mathbb{A})}dv\sum_{\gamma\in N^\theta(F)} f(\omega^{-1}a^{-1}\gamma vubk)\,.$$

We now want to use Poisson summation formula on the Lie algebra of N. For a general group we would have to use the exponential function but on $GL(n)$ we can dispense with it. Indeed, the Lie algebra of N noted $\mathfrak{n}$ can identified with the space of upper triangular matrices with 0 diagonal. The dual vector space ${}^t\mathfrak{n}$ is the space of lower triangular matrices with 0 diagonal. The duality is given by

$$(x,y)\mapsto \mathrm{tr}xy\,.$$

If we use the standard basis (X_α) of $\mathfrak{n}$ the dual basis is $(X_{-\alpha})$. For $x=\sum x_\alpha X_\alpha$, $y=\sum y_{-\alpha}X_{-\alpha}$ we have $\mathrm{tr}xy=\sum x_\alpha y_{-\alpha}$. Similarly, the Lie algebra $\mathrm{Lie}(N^\theta)=\mathfrak{n}^\theta$ and $\mathrm{Lie}(V^\theta)=\mathfrak{v}^\theta$ are vector subspaces of $\mathfrak{n}$, and we have a direct sum decomposition

$$\mathfrak{n}=\mathfrak{v}^\theta\oplus\mathfrak{n}^\theta$$

and an orthogonal decomposition of the dual vector space

$${}^t\mathfrak{n}={}^t\mathfrak{v}^\theta\oplus{}^t\mathfrak{n}^\theta\,.$$

Our alternating sum can also be written as

$$\mathrm{Alt}=\sum_\theta(-1)^{|\theta|}\int_{\mathfrak{v}^\theta(\mathbb{A})}\sum_{\xi\in\mathfrak{n}^\theta(F)}\int f(\omega^{-1}a^{-1}(1+\xi)(1+X)ubk)dX\,.$$

But

$$(1+\xi)(1+X)=1+\xi+(1+\xi)X$$

and we can change X into $(1+\xi)^{-1}X$. So we get

$$\mathrm{Alt} = \sum_\theta (-1)^{|\theta|} \int_{\mathrm{Lie}V^\theta(\mathbb{A})} \sum_{\xi \in \mathrm{Lie}N^\theta(F)} \int f(\omega^{-1}a^{-1}(1+\xi+X)ubk)dX \,.$$

Now let us now introduce a Fourier transform. It is a function on ${}^t\mathfrak{n}(\mathbb{A})$:

$$Y \mapsto \int_{\mathfrak{n}(\mathbb{A})} f(\omega^{-1}a^{-1}(1+X)ubk)\psi(\mathrm{tr}XY)dX \,.$$

Using Poisson summation formula we get

$$\mathrm{Alt} = \sum_\theta (-1)^{|\theta|} \sum_{\lambda \in {}^t\mathfrak{n}^\theta(F)} \int_{\mathfrak{n}(\mathbb{A})} f(\omega^{-1}a^{-1}(1+X)ubk)\psi(\mathrm{tr}X\lambda)dX \,.$$

After taking into account the cancellation we find this reduces to

$$\mathrm{Alt} = \sideset{}{^\bullet}\sum_{\lambda \in {}^t\mathfrak{n}(F)} \int_{\mathfrak{n}(\mathbb{A})} f(\omega^{-1}a^{-1}(1+X)ubk)\psi(\mathrm{tr}X\lambda)dX \,,$$

where the $\bullet$ means we sum only for those λ which do not belong to some ${}^t\mathfrak{n}^\theta$ with $\theta \neq \emptyset$. If we write

$$\lambda = \sum X_{-\alpha}\lambda_{-\alpha}$$

we sum only for those λ such that

$$\sum_{\lambda_{-\alpha}\neq 0} \alpha \succeq \sum_{1\leq i\leq n-1} \alpha_i \,.$$

For instance, for $n=3$, the sum is over the elements

$$\lambda = X_{-\alpha_1}\lambda_{-\alpha_1} + X_{-\alpha_3}\lambda_{-\alpha_2} + X_{-\alpha_1-\alpha_2}\lambda_{-\alpha_1-\alpha_2}$$

such that

$$\lambda_{-\alpha_1} \neq 0 \text{ and } \lambda_{-\alpha_2} \neq 0$$

or

$$\lambda_{-\alpha_1-\alpha_2} \neq 0 \,.$$

Now let us majorize

$$\delta(b)^{-1}\mathrm{Alt} = \delta(b)^{-1} \sideset{}{^\bullet}\sum_{\lambda \in {}^t\mathfrak{n}(F)} \int_{\mathfrak{n}(\mathbb{A})} f(\omega^{-1}a^{-1}(1+X)ubk)\psi(\mathrm{tr}X\lambda)dX$$

It can be written as

$$\delta(b)^{-1} \sideset{}{^\bullet}\sum_{\lambda \in {}^t\mathfrak{n}(F)} \int_{\mathfrak{n}(\mathbb{A})} f(\omega^{-1}(1+a^{-1}Xa)a^{-1}uaa^{-1}bk)\psi(\mathrm{tr}X\lambda)dX \,.$$

After changing variables, and recalling that $\mathrm{tr} aXa^{-1}\lambda = \mathrm{tr} X(a^{-1}\lambda a)$, we find

$$\delta(b)^{-1}\mathrm{Alt} =$$

$$\delta(ab^{-1}) \sum_{\lambda \in {}^t\mathfrak{n}(F)}^{\bullet} \int_{\mathfrak{n}(\mathbb{A})} f(\omega^{-1}(1+X)a^{-1}uaa^{-1}bk)\psi(\mathrm{tr} Xa^{-1}\lambda a)dX\,.$$

Since ω, $a^{-1}ua$, $a^{-1}b$ and K remain in compact sets the functions

$$X \mapsto \delta(ab^{-1})f(\omega^{-1}(1+X)a^{-1}uaa^{-1}bk)\delta(ab^{-1})$$

remain in a compact set of the space of Schwartz-Brunat functions. So do their Fourier transforms. We now appeal to the following lemma:

Lemma 8.2. *Suppose B is a compact set of the space of Schwartz-Bruhat functions on ${}^t\mathfrak{n}(\mathbb{A})$ and let $m > 1$. Then there exist $c > 0$ such that, for all $\Phi \in B$ and $a \in A(t)$,*

$$\left| \sum_{\lambda \in {}^t\mathfrak{n}(F))}^{\bullet} \Phi(a^{-1}\lambda a) \right| \leq c\beta(a)^{-m}\,.$$

Taking the lemma for granted at the moment, we have

$$\delta(b)^{-1}\mathrm{Alt} = \prec \beta(a)^{-m}$$

and

$$\phi(a\omega) = \int_{N(F)\backslash N(\mathbb{A})\times A^1\times K} \mathrm{Alt}\cdot\delta^{-1}(b)\cdot\phi(ubk)\cdot dudbdk$$

Now u, ba^{-1} and K are in compact sets we have

$$|\phi(ubk)| \prec ||b||^{m_0} \prec ||a||^{m_0}$$

for some $m_0 > 0$. Thus we find

$$|\phi(a\omega)| \prec \beta^{-m}(a)||a||^{m_0}$$

for some m_0 and any $m > 0$. Since a is in $A(t)$ by taking m large enough we obtain

$$|\phi(a\omega)| \prec \beta^{-m_1}(a)$$

for any m_1.

It remains to prove the lemma. We may assume that

$$\Phi = \Phi_\infty \prod_{v \mathrm{finite}} \Phi_v$$

where each Φ_v is the characteristic function of

$$\omega_v^{-r_v} M(n \times n, \mathcal{O}_v)$$

and Φ_∞ remain in a compact set. Then the sum takes the form

$$\sum_{\lambda\in\Lambda}^{\bullet} \Phi_\infty(a^{-1}\lambda a)\,,$$

where Λ is a lattice in $M(n\times n, F)$ and Φ_∞ a Schwartz function which remain in a compact set. For any Archimedean place v, there is $c_v > 0$ such that for $\lambda\in\Lambda$ and $\lambda_{-\alpha}\neq 0$ we have

$$|\lambda_{-\alpha}|_v \geq c_v\,.$$

There is also a constant d_v such that, for $a \in A(t)$ and all positive root α

$$|\alpha(a)|_v \geq d_v\,.$$

Now we have

$$|\Phi_\infty(x)| \leq C \prod_{v\,\mathrm{real}} \frac{1}{\left(1+x_{-\alpha,v}^2\right)^{2m}} \prod_{v\,\mathrm{complex}} \frac{1}{(1+(x_{-\alpha,v}\overline{x_{-\alpha,v}})^2)^{2m}}\,.$$

Now take

$$x = a^{-1}\lambda a = \sum_\alpha \alpha(a)\lambda_{-\alpha}X_{-\alpha}\,.$$

which appears in our $\bullet$ sum. For v real, we have

$$(1+\alpha(a)_v^2\lambda_{-\alpha,v}^2)^{2m} \geq |\alpha(a)|_v^m c_v^m \cdot (1+d_v^2\lambda_{-\alpha,v}^2)^m$$

For v complex, we have

$$(1+\alpha(a)_v^4(\lambda_{-\alpha v}\overline{\lambda_{-\alpha,v}})^2)^{2m}$$
$$\geq |\alpha(a)|_v^m c_v^m \cdot (1+d_v^2(\lambda_{-\alpha v}\overline{\lambda_{-\alpha,v}})^2)^m\,.$$

So for λ in our sum we get

$$|\Phi_\infty(a^{-1}\lambda a)| \prec$$
$$\prod_\alpha |\alpha(a)|^{-2m} \prod_{v\,\mathrm{real}} \frac{1}{(1+d_v^2\lambda_{-\alpha,v}^2)^m} \prod_{v\,\mathrm{complex}} \frac{1}{(1+d_v^2(\lambda_{-\alpha v}\overline{\lambda_{-\alpha,v}})^2)^m}\,.$$

The first product is over those α for which $\lambda_{-\alpha}\neq 0$. By assumption, summing over those α we have

$$\sum_\alpha \alpha \succeq \sum_{1\leq i\leq n-1} \alpha_i\,.$$

and thus $\prod|\alpha(a)| \succeq \beta(a)$. Finally, we find

$$\left|\sum^{\bullet} \Phi_\infty(a^{-1}\lambda a)\right| \prec$$

$$\beta(a)^{-m}\sum_{\lambda\in\Lambda}\prod_{v\,\mathrm{real}}\frac{1}{(1+d_v^2\lambda_{-\alpha,v}^2)^m}\prod_{v\,\mathrm{complex}}\frac{1}{(1+d_v^2(\lambda_{-\alpha v}\overline{\lambda_{-\alpha,v}})^2)^m}\,,$$

where in the new sum we have no restriction on λ. For m large emough this sum is finite and we are done. □

Thus the space of automorphic cusp forms of a given type and invariant under $Z_{>0}$ is a closed subspace of $L^2(G(F)\backslash G^1)$ whose members are continuous bounded functions.. Hence it is finite dimensional by lemma 7.1.

This result can be easily extended to any space of cuspidal automorphic forms of a given type.

Suppose f is an automoprhic form on $G(\mathbb{A})$ of a given type $\mathfrak{i}$, θ. Let $P = MU$ be a proper parabolic subgroup of G. We claim that for any $k \in K$ the function

$$m \mapsto f_U(mk) := \int_{U(F)\backslash U(\mathbb{A})} f(umk)\,du$$

on $M(\mathbb{A})$ is an automorphic form. (We have to extend the discussion to the case of a product of linear groups). Indeed it is invariant under $M(F)$. It is of moderate growth since

$$|f_U(mk)| \le c\int_{U(F)\backslash U(\mathbb{A})} ||umk||^r du \prec ||m||^r \int_{U(F)\backslash U(\mathbb{A})} du\,.$$

It is $K \cap M(\mathbb{A})$ finite of a type determined by θ. Finally we have

$$Z(\mathfrak{g}) \subset Z(\mathfrak{m}) + \mathfrak{u}U(\mathfrak{g})\,.$$

For $X \in Z(\mathfrak{g})$ call $r(X)$ its projection on $Z(\mathfrak{m})$. Then r is an homomorphism and each function $m \mapsto f_U(mk)$ is annihilated by $r(\mathfrak{i})$. In addition $Z(\mathfrak{m})$ is a $Z(\mathfrak{g})$ module of finite type. Thus $r(\mathfrak{i})$ is an ideal of finite codimension in $Z(\mathfrak{m})$. A simple inductive argument shows that the dimension of the space of automorphic forms is a given type is finite ([20]).

Finally, let us consider the space $L^2_{\mathrm{cusp}}(Z_{>0}G(F)\backslash G(\mathbb{A}))$.

Theorem 8.3. *Suppose f is a smooth function of compact support on $G(\mathbb{A})$. For any $\phi \in L^2_{\mathrm{cusp}}(Z_{>0}G(F)\backslash G(\mathbb{A}))$ the function $\rho(f)\phi$ is a bounded continuous function, in fact a rapidly decreasing function on $G(F)\backslash G^1$.*

Proof. We use the notations of the proof of lemma 8.1. It suffices to estimate $\rho(f)\phi(a\omega)$ where $a \in A(t)$ and ω is in a compact set. We have

$$\rho(f)\phi(g) = \int_{G^1} f^1(h)\phi(gh)dh$$

where $f^1(g) = \int_{Z_{>0}} f(zg)dz$. We find

$$\rho(f)\phi(a\omega) = \int \mathrm{Alt} \cdot \delta(b)^{-1}\phi(ubk)dudbdk$$

where

$$|\mathrm{Alt}| \prec \delta(a)\beta^{-m}(a)$$

and $a^{-1}b$ is in a compact set. Thus ubk is in a Siegel set which is itself contained in a finite union of translates by elements of $G(F)$ of a section $\mathfrak{S}'$ of $G(F)\backslash G^1$. Thus the above expression is majorized by

$$\begin{aligned}\delta(a)\beta(a)^{-m} \int |\phi(ubk)|\delta(b)^{-1}dudbdk \\ \prec \delta(a)\beta(a)^{-m} \int_{G(F)\backslash G^1} |\phi|(g)dg \\ \leq \delta(a)\beta(a)^{-m}\mathrm{Vol}(G(F)\backslash G^1)^{1/2}||\phi||_2 \,.\end{aligned}$$

Taking m large enough $\delta(a)\beta(a)^{-m}$ is bounded independently of a and our assertion follows. ☐

Lemma 7.2 implies that the operator $\rho(f)$ on $L^2_{\mathrm{cusp}}(Z_{>0}G(F)\backslash G(\mathrm{A}))$ is a compact operator. It follows that this space decomposes as a discrete sum of unitary irreducible representations, each occurring with finite multiplicity. In fact for $GL(n)$ the multiplicity is (at most) 1 but we will not need this fact.

We also the following result.

Lemma 8.4. *Let V be the space of smooth vectors in $L^2(Z_{>0}G(F)\backslash G(\mathbb{A}))$. Every ϕ in V is bounded (in fact rapidly decreasing on a Siegel set as in the proof of 8.1).*

Proof. The proof is similar to the proof of the previous theorem using lemma 6.2 ☐

9. Global Theory of L–functions for cusp forms

Theorem 9.1. *Let π be a unitary irreducible representation of $G(\mathbb{A})$. Suppose π occurs in $L^2_{\mathrm{cusp}}(Z_{>0}G(F)\backslash G(\mathbb{A})$. Define*

$$L(s,\pi) = \prod_v L(s,\pi_v)\,, \qquad \epsilon(s,\pi) = \prod_v \epsilon(s,\pi_v,\psi_v)\,.$$

Then the Eulerian product $L(s,\pi)$ converges absolutely for $\mathrm{Re}(s) >> 0$, can be analytically continued as an entire function of s bounded at infinity in vertical strips. As such, it satisfies the functional equation

$$L(1-s,\widetilde{\pi}) = \epsilon(s,\pi)L(s,\pi)\,.$$

Proof. We first observe that the contragredient $\widetilde{\pi}$ is the imaginary conjugate of π and occurs in the space of cusp forms. Moreover because the central character ω of π is automorphic, the factor $\epsilon(s,\pi)$ does not depend on the choice of ψ, which justifies the notation.

For the proof we consider a matrix coefficient f of π given by the formula

$$f(g) = \int_{G(F)\backslash G^1} \phi(hg)\widetilde{\phi}(h)dh$$

where ϕ and $\widetilde{\phi}$ are K–finite vectors (or even smooth vectors) in the space of π and $\widetilde{\pi}$ respectively. Then we consider the global Zeta integral

$$Z(\Phi, f, s) := \int_{G(\mathbb{A})} \Phi(g) f(g) \, |\det g|^{s+\frac{n-1}{2}} dg \, .$$

where Φ is a Schwartz-Bruhat function on $M(n \times n, \mathbb{A})$. We assume that Φ is a product

$$\Phi(g) = \prod_v \Phi_v(g_v)$$

where Φ_v is the characteristic function of $M(n \times n, \mathcal{O}_v)$ for almost all v. We will see that this integral converges for $\mathrm{Re}(s) >> 0$.

Replacing f by its definition we find

$$\int_{G(\mathbb{A})} \Phi(g) \left(\int_{G(F)\backslash G^1} \phi(hg)\widetilde{\phi}(h)dh \right) |\det g|^{s+\frac{n-1}{2}} dg \, .$$

Exchanging the order of integration and changing g to $h^{-1}g$ we find

$$\int_{G(F)\backslash G^1} \widetilde{\phi}(h) \left(\int_{G(\mathbb{A})} \Phi(h^{-1}g) \, \phi(g) \, |\det g|^{s+\frac{n-1}{2}} dg \right) dh \, .$$

We further decompose the integral over g and we find

$$\int_{G(F)\backslash G^1 \times G(F)\backslash G^1} \widetilde{\phi}(h_1)\phi(h_2) dh_1 dh_2 \int_{Z_{>0}} \sum_{\gamma \in G(F)} \Phi\left(h_1^{-1} z \gamma h_2\right) |z|^{ns+\frac{n(n-1)}{2}} d^\times z \, .$$

Here z has the form

$$z = xI_n \, , \; x = (y, y, \ldots, y, 1, 1, \ldots) \, , \; y > 0 \, , \; |z| = y^{r+2c} \, .$$

We need to majorize the sum over γ for h_1 and h_2 in a Siegel set. Thne

$$h_1 = a\omega_1 \, , \qquad h_2 = b\omega_2 \, ,$$

where ω_1 and ω_2 remains in compact sets while a and b are in $A(t)$.

We need a Lemma.

Lemma 9.2. *Let $1 \le r \le n$, be an integer. With the previous notations, we have the following majorizations.*

(i) There is a constant c such that

$$\left| \sum_{\mathrm{rank}(\gamma)=r} \Phi(h_1^{-1} z\gamma h_2) \right| \le c||a|| \cdot ||b|| \cdot |z|^{-n^2}$$

for $|z| \le 1$.

(ii) For every $M >> 0$ there is a constant c_M such that

$$\left| \sum_{\mathrm{rank}(\gamma)=r} \Phi(h_1^{-1} z\gamma h_2) \right| \le c_M||a||^{n^2(M+1)} \cdot ||b||^{n^2(M+1)} \cdot |z|^{-M}$$

for $|z| \ge 1$.

Proof. The functions

$$X \mapsto \Phi(\omega_1^{-1} X \omega_2)$$

remain in a compact set of the space of Schwartz-Bruhat functions. Thus are dominated in absolute value by a fixed Schwartz-Bruhat function $\Phi_0 \ge 0$. Thus it suffices to estimate the sums

$$\sum_{\mathrm{rank}(\gamma)=r} \Phi(a^{-1} z\gamma b)$$

with $\Phi \ge 0$. Each one of these sums is bounded by

$$\sum_{\gamma \in M(n \times n, F) \neq 0} \Phi(a^{-1} z\gamma b)\,.$$

In turn we may assume Φ is majorized by a sum of decomposable functions. So we may as well assume that

$$\Phi(x) = \prod_{(i,j) \in [1,n] \times [1,n]} \phi_{ij}(x_{ij})$$

with $\phi_{ij} \ge 0$. The sum is then equal to a sum over all non empty subsets S of the product $[1,n] \times [1,n]$:

$$(1) \qquad \sum_S \prod_{(i,j) \in S} \left(\sum_{\xi \in F^\times} \phi_{i,j}(a_i^{-1} \xi z b_j) \right) \prod_{(i,j) \notin S} \phi_{ij}(0)\,.$$

In general if $\phi \ge 0 \in \mathcal{S}(\mathbb{A})$, and $y \in \mathbb{A}_{>0}$ then, for any $M > 0$, we have, for a suitable $c > 0$,

$$\sum_{\xi \in F^\times} \phi(y\xi) \le c \frac{|y|^{-1}}{1 + |y|^M}\,.$$

Thus the term corresponding to a subset S in (1) is bounded by a constant times

$$|z|^{-|S|}\prod_{(i,j)\in S}|a_i||b_j|^{-1}$$

and by a constant times

$$|z|^{-|S|-M|S|}\prod_{(i,j)\in S}|a_i|^{1+M}|b_j|^{-1-M}\,.$$

Now

$$|a_i|\prec||a||\,,\ |a_i|^{-1}\prec||a||\,,\ |b_j|\prec||b||\,,\ |b_j|^{-1}\prec||b||\,.$$

So the sums of the terms corresponding to S are dominated by a constant times

$$|z|^{-|S|}\cdot||a||\cdot||b||$$

for $|z|\leq 1$ or since $|S|\leq n^2$

$$|z|^{-n^2}\cdot||a||\cdot||b||\,.$$

For $|z|\geq 1$ the sums of the terms corresponding to S are dominated by a constant times

$$|z|^{-|S|(M+1)}||a||^{|S|(1+M)}||b||^{|S|(1+M)}\leq|z|^{-M}||a||^{n^2(1+M)}||b||^{n^2(1+M)}\,.$$

Our assertion follows. □

Thus for $|z|\leq 1$ we have

$$\left|\widetilde{\phi}(h_1)\phi(h_2)\sum_{\gamma\in G(F)}\Phi(h_1^{-1}z\gamma h_2)\right|\prec\beta(a)^{-M}\beta(b)^{-M}\cdot||a||\cdot||b||\cdot|z|^{-n^2}$$

where M is arbitrary large. On the other hand $||a||.||b||\prec\beta(a)^{m_1}\beta(b)^{m_1}$ for some m_1. We conclude that, for $|z|\leq 1$,

$$\left|\widetilde{\phi}(h_1)\phi(h_2)\sum_{\gamma\in G(F)}\Phi(h_1^{-1}z\gamma h_2)\right|\prec|z|^{-n^2}\,.$$

So the integral over $|z|\leq 1$ converges for $\mathrm{Re}(s)>>0$.

Om the other hand for $|z|\geq 1$ we have

$$\left|\widetilde{\phi}(h_1)\phi(h_2)\sum_{\gamma\in G(F)}\Phi(h_1^{-1}z\gamma h_2)\right|\prec\beta(a)^{-M}\beta(b)^{-M}||a||^{M_2}\cdot||b||^{M_2}\cdot|z|^{-M_1}$$

where M and M_1 are arbitrarily large but independent while M_2 depends on M_1. In turn this is dominated by

$$\beta(a)^{-M}\beta(b)^{-M}\beta(a)^{M_2m_1}\beta(b)^{M_2m_1}\cdot|z|^{-M_1}\,.$$

We conclude that, for $|z| \leq 1$,

$$\left| \widetilde{\phi}(h_1)\phi(h_2) \sum_{\gamma \in G(F)} \Phi(h_1^{-1} z \gamma h_2) \right| \prec |z|^{-M_1} .$$

So the integral for $|z| \geq 1$ converges for all s.

Now we apply Poisson summation formula. We have

$$\begin{aligned} \sum_{\gamma \in G(F)} \Phi(h_1^{-1}\gamma h_2) = & \sum_{\gamma \in G(F)} \widehat{\Phi}(h_2^{-1}\gamma z^{-1} h_1)|z|^{-n^2} \\ & + \sum_{1 \leq r \leq n-1} \sum_{\mathrm{rank}(\gamma)=r} \widehat{\Phi}(h_2^{-1}\gamma z^{-1} h_1)|z|^{-n^2} \\ & - \sum_{1 \leq r \leq n-1} \sum_{\mathrm{rank}(\gamma)=r} \Phi(h_1^{-1}\gamma z h_1) \\ & + \widehat{\Phi}(0)|z|^{-n^2} - \Phi(0) . \end{aligned}$$

We integrate this expression against

$$\widetilde{\phi}(h_2)\phi(h_1)|z|^{s+\frac{n(n-1)}{2}}$$

for $|z| \leq 1$. Using the same argument as before we see that the integral of the first term (over $\gamma \in G(F)$) converges for all s. Similarly, the integral over matrices of rank r for $\widehat{\Phi}$ converges for all s. The integral over matrices of rank r for Φ converges for $\mathrm{Re}(s) >> 0$. Finally the integral of the term for $\Phi(0)$ and $\widehat{\Phi}(0)$ converge for $\mathrm{Re}(s) >> 0$.

Because

$$\int \widetilde{\phi}_2(h_2)\phi(h_1) dh_2 dh_1 = 0 ,$$

the terms containing $\Phi(0)$ and $\widehat{\Phi}(0)$ give a zero integral. We claim that the integral

$$\int_{G(F)\backslash G^1 \times G(F)\backslash G^1} \widetilde{\phi}(h_1)\phi(h_2) dh_1 dh_2 \sum_{\mathrm{rank}(\gamma)=r} \Phi(h_1^{-1} z \gamma h_2)$$

is 0. Indeed, the matrices of rank r can be written as

$$\gamma = \gamma_1^{-1} \begin{pmatrix} 1_r & 0 \\ 0 & 0_{n-r \times n-r} \end{pmatrix} \gamma_2$$

with $\gamma_1, \gamma_2 \in G(F)$. Call M the group of pairs (γ_1, γ_2) such that

$$\gamma_1^{-1} \begin{pmatrix} 1_r & 0 \\ 0 & 0_{n-r \times n-r} \end{pmatrix} \gamma_2 = \begin{pmatrix} 1_r & 0 \\ 0 & 0_{n-r \times n-r} \end{pmatrix} .$$

Then M is the set of pairs (h_2, h_1) of the form

$$h_2 = u \begin{pmatrix} a & 0 \\ 0 & a_2 \end{pmatrix}, \, h_1 = v \begin{pmatrix} a & 0 \\ 0 & a_1 \end{pmatrix}$$

where $a \in GL(r)$, $a_1, a_2 \in GL(n-r)$ while u is in the group

$$U = \left\{ \begin{pmatrix} 1_r & * \\ 0 & 1_{n-r} \end{pmatrix} \right\},$$

and v in the group

$$V = \left\{ \begin{pmatrix} 1_r & 0 \\ * & 1_{n-r} \end{pmatrix} \right\}.$$

The integral

$$\int_{G(F)\backslash G^1 \times G(F)\backslash G^1} \widetilde{\phi}(h_1)\phi(h_2) dh_1 dh_2 \sum_{\text{rank}(\gamma)=r} \Phi(h_1^{-1} z \gamma h_2)$$

becomes the integral

$$\int_{M(F)\backslash G^1 \times G^1} \widetilde{\phi}(h_1)\phi(h_2)\Phi\left(h_1^{-1} z \begin{pmatrix} 1_r & 0 \\ 0 & 0_{n-r} \end{pmatrix} h_2\right) dh_1 dh_2 \, .$$

This integral factors trough an integral over $M(F)\backslash M(\mathbb{A})$ against the left invaraint measure on $M(\mathbb{A})$. Becauee the group $U \times V$ is a normal subgroup of M, in turn, factors this integral factors trough an integral over $(U(F)\backslash U(\mathbb{A})) \times V(F)(\backslash V(\mathbb{A}))$. That is to compute our integral we first compute the integral

$$\int\int_{(U(F)\backslash U(\mathbb{A}))\times V(F)(\backslash V(\mathbb{A}))} \widetilde{\phi}(vh_1)\phi(uh_2)\Phi\left(h_1^{-1} z \begin{pmatrix} 1_r & 0 \\ 0 & 0_{n-r} \end{pmatrix} h_2\right) dudv$$

and then further integrate over (h_1, h_2) against certain measures. Because ϕ and $\widetilde{\phi}$ are cuspidal the integral over u and v are 0 ,which proves our claim. Similarly the terms containing $\widehat{\Phi}$ and matrices of rank r give a 0 integral.

Finally we see

$$\int_{G(\mathbb{A})} \Phi(g) f(g) \left|\det g\right|^{s+\frac{n-1}{2}} dg$$

$$= \int_{|z|\geq 1} \int_{G(F)\backslash G^1 \times G(F)\backslash G^1} \widetilde{\phi}(h_1)\phi(h_2) dh_1 dh_2 \sum_{\gamma \in G(F)} \Phi\left(h_1^{-1} z \gamma h_2\right) |z|^{ns+\frac{n(n-1)}{2}} d^\times z$$

$$+ \int_{|z|\geq 1} \int_{G(F)\backslash G^1 \times G(F)\backslash G^1} \widetilde{\phi}(h_1)\phi(h_2) dh_1 dh_2 \sum_{\gamma \in G(F)} \widehat{\Phi}\left(h_2^{-1} z \gamma h_1\right) |z|^{n(1-s)s+\frac{n(n-1)}{2}} d^\times z\,.$$

We have changed z into z^{-1} on the second integral.

In this expression both integrals converge for all s. This shows that they represent entire of s. The proof also show that these functions of s are bounded at infinity in vertical strips. Moreover, we have clearly the functional equation

$$\int_{G(\mathbb{A})} \Phi(g) f(g) \,|\det g|^{s+\frac{n-1}{2}} dg = \int_{G(\mathbb{A})} \widehat{\Phi}(g) \check{f}(g) \,|\det g|^{1-s+\frac{n-1}{2}} dg\,.$$

Now we are ready to use the local theory of L–functions. First we write the Haar measure on $G(\mathbb{A})$ as a tensor product of local Haar measure, being understood that for almost all (or even for all) finite places v the measure of $GL(n, \mathcal{O}_v)$ is 1. Note that here we do not need to normalize the Haar measure because the **same** Haar measure appears on both sides of our functional equation. We can take ϕ and $\widetilde{\phi}$ to be pure tensors. Then

$$f(g) = \prod_v f_v(g_v)\,,$$

where for all v the function f_v is a matrix coefficient of π_v and, for almost all finite v, the function f_v is the spherical coefficient of π_v, and in particular takes the value 1 at e. Formally we have

$$Z(\Phi, f, s) = \prod_v Z(\Phi_v, f, s)\,.$$

If we take Res $> \frac{n-1}{2}$ each local integral converges. Moreover we have seen that the integral on the left converges for $\mathrm{Re}(s) >> 0$. This implies that the infinite product on the right converges absolutely for $\mathrm{Re}(s) >> 0$. (by replacing f by the constant function one can see the infinite product converges for $\mathrm{Re}(s) > 1 + \frac{n-1}{2}$). Almost all factors in the product are equal to $L(s, \pi_v)$.

Now using the local theory we can choose the functions Φ_i and matrix coefficients f_i so that

$$L(s,\pi)=\sum_{i=1}^{r} Z(\Phi_i,f_i,s)\,.$$

This shows that $L(s,\pi)$ has an analytic continuation as an entire function of s. Then

$$\epsilon(s,\pi)L(1-s,\widetilde{\pi})=\sum_{i=1}^{r} Z(\widehat{\Phi}_i,\check{f}_i,1-s)$$

and our assertion follows. □

10. General automorphic forms

We can also define the notion of irreducible automorphic representation [6]. Such a representation is really a representation of $(\mathfrak{g},K_\infty)$ and a representation $G(\mathbb{A}_F)$ commuting to one another on a complex vector space V. The space V has no non-trivial invariant subspace. Furthermore the representation is admissible in the sense that an irreducible representation of K appears with finite multiplicity. We say that such a representation is autormorphic if there exists two invariant subspaces $V_0\subset V_1$ of the space of automorphic forms such that the representation π is the representation on the quotient V_1/V_0.

One can show (Langlands, [6]) that any such π is an irreducible of an induced representation

$$I(G,P;\pi_1,\pi_2,\dots,\pi_r)$$

where each representation π_i is automorphic and cuspidal. Thai means in fact that for any place v, the representation π_v is a component of the induced representation

$$I(G_v,P_v;\pi_{1,v},\pi_{2,v},\dots,\pi_{r,v})\,.$$

Furthermore for almost all finite v, the induced representation has a unique irreducible component with a vector fixed by K_v and π_v is this irreducible component. This implies that

$$L(s,\pi):=\prod_v L(s,\pi_v)$$

is equal to

$$P(s)\prod_{1\le i\le r} L(s,\pi_i)$$

where

$$P(s) = \prod_v P_v(s)\,,$$

and $P_v(s)$ is a polynomial in s if v is infinite, a polynomial in q^{-s} if v is finite and $P_v = 1$ for almost all v. Similarly for the contragredient representations. On the other hand

$$\gamma(s,\pi,\psi_v) = \prod_{1\le i\le r} \gamma(s,\pi_i,\psi_v)$$

for all v. We conclude that $L(s,\pi)$ is meromorphic with the functional equation

$$L(1-s,\widetilde{\pi}) = \epsilon(s,\pi)L(s,\pi)\,.$$

Finally the theory of Eisenstein series (Langlands, [6]) shows that any irreducible component of such an induced representation is automorphic.

11. $GL(2)$ Examples

The earliest examples of automorphic forms were holomorphic modular forms for $G = GL(2)$. Let $K = O(2,\mathbb{R})$ be the maximal compact subgroup of $G(\mathbb{R})$. By the Iwasawa decomposition, the upper half-plane

$$\mathfrak{h}^2 := \{x + iy \mid x \in \mathbb{R},\ y > 0\}$$

can be identified with

$$\mathfrak{h}^2 \cong G(\mathbb{R})/(K\cdot\mathbb{R}^\times) \cong \left\{ \begin{pmatrix} y & x\\ 0 & 1\end{pmatrix} \;\middle|\; x\in\mathbb{R},\ y>0\right\}.$$

Indeed, under the action of $GL(2,\mathbb{R})$ on the upper half plane given by

$$gz := \frac{az+b}{cz+d}, \qquad \Big(\text{for } g = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in GL(2,\mathbb{R}),\ z\in\mathfrak{h}^2\Big),$$

we see that $\left(\begin{smallmatrix} y & x\\ 0 & 1\end{smallmatrix}\right) i = x+iy$ establishes that $\mathfrak{h}^2 \cong G(\mathbb{R})/(K\cdot\mathbb{R}^\times)$.

One of the most famous examples of a classical holomorphic modular form is the Ramanujan cusp form of weight 12 given by:

$$\begin{aligned}\Delta(z) &:= e^{2\pi iz}\prod_{n=1}^{\infty}\left(1-e^{2\pi inz}\right)^{24}\\ &= e^{2\pi iz} - 24e^{4\pi iz} + 252e^{6\pi iz} - 1472e^{8\pi iz} + \cdots\end{aligned}$$

for $z = x+iy \in \mathfrak{h}^2$. The Ramanujan cusp form statisfies the modular relations

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z)$$

for all $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$.

We would like to define a modular form purely in group theoretic terms. For modular forms for the group $SL(2,\mathbb{Z})$ one might make the following definition. Define an automorphic form for $SL(2,\mathbb{Z})$ as a function

$$\phi : G \to \mathbb{C}$$

which is invariant under $SL(2,\mathbb{Z})$ on the left, K-invariant on the right, and is invariant under the center $\mathbb{R}^\times$ of $G(\mathbb{R})$. Further, we demand that $\phi\left(\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)\right)$ is $\mathbb{C}^\infty$ and has moderate growth, that is

$$\left|\phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)\right| \le c \cdot y^M$$

for some $c, M > 0$, and $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$ in a Siegel set, i.e., $0 \le x < 1,\ y > \frac{\sqrt{3}}{2}$. We term this the *"group theoretic upper half plane model"*.

Note that Δ does not satisfy the above definition since Δ is not invariant on the left under $SL(2,\mathbb{Z})$. To get around this difficulty we need to make the following modification.

We introduce the cocycle $j : GL(2,\mathbb{R}) \times \mathbb{C} \to \mathbb{C}$ which is defined by

$$j(\gamma, \tau) := c\tau + d, \qquad \left(\text{for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,\mathbb{R}),\ \ \tau \in \mathfrak{h}^2\right).$$

One easily checks that j satisfies the cocycle relation

$$j(\gamma\gamma', \tau) = j(\gamma, \gamma'\tau) \cdot j(\gamma', \tau).$$

Define

$$\phi(gkd) := \Delta(g\,i) \cdot j(g, i)^{-12}$$

for all $g = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathfrak{h}^2$, all $k \in K = O(2,\mathbb{R})$, and all $d = \left(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}\right)$ with $r \in \mathbb{R}^\times$.

Clearly

$$\phi(g) = \Delta(x + iy) = \Delta(z)$$

for $g = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathfrak{h}^2$. Note that we are forcing ϕ to be K-invariant and also invariant under the center of $GL(2,\mathbb{R})$ to conform with the upper half-plane model.

It follows from the Iwasawa decomposition that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ and $g = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathfrak{h}^2$, with $z = x + iy$, we have

$$\begin{aligned} \phi(\gamma g) &= \Delta(\gamma z) \cdot j(\gamma g, i)^{-12} \\ &= (cz + d)^{12} \cdot \Delta(z) \cdot j(\gamma,\ g\,i)^{-12} \cdot j(g, i)^{-12} \\ &= (cz + d)^{12} \Delta(z) j(\gamma, z)^{-12} j(g, i)^{-12} \\ &= \Delta(z) j(g, i)^{-12} \\ &= \phi(g). \end{aligned}$$

This shows that $\phi(g)$ is invariant under $SL(2,\mathbb{Z})$ on the left. Furthermore, by the Fourier expansion it is clear that ϕ is $\mathbb{C}^\infty$ and that for $y > \frac{\sqrt{3}}{2}$ and $0 < x < 1$ we have

$$|\phi(g)| \ll e^{-2\pi y} \cdot y^{-12},$$

so ϕ has moderate growth. Therefore, ϕ is a modular form for $SL(2,\mathbb{Z})$ for the group theoretic upper half-plane model.

More generally, if

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$$

is a classical holomorphic modular form of weight $\ell \equiv 0 \pmod 2$ (with an integer $\ell \geq 0$) for $SL(2,\mathbb{Z})$, then the function $\phi : \mathfrak{h}^2 \to \mathbb{C}$ defined by

$$\phi(g) := f(g\, i) \cdot j(g,i)^{-\ell}$$

will satisfy the general definition of an automorphic form for $SL(2,\mathbb{Z})$ in the group theoretic upper-half plane model.

We have thus shown that one may replace the classical definition of a holomorphic modular form $f(z)$ (with $z = x + iy$ in the upper half plane) by defining a new function $\phi(g)$ where g is a matrix of the form $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$. Unfortunately, this definition is too restrictive and loses information. We, therefore, drop the assumption that ϕ be K-invariant and replace it with another function $\widetilde{\phi}$ which will turn out to be both K-finite and invariant under the center $\mathbb{R}^\times$ of $GL(2,\mathbb{R})^+$, where the $+$ indicates that the matrices are of positive determinant. In this case we define

$$\widetilde{\phi}(g) := \text{Im}(gi)^{\frac{\ell}{2}} \cdot f(gi) \cdot \left(\frac{j(g,i)}{|j(g,i)|}\right)^{-\ell}$$

for all $g \in GL(2,\mathbb{R})^+$. Here again, we have

$$\widetilde{\phi}(\gamma g) = \widetilde{\phi}(g)$$

for all $\gamma \in GL(2,\mathbb{Z})$. This is because $\text{Im}\left(\frac{az+b}{cz+d}\right) = \frac{(ad-bc)}{|cz+d|^2} \cdot y$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ and $z = x + iy$ in the upper half plane. Note that inserting the ratio $\frac{j(g,i)}{|j(g,i)|}$ ensures that $\widetilde{\phi}$ is invariant under the center of $g \in GL(2,\mathbb{R})^+$.

Now every $g \in GL(2,\mathbb{R})^+$ has a unique Iwasawa decomposition

$$g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

with $x, y, r, \theta \in \mathbb{R},\ y, r > 0$, and $0 \leq \theta < 2\pi$. It follows that

$$\widetilde{\phi}(g) = \big(\cos\theta + i\sin\theta\big)^{\ell} y^{\frac{\ell}{2}} f(x+iy) = e^{i\ell\theta} y^{\frac{\ell}{2}} f(x+iy).$$

Consider the character $\rho_\ell : SO(2,\mathbb{R}) \to \mathbb{C}^\times$ defined by

$$\rho_\ell\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) := (\cos\theta + i\sin\theta)^\ell.$$

We then see that

$$\widetilde{\phi}(gzk) = \rho_\ell(k)\widetilde{\phi}(g)$$

for all $g \in GL(2,\mathbb{R})^+$, all $z \in Z$ (here Z is the center of $GL(2,\mathbb{R})^+$) and all $k \in K$. This establishes that $\widetilde{\phi}$ is Z-invariant and K-finite.

If we assume that f or equivalently that $\widetilde{\phi}$ is an eigenfunction of the Hecke operators, then associated to $\widetilde{\phi}$ one has the Hecke L-function [32]

$$L(s,\widetilde{\phi}) := \sum_{n=1}^{\infty} a_n n^{-s} = \prod_p \left(1 - a_p p^{-s} + p^{k-1-2s}\right)^{-1},$$

the product ranging over all rational primes, where for every prime p, the complex number a_p is the eigenvalue of the p^{th} Hecke operator. The above series and product converge absolutely for $\Re(s) > (k+1)/2$ by the work of Deligne [8] who proved the Ramanujan conjecture that

$$|a_p| \le 2p^{\frac{k-1}{2}}.$$

It is well known that $L(s,\widetilde{\phi})$ has meromorphic continuation to all $s \in \mathbb{C}$ with at most a simple pole at $s = 1$ (only if $a_0 \neq 0$) and satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)L(s,\widetilde{\phi}) = \pm(2\pi)^{-(k-s)}\Gamma(k-s)L(k-s,\phi).$$

In addition to holomorphic modular forms there are also infinitely many non-holomorphic forms first found by Maass [27]. The simplest examples are of weight zero. A Maass form of weight zero is an automorphic form $f : \mathfrak{h}^2 \to \mathbb{C}$ which is left invariant under $GL(2,\mathbb{Z})$ and is also an eigenfunction of the Laplacian with Laplace eigenvalue $v(1-v)$ ($v \in \mathbb{C}$). For $z = x + iy \in \mathfrak{h}^2$, the Maass form has Fourier expansion of the form

$$f(z) = \sum_{n\neq 0} a_n \sqrt{2\pi y} K_{v-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x},$$

where for $v \in \mathbb{C}$ and $y > 0$,

$$K_v(y) = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\cdot y\left(u+u^{-1}\right)} u^v \, du$$

is the modified Bessel function of the second kind.

As before we may lift the Maass form f to a function $\widetilde{\phi} : GL(2,\mathbb{R})^+ \to \mathbb{C}$ defined by

$$\widetilde{\phi}(g) := f(g\,i), \qquad \big(g \in GL(2,\mathbb{R})^+\big)\,.$$

If the Maass form $\widetilde{\phi}$ is also an eigenfunction of the Hecke operators then the L-function associated to $\widetilde{\phi}$ is given by

$$L\big(s,\widetilde{\phi}\,\big) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_p \big(1 - a_p p^{-s} + p^{-2s}\big)^{-1}\,, \qquad (\Re(s) > 3/2)\,,$$

where for each prime p, the coefficient $a_p \in \mathbb{C}$ is the eigenvalue of the p^{th} Hecke operator. Furthermore, $L\big(s,\widetilde{\phi}\,\big)$ is an entire function and satisfies the functional equation

$$\Lambda\big(s,\widetilde{\phi}\,\big) := \pi^{-s}\,\Gamma\left(\frac{s-\frac{1}{2}+v}{2}\right)\Gamma\left(\frac{s+\frac{1}{2}-v}{2}\right) L\big(s,\widetilde{\phi}\,\big) = \Lambda\big(1-s,\widetilde{\phi}\,\big).$$

We have now exhibited some simple examples $\widetilde{\phi}$ of automorphic forms for the real group $GL(2,\mathbb{R})^+$. It is then possible to define ([18], §4.12) an adelic automorphic form $\phi_{\text{adelic}}((g_\infty, g_2, g_3, \ldots))$ on $GL(2,\mathbb{A}_\mathbb{Q})$ which is identical to $\widetilde{\phi}(g_\infty)$ when the finite adele $(g_2, g_3, \ldots, g_p, \ldots)$ is just $(I_2, I_2, I_2, \ldots)$ and I_2 is the 2×2 identity matrix.

More generally, one may consider a classical modular form f which has integer weight $\ell \geq 0$, level $N \geq 1$, character $\chi \pmod N$, and is an eigenfunction of the Hecke operators as well as the Laplacian. Then f is a smooth function of moderate growth on the upper half plane $\{z = x+iy \mid x \in \mathbb{R}, y > 0\}$ which satisfies

$$f(\gamma z) = \chi(d)(cz+d)^\ell f(z), \qquad \left(\forall\,\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\right).$$

Again, each of these classical modular forms can be lifted to an adelic form (see [18], §4.12).

One may ask whether the space of adelic automorphic forms for $GL(2,\mathbb{A}_\mathbb{Q})$ contains new objects in addition to the lifts of the classical automorphic forms? We now show that it is also possible to go in the other direction and establish that in every irreducible automorphic cuspidal representation there is a vector which is an idelic lift of a classical modular form of weight ℓ, level N, and character $\chi \pmod N$ as described above.

Fix an integer $N \geq 1$. The Iwahori subgroup $K_0(N) \subset GL(2,\mathbb{A}_\mathbb{Q})$ is defined as $K_0(N) = \prod_p K_0(N)_p$ (with the product ranging over all primes p) where

$$K_0(N)_p = \left\{ \begin{pmatrix} a & b \\ N\cdot c & d \end{pmatrix} \in GL(2,\mathbb{Z}_p) \;\middle|\; c\in\mathbb{Z}_p \right\}.$$

We have the strong approximation theorem ([18], §4.11)

$$GL(2,\mathbb{A}_{\mathbb{Q}}) = GL(2,\mathbb{Q})\, GL(2,\mathbb{R})^+\, K_0(N),$$

where $GL(2,\mathbb{Q}) \cap \big(GL(2,\mathbb{R})^+ K_0(N)\big) = \Gamma_0(N)$.

Recall that an adelic automorphic form ϕ for $GL(2,\mathbb{A}_{\mathbb{Q}})$ with central character ω is left invariant under $GL(2,\mathbb{Q})$, right K-finite, $\mathbb{Z}(\mathfrak{g})$-finite, and has moderate growth. If we assume, in addition, that ϕ is a suitable vector in an irreducible automorphic cuspidal representation then ϕ will be invariant under an Iwahori subgroup and we will have

$$\phi(gk) = \psi(k)\cdot\phi(g)$$

for all $g\in GL(2,\mathbb{A}_{\mathbb{Q}})$ and all $k\in K_0(N)$ for some Iwahori subgroup $K_0(N)$, and where $\psi : K_0(N)\to\mathbb{C}^\times$ is a character of the Iwahori subgroup. Furthermore, at the archimedean place we must have

$$\phi\left(g\cdot\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, I_2, I_2, I_2, \ldots\right)\right) = e^{\pi i\ell}\phi(g)$$

for all $g\in GL(2,\mathbb{A}_{\mathbb{Q}})$, where I_2 is the identity matrix.

We may then define the classical modular form $f:\mathfrak{h}\to\mathbb{C}$ by

$$f(x+iy) := \phi\left(\left(\begin{pmatrix} y^{\frac12} & xy^{-\frac12} \\ 0 & y^{-\frac12} \end{pmatrix}, I_2, I_2, I_2, \quad \ldots\right)\right).$$

which satisfies

$$f(\gamma z) = \chi(d)(cz+d)^{\ell} f(z)$$

for all $z\in\mathfrak{h}$, all $\gamma\in\Gamma_0(N)$, and where χ is a Dirichlet character (mod N) determined by the character ψ of the Iwahori subgroup $K_0(N)$. For the precise determination of the Dirichlet character χ, see ([18], §5.5.6).

If ϕ is a Hecke cuspform on the upper half-plane, then we first lift ϕ to a function $\widetilde{\phi}$ on the real group $GL(2,\mathbb{R})^+$, and then lift this function to an adelic automorphic form ϕ_{adelic} as above. We may then associate to ϕ an irreducible unitary infinite dimensional automorphic representation π_ϕ of $GL(2,\mathbb{A}_{\mathbb{Q}})$. This can be done as follows. We consider the following actions (denoted $\mathcal{A}$) on the adelic automorphic form ϕ_{adelic}.

- *The action of the finite adeles $GL(2,\mathbb{A}_f)$ of $\mathbb{A}_{\mathbb{Q}}$ by right translation.*
- *The action of the universal enveloping algebra $\mathfrak{U}$ by differential operators $D\in\mathfrak{U}$ (at the real place g_∞) .*

Now, define the vector space

$$V_\phi := \left\{ \sum_{\ell=1}^{N} c_\ell \cdot D_\ell \cdot \phi_{\text{adelic}}\big(g \cdot h_\ell\big) \,\middle|\, N \geq 0,\ c_l \in \mathbb{C},\ h_\ell \in GL(2, \mathbb{A}_f),\ D_\ell \in \mathfrak{U} \right\}.$$

Then V_ϕ is clearly invariant under the actions $\mathcal{A}$. The space V_ϕ with the actions $\mathcal{A}$ define the automorphic representation π_ϕ. Further, it can be shown that the Godement-Jacquet L-function $L(s, \pi_\phi) = L(s, \phi)$.

12. $GL(n)$ Examples

We shall now present some examples of $G = GL(n)$ automorphic forms over $\mathbb{A}_\mathbb{Q}$ for $n > 2$. It is enough to present examples for the real group $GL(n, \mathbb{R})$, since these may be lifted to adelic automorphic forms. We may define the generalized upper half plane

$$\mathfrak{h}^n \cong G(\mathbb{R})/(K \cdot \mathbb{R}^\times)$$

where $K = O(n, \mathbb{R})$ is the maximal compact subgroup. By the Iwasawa decomposition, every $g \in \mathfrak{h}^n$ is an element of the form $g = xy$ where

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & & x_{1,n} \\ & 1 & x_{2,3} & \cdots & & x_{2,n} \\ & & \ddots & & & \vdots \\ & & & & 1 & x_{n-1,n} \\ & & & & & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & \ddots & & & \\ & & y_1 y_2 & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

with $x_{i,j} \in \mathbb{R}$ for $1 \leq i < j \leq n$ and $y_i > 0$ for $1 \leq i \leq n-1$.

If we consider the discrete subgroup $G(\mathbb{Z})$, then an automorphic form is a function

$$\phi : G \to \mathbb{C}$$

which is invariant under $G(\mathbb{Z})$ on the left, K-invariant on the right, and is invariant under the center $\mathbb{R}^\times$ of $G(\mathbb{R})$. Further, we demand that ϕ is $\mathbb{C}^\infty$ and has moderate growth, that is

$$\big|\phi(xy)\big| \;\leq\; c \prod_{i=1}^{n-1} y_i^M$$

for some $c, M > 0$, and xy in a Siegel set, i.e., $0 \leq x_{\ell,j} < 1,\ y_i > \frac{\sqrt{3}}{2}$, for $1 \leq \ell < j \leq n$ and $1 \leq i < n$.

The space $\mathfrak{h}^n$ does not have a complex structure for $n > 2$, so there will be no holomorphic automorphic forms. There will, however, be Maass forms which we now describe. A Maass form is defined to be a complex valued function $\phi : \mathfrak{h}^n \to \mathbb{C}$ which is an automorphic form (as defined above), and in addition, is an eigenfunction of the center of the universal enveloping algebra of $\mathfrak{g}$ (denoted by $\mathcal{D}^n$) which is just the ring of $GL(n,\mathbb{R})$ invariant differential operators on $\mathfrak{h}^n$.

Since $\mathcal{D}^n$ is commutative we may construct a basis of simultaneous eigenfunctions of all $\delta \in \mathcal{D}^n$. The eigenvalues of such eigenfunctions can be expressed in terms of Langlands parameters

$$\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$$

with $\sum_{i=1}^{n} \alpha_i = 0$. We shall now explicitly describe the representation of eigenvalues of $\mathcal{D}^n$ in terms of Langlands parameters.

Let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$ denote a set of Langlands parameters. We define a character $I_\alpha : U_n(\mathbb{R})\backslash\mathfrak{h}^n \to \mathbb{C}$ by

$$I_\alpha(g) := \prod_{i=1}^{n-1}\prod_{j=1}^{n-1} y_i^{b_{i,j}\frac{\alpha_j-\alpha_{j+1}}{n}}, \qquad b_{i,j} = \begin{cases} ij & \text{if } i+j \le n, \\ (n-i)(n-j) & \text{if } i+j \ge n. \end{cases}$$

Here, the powers of the y_i are chosen to simplify later formulae.

Then I_α is an eigenfunction of all $\delta \in \mathcal{D}^n$, so we may write

$$\delta\, I_\alpha = \lambda_\delta \cdot I_\alpha,$$

where λ_δ denotes the Harish Chandra character. The Laplace eigenvalue λ_Δ can be represented in the form (see [31])

$$\lambda_\Delta = \frac{n^3-n}{24} - \frac{\alpha_1^2+\alpha_2^2+\cdots\alpha_n^2}{2}.$$

Consider Maass forms $\phi : \mathfrak{h}^n \to \mathbb{C}$. Since $\mathcal{D}^n$ is a commutative ring, we may take a basis of Maass forms consisting of Laplace eigenfunctions which are also common eigenfunctions of all $\delta \in \mathcal{D}^n$. Then ϕ will be an eigenfunction of the Laplacian Δ for $\mathfrak{h}^n$, i.e.,

$$\Delta\phi = \lambda_\Delta\phi, \qquad (\text{for some } \lambda_\Delta \in \mathbb{C}).$$

Each such Maass form ϕ will have an associated Langlands parameter $\alpha \in \mathcal{C}^n$ with associated Laplace eigenvalue $\lambda_\Delta = \frac{n^3-n}{24} - \frac{\alpha_1^2+\alpha_2^2+\cdots\alpha_n^2}{2}$.

Given Langlands parameters $\alpha \in \mathbb{C}^n$ (with Harish Chandra character λ_δ as described above) and a character ψ of the unipotent subgroup $U_n(\mathbb{R}) \subset G(R)$ then there exists a unique (up to a constant multiple) Whittaker function

$$W_\alpha : \mathfrak{h}^n \to \mathbb{C}$$

which satisfies the following properties

- $\delta W_\alpha = \lambda_\delta \cdot W_\alpha, \quad (\forall\, \delta \in \mathcal{D}^n)$,
- $W_\alpha(ug) = \psi(u)\cdot W_\alpha(g), \quad (\forall\, u \in N(\mathbb{R}),\, g \in GL(n,\mathbb{R}))$,
- W_α is invariant under all permutations of $\alpha = \{\alpha_1, \dots, \alpha_n\}$,
- W_α has holomorphic continuation to all $\alpha \in \mathbb{C}^n$,
- $W_\alpha(y)$ has rapid decay in $y_i \to \infty$ where $y = \mathrm{diag}(y_1, y_2, \cdots y_n)$.

Let $M = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$, $\Gamma_{n-1} = \mathrm{SL}(n-1,\mathbb{Z})$, and $U_{n-1} = U_{n-1}(\mathbb{Z})$. It was proved by Shalika and Piatetski-Shapiro (see [17], (9.1.2)) that every Maass form with Langlands parameter α has a Fourier-Whittaker expansion of type

$$\phi(g) = \sum_{\gamma \in U_{n-1}\backslash\Gamma_{n-1}} \sum_{M \neq 0} \frac{A(M)}{\prod\limits_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}}\, W_\alpha\left(M^* \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

where $g \in \mathfrak{h}^n$ and

$$M^* = \begin{pmatrix} m_1\cdots m_{n-2}|m_{n-1}| & & & \\ & \ddots & & \\ & & m_1 & \\ & & & 1 \end{pmatrix}.$$

Here $A(m_1, \dots, m_{n-1}) \in \mathbb{C}$ is called the M^{th} Fourier coefficient of ϕ.

We may associate to ϕ the Godement-Jacquet L-function

$$L(s,\phi) = \sum_{m=1}^{\infty} \frac{A(m,1,\dots,1)}{m^s}.$$

If the Maass form ϕ is also an eigenfunction of the Hecke operators then it has the Euler product representation (see [17])

$$\prod_p \Bigg(1 - \frac{A(p,1,\dots,1)}{p^s} + \frac{A(1,p,1,\dots,1)}{p^{2s}} - \frac{A(1,1,p,\dots,1)}{p^{3s}} + \quad \cdots \quad + (-1)^{n-1}\frac{A(1,,\dots,1,p)}{p^{(n-1)s}} + \frac{(-1)^n}{p^{ns}}\Bigg)^{-1}.$$

Now $L(s,\phi)$ is a degree n L-function which means the completed L-function has n local factors at every place and satisfies the following functional equation (see [17], Theorem 12.3.6):

$$\Lambda(s,\phi) := \pi^{-\frac{ns}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{s-\alpha_i}{2}\right) L(s,\phi) = \Lambda\big(1-s, \widetilde{\phi}\big).$$

where $\widetilde{\phi}$ denotes the dual form which has M^{th} Fourier coefficient (for $M = (m_1, m_2, \ldots, m_{n-1})$) given by $A(m_{n-1}, m_{n-2}, \ldots, m_1)$.

More generally, we may also consider automorphic forms of arbitrary weight, level, and character for the real group $GL(n,\mathbb{R})^+$ which acts on $\mathfrak{h}^n$ by left matrix multiplication. This action determines a function

$$\kappa : GL(n,\mathbb{R})^+ \times \mathfrak{h}^n \longrightarrow SO(n,\mathbb{R})$$

as follows.

By the Iwasawa decomposition every $g \in GL(n,\mathbb{R})^+$ has a unique decompostion

$$g = \widetilde{g} \cdot d \cdot k$$

with $\widetilde{g} \in \mathfrak{h}^n$, $d = r \cdot I_n (r > 0)$, and $k \in K = SO(n,\mathbb{R})$. Then for any $\gamma \in GL(n,\mathbb{R})^+$ and $g \in GL(n,\mathbb{R})$, we define $\kappa(\gamma, g)$ by

$$\gamma g = \widetilde{\gamma g} \cdot d \cdot \kappa(\gamma, g)$$

where $d = rI_n$ for some real number $r > 0$. Then $\kappa(\gamma, g)$ satisfies the cocycle identity

$$\kappa(\gamma'\gamma,\, g) = \kappa\big(\gamma,\, \widetilde{\gamma' g}\,\big) \cdot \kappa(\gamma', g).$$

One would like to generalize the notion of *"weight"* to the higher rank situation of $GL(n,\mathbb{R})^+$ with $n > 2$. In this case, the *"weight"* may be realized as a finite irreducible representation ρ of $SO(n,\mathbb{R})$ which generalizes the $GL(2)$-weight which corresponds to an irreducible representation of $SO(2,\mathbb{R})$. Of course, since $SO(2,\mathbb{R})$ is abelian, then it can only have one dimensional representations, i.e., characters.

Let $\rho : SO(n,\mathbb{R}) \to GL(r,\mathbb{C})$ be an irreducible representation. We define a function $J_\rho : GL(n,\mathbb{R})^+ \times \mathfrak{h}^n \to GL(r,\mathbb{C})$ as follows. Let $\gamma \in GL(n,\mathbb{R})^+$ and $g \in \mathfrak{h}^n$. Then we define

$$J_\rho(\gamma, g) \;:=\; \rho\Big(\kappa(\gamma, g)^{-1}\Big).$$

We now prove that J_ρ is a one-cocycle satisfying

$$J_\rho(\gamma\gamma',\, g) \;=\; J_\rho(\gamma', g)\, J_\rho\big(\gamma,\, \widetilde{\gamma' g}\,\big)$$

for all $\gamma, \gamma' \in GL(n,\mathbb{R})^+$ and all $g \in \mathfrak{h}^n$.

Proof. We have

$$\begin{aligned} J_\rho(\gamma\gamma',\, g) &= \rho\left(\kappa(\gamma\gamma',\, g)^{-1}\right) \\ &= \rho\left(\kappa(\gamma',\, g)^{-1}\cdot\kappa\left(\gamma,\, \widetilde{\gamma' g}\right)^{-1}\right) \\ &= \rho\left(\kappa(\gamma',\, g)^{-1}\right)\cdot\rho\left(\kappa\left(\gamma,\, \widetilde{\gamma' g}\right)^{-1}\right) \\ &= J_\rho(\gamma', g)\, J_\rho(\gamma,\, \widetilde{\gamma' g}). \end{aligned}$$

□

Since the *"weight"* is a representation into $GL(r,\mathbb{C})$ it is necessary to consider vector valued automorphic forms of the type

$$\Phi(g) := \begin{pmatrix}\phi_1(g)\\ \vdots \\ \phi_r(g)\end{pmatrix}, \qquad (g\in\mathfrak{h}^n)\,,$$

where each $\phi_i : \mathfrak{h}^n \to \mathbb{C}$, $(1\le i\le r)$ is smooth. We say Φ has weight ρ for a discrete subgroup $\Gamma\subset GL(n,\mathbb{R})^+$ if

$$\Phi(\gamma g) = J_\rho(\gamma, g)\cdot\Phi(g)$$

for all $\gamma\in\Gamma$ and all $g\in\mathfrak{h}^n$.

Next, we consider vector valued automorphic functions for the real group $GL(n,\mathbb{R})^+$ with level N and character. For an integer $N\ge 2$, we define the congruence subgroup $\Gamma_0(N)\subset SL(n,\mathbb{Z})$ to be the multiplicative group of all matrices of the form:

$$\begin{pmatrix}A & B\\ C & d\end{pmatrix} \text{ with } \begin{cases} A \textit{ is an } (n-1)\times(n-1) \textit{ matrix with entries in } \mathbb{Z}, \\ B \textit{ is a column vector with entries in } \mathbb{Z}, \\ C \textit{ is a row vector with entries in } N\cdot\mathbb{Z}, \\ d\in\mathbb{Z}. \end{cases}$$

In addition, we define $\Gamma_0(1) := SL(n,\mathbb{Z})$.

We call N the *"level."* For a given level N we may consider introduce a *"character"* which we take to be a Dirichlet character $\chi \pmod N$. We say a vector valued automorphic function of the type Φ above has weight ρ, level N, and character χ if

$$\Phi(\gamma g) = \chi(d)\, J_\rho(\gamma, g)\,\Phi(g)$$

for all

$$\gamma = \begin{pmatrix}A & B\\ C & d\end{pmatrix}\in\Gamma_0(N).$$

Next, consider a vector valued automorphic function Φ on the real group $GL(n,\mathbb{R})^+$ of weight ρ, level N, and character χ for some r-dimensional representation of $SO(n,\mathbb{R})$, which is Z-finite, $\mathbb{Z}(\mathfrak{g})$-finite, and has moderate growth. We will show that Φ can be lifted to an adelic automorphic form on $GL(n,\mathbb{A}_\mathbb{Q})$. One immediate problem that arises is the fact that a vector valued automorphic function takes values in $\mathbb{C}^r$ while an adelic automorphic form always takes values in $\mathbb{C}$.

Fix an integer $N \geq 1$. The Iwahori subgroup $K_0(N) \subset GL(n,\mathbb{A}_\mathbb{Q})$ is defined as $K_0(N) = \prod_p K_0(N)_p$ (with the product ranging over all primes p) where

$$K_0(N)_p = \left\{ \begin{pmatrix} A & B \\ N\cdot C & d \end{pmatrix} \in GL(2,\mathbb{Z}_p) \right\},$$

where A is an $(n-1)\times(n-1)$ matrix with entries in $\mathbb{Z}_p$, where B is a column vector with entries in $\mathbb{Z}_p$, while C is a row vector with entries in Z_p, and $d \in \mathbb{Z}_p$.

We have the strong approximation theorem ([19], Proposition 13.3.3)

$$GL(n,\mathbb{A}_\mathbb{Q}) = GL(n,\mathbb{Q})\, GL(n,\mathbb{R})^+\, K_0(N),$$

where $GL(n,\mathbb{Q}) \cap \big(GL(n,\mathbb{R})^+ K_0(N)\big) = \Gamma_0(N)$.

Strong approximation can be used to define the adelic lift

$$\Phi_{\text{adelic}} : GL(n,\mathbb{A}_\mathbb{Q}) \to \mathbb{C}^r$$

given by

$$\Phi_{\text{adelic}}(\gamma g_\infty k) := \psi(k)\, J_\rho(g_\infty, I_n)\, \Phi(g_\infty), \qquad (\text{for all } g_\infty \in GL(n,\mathbb{R})^+)$$

where $k \in K_0(N)$ and $\gamma = (\alpha,\alpha,\alpha,\ \dots\ ) \in GL(n,\mathbb{A}_\mathbb{Q})$ where we have $\alpha \in GL(n,\mathbb{Q})$. Here ψ will be a character of the Iwahori subgroup $K_0(N)$. One may show that (see [19], Lemma 13.4.8) that

$$\Phi_{\text{adelic}}(g) = \begin{pmatrix} \phi_1^*(g) \\ \vdots \\ \phi_r^*(g) \end{pmatrix}, \qquad \big(\text{for all } g \in GL(n,\mathbb{A}_\mathbb{Q})\big),$$

where each ϕ_i^* $(i = 1, 2, \dots, r)$ is an adelic automorphic form.

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FIGURE 1. Hervé Jacquet and Robert Langlands
(Courtesy of the Simons Foundation)