Singularities and perfectoid geometry

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$6accdae13eff7i3l9n4o4qrr4s8t12ux$

This is Newton's anagram, from his second letter to Leibniz (1677):
... The foundation of these operations is evident enough, in fact; but because I cannot proceed with the explanation of it now, I have preferred to conceal it thus: 6accdae13eff7i3l9n4o4qrr4s8t12ux.

decoded as:
Data aequatione quotcunque fluentes quantitates involvente, fluxiones invenire; et vice versa.
[Given an equation involving any number of fluent quantities to find the fluxions, and vice versa.]

and usually grossly translated as:

It is useful to solve differential equations.

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300 years later, new paradigm:

It is (also) useful *not to solve* differential equations
... but study their structure.

Grothendieck: classical resolvant as descent datum, crystals

Sato, Kashiwara: solutions and cosolutions on equal footing;

algebraic analysis as *homological theory* of differential modules.

Similar situation in algebraic singularity theory

Different perspectives on singularities:

- viewed as nuisances: get rid of them -> *resolution* of singularities.

- viewed as jewels of commutative algebra: cultivate and classify them -> *homological theory* of singularities.

Natural dichotomy between “mild" singularities and others:

Cohen-Macaulay singularities versus non-CM singularities.

CM singularities satisfy Serre duality, allow concrete calculations of syzygies etc.

Homological characterization: $(S, \mathfrak{m})$: local ring.

[Auslander-Serre] S is regular $\Leftrightarrow$ every finite S-module has a finite free resolution $\Leftrightarrow$ $S/\mathfrak{m}$ has a finite free resolution.

[Peskine-Szpiro-Roberts] S is CM $\Leftrightarrow$ some nonzero S-module of finite length has a finite free resolution.

Reminder. $(S, \mathfrak{m})$: local ring, M: S-module.

$x_1, \ldots, x_d \in \mathfrak{m}$
- is a system of parameters (s.o.p.) if $S/\underline{x}S$ has Krull dimension 0
- is an M-regular sequence if
$M/(x_1, \ldots, x_{i-1})M \stackrel{\cdot x_i}{\to} M/(x_1, \ldots, x_{i-1})M$ injective $(i = 1, \ldots, d)$, and $M \neq \underline{x}M$.

M is a *CM module* (resp. S is a CM ring) if any system of parameters $\underline{x}$ is M-regular (resp. S-regular).

- *if S is CM*, any CM module is a direct limit of finite CM modules (Holm)

- *if S is regular*, an S-module is CM iff it is faithfully flat.

What to do in front of a non-CM local ring S?

• first attitude - get rid of the problem: CM resolution

Theorem [Faltings ’78, Kawasaki ’00, Cesnavicius ’19]

S: quasi-excellent noetherian ring.
There exists a projective morphism $Y \to X = \mathrm{Spec}\, S$ with Y CM, which is an isomorphism over the CM locus of X.

Corollary [Cesnavicius]: *every proper, smooth scheme over a number field admits a proper, flat, Cohen-Macaulay model over the ring of integers.*

• second attitude: look for (big) CM algebras (Hochster's problem): a not necessarily noetherian S-algebra T which is a CM S-module.

$Y = \mathrm{Spec}\ T \to X = \mathrm{Spec}\ S.$

In the first approach (CM resolution), any s.o.p. on Y is regular, but a s.o.p on X need not become a s.o.p. on Y.

In the second approach, any s.o.p. on X becomes regular on Y, but a s.o.p on Y needs not be regular.

The second approach

- provides an efficient tool to investigate non CM singularities: "ideal closure" theory
I ideal of $S \rightsquigarrow \bar{I} := IT \cap S$.

- replaces to some extent unavailable resolutions of singularities (in residual char. $p > 0$).

Main Theorem [A. ’16, ’18]

(Big) Cohen-Macaulay algebras exist, and are weakly functorial.

Questions:
what does this mean?
how is this proved?
what does this imply about singularities?

What does this mean?

1) for any complete local ring S, there is a CM S-algebra T,

2) for any chain of local homomorphisms $S_1 \to \ldots \to S_n$ of complete local domains, there is a compatible chain
$T_1 \to \ldots \to T_n$
of CM algebras for $S_1, \ldots S_n$ respectively.

(conjectured by Hochster-Huneke, proved by them in equal characteristic.)

Geometric form of 1):
For any regular ring R and any finite extension S, there is an S-algebra T which is faithfully flat over R.

3 applications in commutative algebra:

- *direct summand conjecture* [Hochster '69]:
any finite extension S of a regular ring R splits (as R-module).

- *another direct summand conjecture*:
any ring S which is a direct summand (as S-module) of a regular ring R is Cohen-Macaulay.

- *syzygy conjecture* [Evans-Griffiths '81]:
any n-th syzygy module of a finite module M of projective dimension $> n$ has rank $\geqslant n$.

How is this proved (in mixed characteristic)?

using deep ramification: perfectoid spaces.

Perfectoid valuation rings.
K: complete, non discretely valued field of mixed char. $(0, p)$.
K^o: valuation ring.
$\varpi \in K^o$, $p \in \varpi^p K^o$.

Proposition [Gabber-Ramero]

The following are equivalent:

- $F : K^o/\varpi \stackrel{x \mapsto x^p}{\to} K^o/\varpi^p$ *is an isomorphism*
- $\Omega_{\bar{K}^o/K^o} = 0$.

One then says that K^o is *perfectoid* [Scholze] or *deeply ramified* [Coates-Greenberg].

Ex: $K^o = W(k)\langle p^{1/p^\infty}\rangle, \ \ \varpi = p^{1/p}$.

Perfectoid K^o-algebras.

A: p-adically complete, p-torsionfree K^o-algebra.

Definition [Scholze]

A is *perfectoid* if $F : A/\varpi \overset{x \mapsto x^p}{\to} A/\varpi^p$ is an isomorphism.

Glueing: perfectoid spaces over K.

Tilting: $K^{\flat o} := \lim_F K^o$: complete perfect valuation ring of char. p.
$K^\flat$: its field of fractions. Tilting equivalence (Scholze):

$$\text{perfectoid spaces}/K \leftrightarrow \text{perfectoid spaces}/K^\flat.$$

Ex: $A = p$-adic completion of $K^o[[\underline{x}^{1/p^\infty}]]$: perfectoid K^o-algebra.

Adjoining p^{1/p^∞}-roots of an element $g \in A$:

Theorem [A. '16; improved by Gabber-Ramero '19]

The completed p-root closure of $A[g^{1/p^\infty}]$ is perfectoid and faithfully flat over A.

[p-root closure of a p-adic ring R: elements r of $R[1/p]$ such that $r^{p^j} \in R$ for some $j > 0$.]

Almost algebra (Faltings, Gabber-Ramero): given a commutative ring $\mathfrak{V}$ and an idempotent ideal $\mathfrak{I}$, “neglect" $\mathfrak{V}$-modules killed by $\mathfrak{I}$.

This goes much beyond (Gabriel) categorical localization: notions of almost finite, almost flat, almost etale...

Standard set-up: $(\mathfrak{V}, \mathfrak{I}) = (K^o, p^{\frac{1}{p^\infty}} K^o)$ as above; we say $p^{\frac{1}{p^\infty}}$-almost: “$p^{\frac{1}{p^\infty}}$-almost zero" means “killed by $p^{\frac{1}{p^\infty}}$".

We need a non-standard set-up:
$(\mathfrak{V}, \mathfrak{I}) = (K^o[t^{1/p^\infty}], (pt)^{\frac{1}{p^\infty}} K^o[t^{1/p^\infty}])$ as above; we say $(pt)^{\frac{1}{p^\infty}}$-almost.

$\rightsquigarrow$ notion of $(pt)^{\frac{1}{p^\infty}}$-almost perfectoid algebra.

Perfectoid Abhyankar lemma

Abhyankar's classical lemma: under appropriate assumptions (tameness...), one can achieve etaleness of a given finite extension by adjoining roots of the discriminant (rather than inverting it).

Analog for finite ramified extensions of perfectoid algebras:

Theorem [A. '16]

A: perfectoid $K^o[t^{1/p^\infty}]$-algebra: $t \mapsto g \in A$ nonzero divisor.
B': finite etale $A[1/pg]$-algebra.
B: integral closure of A in B'.
Then B is $(pt)^{\frac{1}{p^\infty}}$-almost perfectoid, and for any $n > 0$, B/p^n is $(pt)^{\frac{1}{p^\infty}}$-almost faithfully flat and almost finite etale over A/p^n.

Application to CM algebras.

- S: complete local domain char. $(0, p)$ with perfect residue field k (for simplicity).

We want to construct a (big) CM S-algebra.

View S as a finite extension of some $R = W(k)[[\underline{x}]]$ (Cohen).
Then an S-algebra is a CM S-algebra iff it is faithfully flat over R.

- $g \in R$ such that $S[1/pg]$ finite etale over $R[1/pg]$.
- $K^o = W(k)\langle p^{1/p^\infty}\rangle$: perfectoid valuation ring.
- A: completed p-root closure of $K^o[[\underline{x}^{1/p^\infty}]][g^{1/p^\infty}]$: perfectoid and faithfully flat over R.
- $B' = A[1/pg] \otimes_R S$: finite etale extension of $A[1/pg]$.
- B: integral closure of A in B'

$\rightsquigarrow$ $(pg)^{\frac{1}{p^\infty}}$-almost perfectoid almost CM S-algebra.

How to get rid of almost? 2 ways:

1) Hochster's modifications.
2) Gabber's trick: $B \rightsquigarrow B^{\mathbb{N}}/B^{(\mathbb{N})} \rightsquigarrow \widetilde{B} = \Sigma^{-1}(B^{\mathbb{N}}/B^{(\mathbb{N})})$,
Σ: multiplicative system $(pg)^{\varepsilon_i}, \varepsilon_i \to 0 \in \mathbb{N}[1/p]$.

B almost perfectoid $(pg)^{\frac{1}{p^\infty}}$-almost CM S-algebra $\Rightarrow \widetilde{B}$ perfectoid CM S-algebra. □

Weak functoriality uses similar techniques, but is more difficult...

Theorem [A. '18]

Any finite sequence $R_0 \xrightarrow{f_1} R_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} R_n$ of local homomorphisms of complete Noetherian local domains, with R_0 of mixed characteristic, fits into a commutative diagram

$$\begin{array}{ccccccc}
R_0 & \xrightarrow{f_1} & R_1 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_n} & R_n \\
\downarrow & & \downarrow & & & & \downarrow \\
R_0^+ & \xrightarrow{f_1{}^+} & R_1^+ & \xrightarrow{f_2{}^+} & \cdots & \xrightarrow{f_n{}^+} & R_n^+ \\
\downarrow & & \downarrow & & & & \downarrow \\
C_0 & \longrightarrow & C_1 & \longrightarrow & \cdots & \longrightarrow & C_n
\end{array} \tag{1}$$

where
R_i^+ is the absolute integral closure of R_i,
C_i is a perfectoid CM R_i-algebra if R_i is of mixed characteristic (resp. a perfect CM R_i-algebra if R_i is of positive characteristic).
Moreover, the f_i^+ can be given in advance.

Kunz' theorem in mixed characteristic

- S: Noetherian ring of char. p. Kunz' classical theorem ('69: beginning of the use of F in commutative algebra):

S is regular $\Leftrightarrow$ $S \xrightarrow{F} S$ is flat $\Leftrightarrow$ there exists a perfect faithfully flat S-algebra.

- S: Noetherian p-adically complete ring.

Theorem [Bhatt-Iyengar-Ma '18]

S is regular $\Leftrightarrow$ there exists a perfectoid faithfully flat S-algebra.*

(Note: such an algebra is a CM S-algebra).

Applications to singularities: symbolic powers

S: Noetherian ring, $\mathfrak{p}$: prime ideal.
Symbolic powers are defined by

$$\mathfrak{p}^{(n)} := (\mathfrak{p}^n S_{\mathfrak{p}}) \cap \mathfrak{p}.$$

If $S =$ f. g. algebra over a field, $\mathfrak{p}^{(n)} =$ ideal of functions which vanish at $V(\mathfrak{p})$ at order at least n (Zariski).

$\mathfrak{p}^{(n)} \supset \mathfrak{p}^n$,
$\mathfrak{p}^{(n)} = \mathfrak{p}^n$ if $\mathfrak{p}$ is generated by a regular sequence.

To compare $\mathfrak{p}^{(n)}$ and $\mathfrak{p}^n$ in general is a classical problem, with applications in complex analysis, interpolation theory (fat points) or transcendental number theory (Waldschmidt constants).

Theorem [Ma, Schwede '18]

S: excellent regular ring of dim. d.
For any prime $\mathfrak{p}$ and any n, $\mathfrak{p}^{(dn)} \subset \mathfrak{p}^n$.

Proved by Ein-Lazarsfeld-Smith in char. 0 using subadditivity of the "multiplier ideal"; by Hochster in char. p.
In mixed characteristic: new notion of multiplier ideal in which the complex $R\Gamma(Y, \mathcal{O}_Y)$ attached to a resolution of $V(\mathfrak{p})$ is replaced by a perfectoid Cohen-Macaulay algebra for $S_{\mathfrak{p}}$.

Applications to singularities: rational singularities

Slogan: perfectoid CM algebras play somehow the role of resolution of singularities in char. 0.

$(S, \mathfrak{m})$: local domain, essentially of finite type over $\mathbb{C}$.
$\pi : Y \to \operatorname{Spec} S$: resolution of singularities.
Grauert-Riemenschneider: $R^i\Gamma(Y, \omega_Y) = 0$ for $i > 0$,
Local duality: $\mathbb{H}^j_{\mathfrak{m}}(R\Gamma(Y, \mathcal{O}_Y)) = 0$ for $j < \dim S$.
$R\Gamma(Y, \mathcal{O}_Y) \in D^b(S)$: "derived avatar" of a CM algebra.
In mixed characteristic or in char. p, replace this object by suitable (big) Cohen-Macaulay S-algebras.

Reminder: S (as before) “is" a *rational singularity* if and only if $R\Gamma(Y, \mathcal{O}_Y) \cong S$.

(Grauert-Riemenschneider+duality: any rational singularity is CM).

Question: how to check that a singularity is rational
without computing a resolution?

Criteria by reduction mod. p, after spreading out (Hara, Smith, Mehta-Srinivas):

S rational singularity
$\Leftrightarrow$ (S mod. p) F-rational singularity for all $p >> 0$
(i.e. CM + top local cohomology = simple Frobenius module).

Theorem [Ma, Schwede]

S rational singularity $\Leftrightarrow$ (S mod. p) F-rational singularity for some p.

For small p, checkable property on Macaulay2.
Perfectoid CM S-algebras (existence and weak functoriality) serve here as a bridge between char. p and char. 0, to prove that the algorithm works
(application of p-adic techniques to complex algebraic geometry).