Recent progress on Beilinson–Bloch–Kato conjecture

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- the ($\mathbb{Q}_\ell$-)Selmer group

$$\mathrm{Sel}_{\mathbb{Q}_\ell}(A) := \left(\varprojlim_n \mathrm{Sel}_{\ell^n}(A)\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

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Conjecturally, $L(s, A)$ has an analytic continuation to the entire complex plane and satisfies a functional equation with center $s = 1$. This is known in many cases when F is totally real.

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We have

$$\operatorname{rank} A(F) = \operatorname{ord}_{s=1} L(s, A) = \dim_{\mathbb{Q}_\ell} \operatorname{Sel}_{\mathbb{Q}_\ell}(E)$$

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- *(Rubin, Skinner–Urban, Skinner, W. Zhang, Burungale–Skinner–Tian) If $\dim_{\mathbb{Q}_\ell} \mathrm{Sel}_{\mathbb{Q}_\ell}(A) \leq 1$ for some ℓ, then the above conjecture holds.*

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If we take $M = \mathrm{h}^1(A)(1)$ with the canonical polarization from Poincaré duality, then we have $L(s, M) = L(s+1, A)$ and $\mathrm{H}^1_{\mathrm{f}}(F, M_\ell) = \mathrm{Sel}_{\mathbb{Q}_\ell}(A)$.

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Conjecture (Beilinson–Bloch, Bloch–Kato)

Let M be a polarized motive as above. Then we have

$$\mathrm{ord}_{s=0} L(s, M) = \dim_{E_\lambda} \mathrm{H}^1_{\mathrm{f}}(F, M_\lambda) - \dim_{E_\lambda} \mathrm{H}^0(F, M_\lambda)$$

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If we take $M = \mathrm{h}^1(A)(1)$ with the canonical polarization (and $E = \mathbb{Q}$), then we recover the B-SD conjecture for A.

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From now on, we assume that F is CM. Denote by F^+ the maximal totally real subfield contained in F, and $\mathrm{c} \in \mathrm{Gal}(F/F^+)$ the Galois involution.

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$$\mathrm{I}(\arg^{N-1}, \arg^{N-3}, \cdots, \arg^{3-N}, \arg^{1-N}).$$

Here, $\arg\colon \mathbb{C}^\times \to \mathbb{C}^\times$ is the argument character $z \mapsto \frac{z}{\sqrt{z\bar{z}}}$.

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For example, when $N = 1$, Π_v is the trivial character; when $N = 2$, Π_v is the base change of the discrete series of $\mathrm{GL}_2(\mathbb{R})$ of weight 2.

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We will regard

$$M_\Pi := \{\rho_{\Pi,\lambda}\}_\lambda$$

as an automorphic motive over F, with coefficient E.

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Regarding the B-B-K conjecture for the Rankin–Selberg motive M, in a joint work (close to be done) with Yichao Tian (Strasbourg), Liang Xiao (Peking), Wei Zhang (MIT), and Xinwen Zhu (Caltech), we show the following theorem.

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Theorem (LTXZZ)

Let F/F^+ be a CM extension. Let Π_n and Π_{n+1} be relevant representations of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{GL}_{n+1}(\mathbb{A}_F)$, respectively. Assume $F^+ \neq \mathbb{Q}$ if $n > 2$.

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If $L(\frac{1}{2}, \Pi_n \times \Pi_{n+1}) \neq 0$, then we have

$$\mathrm{H}^1_{\mathrm{f}}(F, \rho_{\Pi_n,\lambda} \otimes_{E_\lambda} \rho_{\Pi_{n+1},\lambda}(n)) = 0$$

for every admissible prime λ of E.

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  Moreover, we have $\mathrm{H}^1_{\mathrm{f}}(F_v, M_\lambda) = 0$ if v has different residue characteristic with λ, which is a consequence of the purity property. Thus, $\mathrm{H}^1_{\mathrm{f}}(F, M_\lambda)$ is the subspace of $\mathrm{H}^1(F, M_\lambda)$ consisting of elements α satisfying

$$\mathrm{loc}_v(\alpha) \in \ker\left(\mathrm{H}^1(F_v, M_\lambda) \to \mathrm{H}^1(F_v, M_\lambda \otimes \mathbb{B}_\lambda^{\mathrm{cris}})\right)$$

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  for every prime v of F of same residue characteristic with λ.
- Heuristically, it is believed that for "generic" Π_n and Π_{n+1}, all but finitely prime λ of E should be admissible. However, due to the lack of knowledge on the Galois image of Rankin–Selberg automorphic motives, we do not even know the existence of a single admissible prime, except for the situation in the following theorem.

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Theorem (LTXZZ)

Let F/F^+ be a CM extension and $n \geq 1$ an integer. Let A_1 and A_2 be two modular elliptic curves over F^+ geometrically without complex multiplication, and geometrically non-isogenous. Assume both $\mathrm{Sym}^{n-1} A_1$ and $\mathrm{Sym}^n A_2$ are modular. Assume $F^+ \neq \mathbb{Q}$ if $n > 2$.

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If (the central critical value)

$$L(n, \mathrm{Sym}^{n-1} A_{1,F} \times \mathrm{Sym}^n A_{2,F}) \neq 0,$$

then we have

$$\mathrm{H}^1_{\mathrm{f}}(F, \mathrm{Sym}^{n-1} \mathrm{V}_\ell(A_1) \otimes_{\mathbb{Q}_\ell} \mathrm{Sym}^n \mathrm{V}_\ell(A_2)(1-n)) = 0$$

for all but finitely many rational prime ℓ.

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- $\mathrm{Sym}^3 A$ is modular (Kim–Shahidi, 2002),
- $\mathrm{Sym}^4 A$ is modular (Kim, 2002),
- $\mathrm{Sym}^5 A$ and $\mathrm{Sym}^6 A$ are modular if $\mathbb{K}$ is totally real and linearly disjoint from $\mathbb{Q}(\zeta_5)$ over $\mathbb{Q}$ (Clozel–Thorne, 2015),
- $\mathrm{Sym}^7 A$ is modular if $\mathbb{K}$ is totally real and linearly disjoint from $\mathbb{Q}(\zeta_{35})$ over $\mathbb{Q}$ (Clozel–Thorne, 2015),
- $\mathrm{Sym}^8 A$ is modular if $\mathbb{K}$ is totally real and linearly disjoint from $\mathbb{Q}(\zeta_7)$ over $\mathbb{Q}$ (Clozel–Thorne, 2015).

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such that the functional

$$\mathcal{P}(f_n, f_{n+1}) \coloneqq \int_{G_n(F^+)\backslash G_n(\mathbb{A}_{F^+}^\infty)} f_n(g) f_{n+1}(g) \mathrm{d}g$$

on $f_n \in \pi_n$ and $f_{n+1} \in \pi_{n+1}$ is nonzero. Here, we regard G_n as a subgroup of G_{n+1} in the natural way.

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Put $G'_N := \mathrm{U}(\mathrm{V}'_N)$. We also obtain an open compact subgroup K' of $G'_N(\mathbb{A}^\infty_{F^+})$ from K by changing (the hyperspecial subgroup) $K_\mathfrak{p}$ to the stabilizer $K'_\mathfrak{p}$ of a nearly self-dual lattice, which is a special maximal subgroup.

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We obtain a natural integral model $\mathcal{X}$ of the Shimura variety $\mathrm{Sh}(G'_N, K'_N)$ (with the reflex field F) over $O_{F_{\mathfrak{p}}} \simeq \mathbb{Z}_{p^2}$.

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In other words, there exists an automorphic representation π'_N of $G'_N(\mathbb{A}_{F^+})$ appearing in the middle cohomology that is congruent to π_N modulo λ. Moreover, π'_N is not unramified at $\mathfrak{p}$.

Thank you!