The arithmetic fundamental lemma for the diagonal cycles

Wei Zhang

Massachusetts Institute of Technology

The first JNT Biennial, Cetraro, 2019

Part I

Arithmetic GGP conjecture

“Local-to-Global principle"

Let E be elliptic curve over $\mathbb{Q}$. B-SD conjecture:

$$\prod_{p<x} \frac{\#E(\mathbb{F}_p)}{p} \to \infty \Longrightarrow \#E(\mathbb{Q}) = \infty.$$

[comp.

$$\sum_{p<x} \frac{\#\text{“Sym}^n E\text{''}(\mathbb{F}_p)}{p^{n/2}} \sim \begin{cases} x/\log x, & n = 0 \\ o(x/\log x), & n \geq 1. \end{cases}$$

implies the Sato–Tate conjecture on the distribution of Frob eigenvalues.]

“Local-to-Global principle"

Let E be elliptic curve over $\mathbb{Q}$. B-SD conjecture:

$$\prod_{p<x} \frac{\#E(\mathbb{F}_p)}{p} \to \infty \Longrightarrow \#E(\mathbb{Q}) = \infty.$$

[comp.

$$\sum_{p<x} \frac{\#\text{“Sym}^n E\text{''}(\mathbb{F}_p)}{p^{n/2}} \sim \begin{cases} x/\log x, & n = 0 \\ o(x/\log x), & n \geq 1. \end{cases}$$

implies the Sato–Tate conjecture on the distribution of Frob eigenvalues.]

Higher dim case: Hasse–Weil L-functions and Chow groups

Let $X/\mathbb{Q}$ be a smooth projective variety of odd dimension $2m-1$. For good primes p,

$$\zeta_{X,p}(s) = \exp\left(\sum_{k\geq 1} \frac{\#X(\mathbb{F}_{p^k})}{k\, p^{ks}}\right),$$

Hasse–Weil zeta & L-functions

$$\zeta_X(s) = \prod_{p,\text{ good}} \zeta_{X,p}(s)$$
$$= \prod_{i=0}^{2\dim X} L(s, H^i(X))^{(-1)^i}.$$

Higher dim case: Hasse–Weil L-functions and Chow groups

Let

$$\mathrm{Ch}^*(X)_0 \subset \mathrm{Ch}^*(X)_{\mathbb{Q}}$$

be the Chow group of homological trivial cycles and Chow group, resp..

Conjecture (B-SD, Beilinson, Bloch)

$$\mathrm{ord}_{s=\mathrm{center}} L(s, H^{2m-1}(X)) = \dim \mathrm{Ch}^m(X)_0$$

This is may be viewed as a "Local-to-Global principle" for $\mathrm{Ch}(X)_0$.

Higher dim case: Hasse–Weil L-functions and Chow groups

Let

$$\mathrm{Ch}^*(X)_0 \subset \mathrm{Ch}^*(X)_{\mathbb{Q}}$$

be the Chow group of homological trivial cycles and Chow group, resp..

Conjecture (B-SD, Beilinson, Bloch)

$$\mathrm{ord}_{s=\mathrm{center}} L(s, H^{2m-1}(X)) = \dim \mathrm{Ch}^m(X)_0$$

This is may be viewed as a “Local-to-Global principle" for $\mathrm{Ch}(X)_0$.

Higher dim case: Hasse–Weil L-functions and Chow groups

Let

$$\mathrm{Ch}^*(X)_0 \subset \mathrm{Ch}^*(X)_{\mathbb{Q}}$$

be the Chow group of homological trivial cycles and Chow group, resp..

Conjecture (B-SD, Beilinson, Bloch)

$$\mathrm{ord}_{s=\mathrm{center}} L(s, H^{2m-1}(X)) = \dim \mathrm{Ch}^m(X)_0$$

This is may be viewed as a “Local-to-Global principle" for $\mathrm{Ch}(X)_0$.

A modest goal motivated by the Gross–Zagier formula

For X whose L-functions are known to be analytic, we hope to show

$$\text{“}\mathrm{ord}_{s=\mathrm{center}} L(s, H^{\mathrm{mid}}(X)) = 1 \Longrightarrow \dim \mathrm{Ch}^m(X)_0 \neq 0.\text{”}$$

For a Shimura datum $(\mathrm{G}, \mathcal{D}_{\mathrm{G}})$, the cohomology of the Shimura variety $X = \mathrm{Sh}_K(G, \mathcal{D}_G)$ is expected to be

$$\mathrm{H}^*(X) = \bigoplus_{\substack{\pi \\ \text{generic}}} \pi^K \otimes \rho_\pi \bigoplus \{\text{others}\},$$

The modest goal is to show a result of the following type

$$\text{“}\mathrm{ord}_{s=\mathrm{center}} L(s, \pi) = 1 \Longrightarrow \dim \mathrm{Ch}^m(X)_0[\pi] \neq 0\text{”}.$$

A modest goal motivated by the Gross–Zagier formula

For X whose L-functions are known to be analytic, we hope to show

$$\text{“}\mathrm{ord}_{s=\mathrm{center}} L(s, H^{\mathrm{mid}}(X)) = 1 \Longrightarrow \dim \mathrm{Ch}^m(X)_0 \neq 0.\text{”}$$

For a Shimura datum $(\mathrm{G}, \mathcal{D}_{\mathrm{G}})$, the cohomology of the Shimura variety $X = \mathrm{Sh}_K(G, \mathcal{D}_G)$ is expected to be

$$\mathrm{H}^*(X) = \bigoplus_{\substack{\pi \\ \text{generic}}} \pi^K \otimes \rho_\pi \bigoplus \{\text{others}\},$$

The modest goal is to show a result of the following type

$$\text{“}\mathrm{ord}_{s=\mathrm{center}} L(s, \pi) = 1 \Longrightarrow \dim \mathrm{Ch}^m(X)_0[\pi] \neq 0\text{”}.$$

Special subvarieties

A *special pair* of Shimura data is a homomorphism

$$(\mathrm{H}, \mathcal{D}_{\mathrm{H}}) \longrightarrow (\mathrm{G}, \mathcal{D}_{\mathrm{G}})$$

such that

1. the pair (H, G) is *spherical*, and
2. the dimensions (as complex manifolds) satisfy

$$\dim_{\mathbb{C}} \mathcal{D}_{\mathrm{H}} = \frac{\dim_{\mathbb{C}} \mathcal{D}_{\mathrm{G}} - 1}{2}.$$

Example (Gross–Zagier pair)

Let $K = \mathbb{Q}[\sqrt{-D}]$ be an imaginary quadratic field. Let

$$\mathrm{H} = \mathrm{R}_{K/\mathbb{Q}}\mathbb{G}_m \subset \mathrm{G} = \mathrm{GL}_{2,\mathbb{Q}}.$$

Then $\dim \mathcal{D}_{\mathrm{G}} = 1, \dim \mathcal{D}_{\mathrm{H}} = 0.$

Some more examples (over $\mathbb{R}$)

1. Gan–Gross–Prasad pairs

	$G_{\mathbb{R}}$	$H_{\mathbb{R}}$
unitary groups	$U(1, n-2) \times U(1, n-1)$	$U(1, n-2)$
orthogonal groups	$SO(2, n-2) \times SO(2, n-1)$	$SO(2, n-2)$

2. Symmetric pairs

	$G_{\mathbb{R}}$	$H_{\mathbb{R}}$
unitary groups	$U(1, 2n-1)$	$U(1, n-1) \times U(0, n)$
orthogonal groups	$SO(2, 2n-1)$	$SO(2, n-1) \times SO(0, n)$

Some more examples (over $\mathbb{R}$)

1. Gan–Gross–Prasad pairs

	$G_{\mathbb{R}}$	$H_{\mathbb{R}}$
unitary groups	$U(1, n-2) \times U(1, n-1)$	$U(1, n-2)$
orthogonal groups	$SO(2, n-2) \times SO(2, n-1)$	$SO(2, n-2)$

2. Symmetric pairs

	$G_{\mathbb{R}}$	$H_{\mathbb{R}}$
unitary groups	$U(1, 2n-1)$	$U(1, n-1) \times U(0, n)$
orthogonal groups	$SO(2, 2n-1)$	$SO(2, n-1) \times SO(0, n)$

Arithmetic diagonal cycles

- For the *unitary* GGP pair (H, G), we obtain the *arithmetic diagonal cycle*

$$\mathrm{Sh}_{\mathrm{H}} \longrightarrow \mathrm{Sh}_{\mathrm{G}}\ ,$$

(for certain level sugroups $K_{\mathrm{H}}, K_{\mathrm{G}}$).

- Arithmetic GGP conjecture: for generic π,

$$\mathrm{ord}_{s=1/2} L(\pi, s) = 1 \Longrightarrow [\mathrm{Sh}_H]_\pi \neq 0 \in \mathrm{Ch}(X)_0.$$

- $n = 2, \dim \mathrm{Sh}_G = 1$: Gross–Zagier, S. Zhang, Yuan–Zhang–Zhang.
- Exceptional example: Liu's special cycles (for GGP $\mathrm{U}(n) \times \mathrm{U}(n)$).

Arithmetic diagonal cycles

- For the *unitary* GGP pair (H,G), we obtain the *arithmetic diagonal cycle*

$$\mathrm{Sh}_{\mathrm{H}} \longrightarrow \mathrm{Sh}_{\mathrm{G}} \ ,$$

(for certain level sugroups $K_{\mathrm{H}}, K_{\mathrm{G}}$).

- Arithmetic GGP conjecture: for generic π,

$$\mathrm{ord}_{s=1/2} L(\pi, s) = 1 \Longrightarrow [\mathrm{Sh}_H]_\pi \neq 0 \in \mathrm{Ch}(X)_0.$$

- $n = 2, \dim \mathrm{Sh}_G = 1$: Gross–Zagier, S. Zhang, Yuan–Zhang–Zhang.
- Exceptional example: Liu's special cycles (for GGP $\mathrm{U}(n) \times \mathrm{U}(n)$).

Arithmetic diagonal cycles

- For the *unitary* GGP pair (H, G), we obtain the *arithmetic diagonal cycle*

$$\mathrm{Sh}_{\mathrm{H}} \longrightarrow \mathrm{Sh}_{\mathrm{G}}\ ,$$

(for certain level sugroups $K_{\mathrm{H}}, K_{\mathrm{G}}$).

- Arithmetic GGP conjecture: for generic π,

$$\mathrm{ord}_{s=1/2} L(\pi, s) = 1 \Longrightarrow [\mathrm{Sh}_H]_\pi \neq 0 \in \mathrm{Ch}(X)_0.$$

- $n = 2, \dim \mathrm{Sh}_G = 1$: Gross–Zagier, S. Zhang, Yuan–Zhang–Zhang.
- Exceptional example: Liu's special cycles (for GGP $\mathrm{U}(n) \times \mathrm{U}(n)$).

Arithmetic diagonal cycles

- For the *unitary* GGP pair (H, G), we obtain the *arithmetic diagonal cycle*

$$\mathrm{Sh}_{\mathrm{H}} \longrightarrow \mathrm{Sh}_{\mathrm{G}} \ ,$$

(for certain level sugroups $K_{\mathrm{H}}, K_{\mathrm{G}}$).

- Arithmetic GGP conjecture: for generic π,

$$\mathrm{ord}_{s=1/2} L(\pi, s) = 1 \Longrightarrow [\mathrm{Sh}_H]_\pi \neq 0 \in \mathrm{Ch}(X)_0.$$

- $n = 2, \dim \mathrm{Sh}_G = 1$: Gross–Zagier, S. Zhang, Yuan–Zhang–Zhang.
- Exceptional example: Liu's special cycles (for GGP $\mathrm{U}(n) \times \mathrm{U}(n)$).

Part II

Main theorem

Global intersection numbers

- ∃ a *PEL*-type variant of the GGP Shimura varieties, with *nice* integral models defined by moduli space [Rapoport–Smithling–Z. '17], to be recalled later.
- Define through the arithmetic intersection theory

$$\mathrm{Int}(f) = \Big(f * [\mathrm{Sh_H}],\ [\mathrm{Sh_H}]\Big)_{\mathrm{Sh_G}}, \quad f \in \mathscr{H}(\mathrm{G}, K_\mathrm{G}),$$

  where the action is through the Hecke correspondence.
- For *regular* Hecke f, the global intersection localizes:

$$\mathrm{Int}(f) = \sum_{p \leq \infty} \mathrm{Int}_p(f).$$

Global intersection numbers

- ∃ a *PEL*-type variant of the GGP Shimura varieties, with *nice* integral models defined by moduli space [Rapoport–Smithling–Z. '17], to be recalled later.
- Define through the arithmetic intersection theory

$$\mathrm{Int}(f) = \Big(f * [\mathrm{Sh_H}],\ [\mathrm{Sh_H}]\Big)_{\mathrm{Sh_G}}, \quad f \in \mathscr{H}\,(\mathrm{G}, K_{\mathrm{G}})\,,$$

where the action is through the Hecke correspondence.

- For *regular* Hecke f, the global intersection localizes:

$$\mathrm{Int}(f) = \sum_{p \leq \infty} \mathrm{Int}_p(f).$$

Global intersection numbers

- $\exists$ a *PEL*-type variant of the GGP Shimura varieties, with *nice* integral models defined by moduli space [Rapoport–Smithling–Z. '17], to be recalled later.
- Define through the arithmetic intersection theory

$$\mathrm{Int}(f) = \Big(f * [\mathrm{Sh_H}],\ [\mathrm{Sh_H}]\Big)_{\mathrm{Sh_G}}, \quad f \in \mathscr{H}(\mathrm{G}, K_{\mathrm{G}}),$$

  where the action is through the Hecke correspondence.
- For *regular* Hecke f, the global intersection localizes:

$$\mathrm{Int}(f) = \sum_{p \leq \infty} \mathrm{Int}_p(f).$$

L-functions (via Jacquet–Rallis Relative trace formula)

- Consider the Hasse-Weil L-functions, counted with suitable weights

$$\mathbb{J}(f, s) = \sum_{\pi} L(\pi, s + 1/2)\mathbb{J}_{\pi}(f, s).$$

- Its derivative also localizes (for regular f)

$$\begin{aligned}\partial\mathbb{J}(f) :&= \frac{d}{ds}\Big|_{s=0}\mathbb{J}(f, s) \\ &= \sum_{p,\text{ non-split}} \partial\mathbb{J}_p(f).\end{aligned}$$

- The p-th term takes the following form

$$\partial\mathbb{J}_p(f) = \sum_{\gamma} \mathrm{Orb}(\gamma, f^p)\partial\mathrm{Orb}(\gamma, f_p).$$

Main theorem

Theorem (Z. ’19)

If the prime p is unramified, then

$$\mathrm{Int}_p(f) = \partial\mathbb{J}_p(f).$$

Remark

1. This was conjectured by [Z. ’12, Rapoport–Smithling–Z. 17’], based on the relative trace formula approach to the arithmetic GGP conjecture, and is a corollary to the “AFL conjecture" (to be recalled later).
2. To fulfill the modest goal, we still have to prove similar statements for *every ramified p* (including archimedean places).

Part III

Some geometric ingredients

An overview

- Integral models of Shimura varieties (RSZ).
- Two types of algebraic cycles
  - (a) Kudla–Rapoport divisors.
  - (b) (Fat Big) CM cycles (aka. Derived CM cycles).
- Two types of associated invariants.

The Hermitian symmetric domain for $\mathrm{U}(n-1,1)$

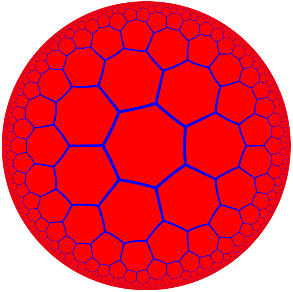
- Hermitian symmetric domain for $\mathrm{U}(n-1,1)$,

$$\mathbb{D}_{n-1} := \{z \in \mathbb{C}^{n-1} : |z| < 1\} \cong \frac{\mathrm{U}(n-1,1)}{\mathrm{U}(n-1) \times \mathrm{U}(1)}.$$

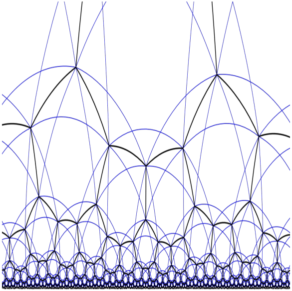
- We have an action

$$\mathrm{U}(n-1,1) \curvearrowright \mathbb{D}_{n-1}.$$

- Notice $\mathbb{D}_1$ is isomorphic to the upper half plane $\mathbb{H}$.



$$\xrightarrow{\sim}$$



The Shimura variety M_n for $\mathrm{U}(n-1,1)$

- $K = \mathbb{Q}(\sqrt{-d})$, an imaginary quadratic field.
- V a hermitian space over K of signature $(n-1,1)$.
- $\mathrm{U}(V)$ the associated unitary group.
- $O_K \subseteq K$ ring of integers.
- $\Lambda \subseteq V$ a self-dual hermitian lattice over O_K.
- $\mathrm{U}(\Lambda) \subseteq \mathrm{U}(V)(\mathbb{R}) = \mathrm{U}(n-1,1)$ a discrete subgroup.
- Shimura variety
$$M_n := \mathrm{U}(\Lambda)\backslash\mathbb{D}_{n-1}.$$
- It has dimension $n-1$ over $\mathbb{C}$.

The Shimura variety $\mathcal{M}_n$ over O_K

- Let $\mathcal{M}_n$ be the moduli stack of tuples $(A, \iota, \lambda, A_0, \iota_0, \lambda_0)$:
- A is an abelian scheme of dimension n.
- $\iota : O_K \hookrightarrow \mathrm{End}(A)$ is an action of O_K on A satisfying the Kottwitz condition of signature $(n-1, 1)$,

$$\det(T - \iota(a)|\mathrm{Lie}A) = (T - a)^{n-1}(T - \overline{a}), \quad a \in O_K.$$

- $\lambda : A \xrightarrow{\sim} A^\vee$ is a principal polarization of A whose Rosati involution induces $a \mapsto \overline{a}$ on $\iota(O_K)$.
- $(A_0, \iota_0, \lambda_0)$ is a triple analogous to (A, ι, λ), but of dimension 1 and signature $(1, 0)$.
- Then $\mathcal{M}_n$ is a Deligne–Mumford stack over O_K, smooth away from ramified characteristics of relative dimension $n-1$.
- $\mathcal{M}_n(\mathbb{C})$ is (a finite disjoint union of various) $M_n(\mathbb{C})$.

Global intersection revisited

Define the integral model of the arithmetic diagonal cycle:

$$\Delta\colon \mathcal{M}_{n-1} \longrightarrow \mathcal{M}_{n-1,n} = \mathcal{M}_{n-1} \times_{\mathrm{Spec}\, O_K} \mathcal{M}_n.$$

and

$$\mathrm{Int}(f) = \left(f * \widehat{\Delta}_{\mathcal{M}_{n-1}},\ \widehat{\Delta}_{\mathcal{M}_{n-1}}\right)_{\mathcal{M}_{n-1,n}}.$$

The Kudla–Rapoport divisor $\mathcal{Z}(m)$

- (KR hermitian lattice) For a geometric point $(A, \iota, \lambda, A_0, \iota_0, \lambda_0) \in \mathcal{M}_n$, the space of homomorphisms

$$V(A_0, A) := \mathrm{Hom}_{O_K}(A_0, A)$$

is a hermitian lattice over O_K. For $x, y \in V(A_0, A)$, the pairing $(x, y) \in O_K$ is given by

$$(A_0 \xrightarrow{x} A \xrightarrow{\lambda} A^\vee \xrightarrow{y^\vee} A_0^\vee \xrightarrow{\lambda_0^{-1}} A_0) \in \mathrm{End}_{O_K}(A_0) = O_K.$$

- Given $m \in \mathbb{Z}_+$, define the Kudla–Rapoport divisor

$$i_m : \mathcal{Z}_m \longrightarrow \mathcal{M}_n$$

to be the moduli stack of tuples $(A, \iota, \lambda, A_0, \iota_0, \lambda_0, x)$, where

$$x \in V(A_0, A) \text{ such that } (x, x) = m.$$

Modularity of generating series of special divisors

Theorem

The generating series

$$c_0 + \sum_{m \geq 1} \mathcal{Z}_m \, q^m \in \mathrm{Ch}^1(M_n)_{\mathbb{Q}}[\![q]\!],$$

where c_0 is a suitable multiple of the first Chern class of the Hodge bundle ω, is a modular form (of weight n and known level).

Remark

1. Replace $\mathrm{Ch}^1(M_n)$ by $H^2(M_n)$: Kudla–Millson.
2. Gross–Kohnen–Zagier ($n = 2$), Borcherds in general (+Liu's thesis).
3. Later proofs by Yuan–Zhang–Zhang, Bruinier.
4. Replace $\mathrm{Ch}^1(M_n)_{\mathbb{Q}}$ by $\widehat{\mathrm{Ch}}^1(\mathcal{M}_n)_{\mathbb{Q}}$: a theorem of Bruinier, Howard, Kudla, Rapoport, and Yang.

An analog

- Replace the signature $(n-1,1)$ by $(n,0)$:

$$\mathrm{Lat_n} = \left\{ \begin{array}{c} \text{hermitian lattices } \Lambda \\ \text{pos. def, self-dual,} \\ \text{rank} = n \end{array} \right\}$$

- Replace M_n by the lattice model $\mathrm{Lat_n}$ and we obtain theta functions as the generating series

$$\sum_{\Lambda \in \mathrm{Lat_n}} \frac{1}{\#\mathrm{Aut}(\Lambda)} \theta_\Lambda,$$

where

$$\theta_\Lambda = \sum_{m \geq 0} \#\{x \in \Lambda \mid (x,x) = m\} q^m.$$

An analog

- Replace the signature $(n-1,1)$ by $(n,0)$:

$$\mathrm{Lat_n} = \left\{ \begin{matrix} \text{hermitian lattices } \Lambda \\ \text{pos. def, self-dual,} \\ \text{rank} = n \end{matrix} \right\}$$

- Replace M_n by the lattice model $\mathrm{Lat_n}$ and we obtain theta functions as the generating series

$$\sum_{\Lambda \in \mathrm{Lat_n}} \frac{1}{\#\mathrm{Aut}(\Lambda)} \theta_\Lambda,$$

where

$$\theta_\Lambda = \sum_{m \geq 0} \#\{x \in \Lambda \mid (x,x) = m\} q^m.$$

A digression: Siegel–Weil, and arithmetic S-W

- Siegel–Weil: the generalized theta function

$$\sum_{\Lambda \in \mathrm{Lat_n}} \frac{1}{\#\mathrm{Aut}(\Lambda)} \sum_{T \in \mathrm{Herm}_n} \#\{\mathbf{x} \in \Lambda^n \mid (x_i, x_j) = T_{i,j}\} q^T$$

  is equal to the central value of Siegel-Eisenstein series on $\mathrm{U}(n, n)$.
- A parallel question is Kudla–Rapoport conjecture (“Intersection number of KR divisors=Fourier coefficients of the central derivative of Siegel-Eisenstein series"), also recently proved by Li–Z. (for good places).

A digression: Siegel–Weil, and arithmetic S-W

- Siegel–Weil: the generalized theta function

$$\sum_{\Lambda \in \mathrm{Lat_n}} \frac{1}{\#\mathrm{Aut}(\Lambda)} \sum_{T \in \mathrm{Herm}_n} \#\{\mathbf{x} \in \Lambda^n \mid (x_i, x_j) = T_{i,j}\} q^T$$

  is equal to the central value of Siegel-Eisenstein series on $\mathrm{U}(n, n)$.
- A parallel question is Kudla–Rapoport conjecture (“Intersection number of KR divisors=Fourier coefficients of the central derivative of Siegel-Eisenstein series"), also recently proved by Li–Z. (for good places).

(Fat Big) CM cycles on $\mathcal{A}_n$

- We consider the "fixed point of the Hecke correspondence $\mathrm{Hecke}_{\mathcal{A}_n}$":

$$\begin{array}{ccc} \mathcal{CM}_n^d & \longrightarrow & \mathrm{Hecke}^d_{\mathcal{A}_n} \\ \downarrow & & \downarrow \\ \mathcal{A}_n & \xrightarrow{\Delta} & \mathcal{A}_n \times \mathcal{A}_n. \end{array}$$

- To a geometric point $(A, \varphi \in \mathrm{End}^\circ(A)) \in \mathcal{CM}_n^d$ one can associate a "characteristic polynomial"

$$\mathrm{char}\colon \ \mathcal{CM}_n^d \longrightarrow \mathbb{Q}[T]_{\deg=2n},$$

and the map induces a disjoint union (of open and closed substacks)

$$\mathcal{CM}_n = \coprod_{a \in \mathrm{Im}(\mathrm{char})} \mathcal{CM}_n(a),$$

(Fat Big) CM cycles on $\mathcal{A}_n$

- We consider the "fixed point of the Hecke correspondence $\mathrm{Hecke}_{\mathcal{A}_n}$":

$$\begin{array}{ccc} \mathcal{CM}_n^d & \longrightarrow & \mathrm{Hecke}^d_{\mathcal{A}_n} \\ \downarrow & & \downarrow \\ \mathcal{A}_n & \xrightarrow{\Delta} & \mathcal{A}_n \times \mathcal{A}_n. \end{array}$$

- To a geometric point $(A, \varphi \in \mathrm{End}^\circ(A)) \in \mathcal{CM}_n^d$ one can associate a "characteristic polynomial"

$$\mathrm{char}\colon \ \mathcal{CM}_n^d \longrightarrow \mathbb{Q}[T]_{\deg=2n},$$

and the map induces a disjoint union (of open and closed substacks)

$$\mathcal{CM}_n = \coprod_{a \in \mathrm{Im}(\mathrm{char})} \mathcal{CM}_n(a),$$

(Fat Big) CM cycles on $\mathcal{M}_n$

- We consider the "fixed point of the Hecke correspondence $\mathrm{Hecke}_{\mathcal{M}_n}$":

$$\begin{array}{ccc} \mathcal{CM}_n^d & \longrightarrow & \mathrm{Hecke}^d_{\mathcal{M}_n} \\ \downarrow & & \downarrow \\ \mathcal{M}_n & \xrightarrow{\Delta} & \mathcal{M}_n \times \mathcal{M}_n. \end{array}$$

- To a geometric point $(A, \varphi \in \mathrm{End}^\circ(A)) \in \mathcal{CM}_n^d$ one can associate a "char. poly."

$$\mathrm{char}_K : \mathcal{CM}_n^d \longrightarrow K[T]_{\deg = n},$$

and the map induces a disjoint union

$$\mathcal{CM}_n^d = \coprod_{a \in \mathrm{Im}(\mathrm{char}_K)} \mathcal{CM}_n^d(a),$$

(Fat Big) CM cycles on $\mathcal{M}_n$

- We consider the "fixed point of the Hecke correspondence $\mathrm{Hecke}_{\mathcal{M}_n}$":

$$\begin{array}{ccc} \mathcal{CM}_n^d & \longrightarrow & \mathrm{Hecke}^d_{\mathcal{M}_n} \\ \downarrow & & \downarrow \\ \mathcal{M}_n & \xrightarrow{\Delta} & \mathcal{M}_n \times \mathcal{M}_n. \end{array}$$

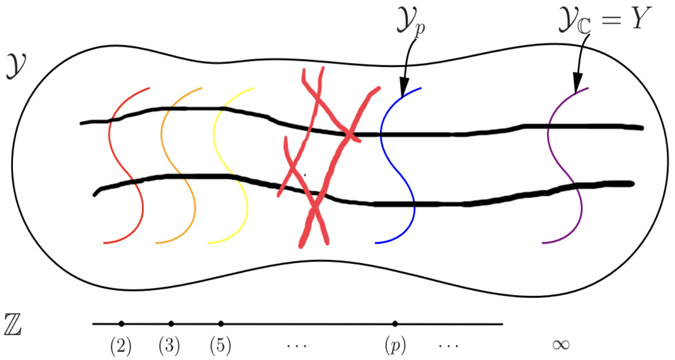
- To a geometric point $(A, \varphi \in \mathrm{End}^\circ(A)) \in \mathcal{CM}_n^d$ one can associate a "char. poly."

$$\mathrm{char}_K : \ \mathcal{CM}_n^d \longrightarrow K[T]_{\mathrm{deg}=n},$$

and the map induces a disjoint union

$$\mathcal{CM}_n^d = \coprod_{a \in \mathrm{Im}(\mathrm{char}_K)} \mathcal{CM}_n^d(a),$$

Example: $n = 3$, a non-flat CM cycle



Intersection theory (I)

$$\begin{array}{ccccc} \coprod \mathcal{CM}_n^d(a) & \xrightarrow{\sim} & \mathcal{CM}_n^d & \longrightarrow & \mathrm{Hecke}^d_{\mathcal{M}_n} \\ & & \downarrow & & \downarrow \\ & & \mathcal{M}_n & \xrightarrow{\Delta} & \mathcal{M}_n \times \mathcal{M}_n. \end{array}$$

Consider the “derived intersection product"

$${}^{\mathbb{L}}\mathcal{CM}_n^d = \sum_{a \in \mathrm{Im}(\mathrm{char}_K)} {}^{\mathbb{L}}\mathcal{CM}_n^d,$$

as classes in

$$\mathrm{Ch}_1(\mathcal{CM}_n^d)_{\mathbb{Q}} = \bigoplus_{a \in \mathrm{Im}(\mathrm{char}_K)} \mathrm{Ch}_1(\mathcal{CM}_n^d(a))_{\mathbb{Q}}.$$

Arithmetic intersection theory

- Arakelov/Gillet–Soulé intersection pairing

$$(\cdot,\cdot): \quad \widehat{\mathrm{Ch}}^1(\mathcal{M}_n) \times Z_{1,c}(\mathcal{M}_n) \longrightarrow \mathbb{R}_D,$$

where

$$\mathbb{R}_D := \mathbb{R}/\{\mathbb{Q}\,\text{span of }\log p, p|D_K\}.$$

- From the modularity, it follows that the generating function

$$c_0 + \sum_{m\geq 1} (\widehat{\mathcal{Z}}_m, {}^{\mathbb{L}}\mathcal{CM}^d(a))\, q^m \in \mathbb{R}_D[[q]],$$

is a modular form.

Arithmetic intersection theory

- Arakelov/Gillet–Soulé intersection pairing

$$(\cdot,\cdot): \quad \widehat{\mathrm{Ch}}^1(\mathcal{M}_n) \times Z_{1,c}(\mathcal{M}_n) \longrightarrow \mathbb{R}_D,$$

where

$$\mathbb{R}_D := \mathbb{R}/\{\mathbb{Q}\,\text{span of } \log p, p|D_K\}.$$

- From the modularity, it follows that the generating function

$$c_0 + \sum_{m\geq 1} (\widehat{\mathcal{Z}}_m, {}^{\mathbb{L}}\mathcal{CM}^d(a))\, q^m \in \mathbb{R}_D[\![q]\!],$$

is a modular form.

The analog revisited

- The “CM cycle" $\mathcal{CM}^d(a)$ on the lattice model $\mathrm{Lat_n}$ is

$$\mathcal{CM}^d(a) = \left\{ \begin{array}{c} (\Lambda, \varphi), \text{ s.t.} \\ \Lambda \in \mathrm{Lat_n},\ \varphi \in \frac{1}{d}\mathrm{End}_{O_K}(\Lambda), \\ \mathrm{char}_K(\varphi) = a\,. \end{array} \right\}$$

- The generating series, for a fixed (irred.) $a \in K[T]_{\deg=n}$

$$\sum_{m \geq 0} \sum_{(\Lambda,\varphi) \in \mathcal{CM}^d(a)} \frac{1}{\#\mathrm{Aut}(\Lambda, \varphi)} \#\{(\Lambda, \varphi, x \in \Lambda) \mid (x, x) = m\} q^m.$$

is a modular form.

The analog revisited

- The "CM cycle" $\mathcal{CM}^d(a)$ on the lattice model $\mathrm{Lat_n}$ is

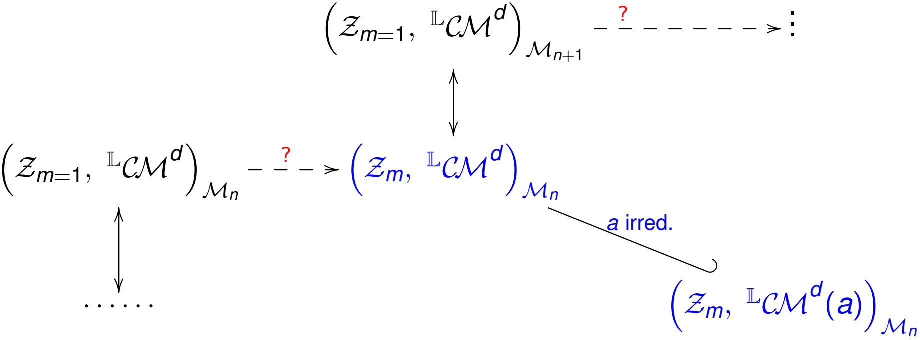
$$\mathcal{CM}^d(a) = \left\{ \begin{array}{c} (\Lambda, \varphi), \text{ s.t.} \\ \Lambda \in \mathrm{Lat_n},\ \varphi \in \frac{1}{d}\mathrm{End}_{O_K}(\Lambda), \\ \mathrm{char}_K(\varphi) = a\,. \end{array} \right\}$$

- The generating series, for a fixed (irred.) $a \in K[T]_{\mathrm{deg}=n}$

$$\sum_{m \geq 0} \sum_{(\Lambda,\varphi) \in \mathcal{CM}^d(a)} \frac{1}{\#\mathrm{Aut}(\Lambda, \varphi)} \#\{(\Lambda, \varphi, x \in \Lambda) \mid (x, x) = m\} q^m.$$

is a modular form.

The induction process (above $p \nmid dD$)



- $$\begin{array}{ccccc} \coprod \Delta^d_{\mathcal{Z}_m}(a,b) & \xrightarrow{\sim} & \Delta^d_{\mathcal{Z}_m} & \longrightarrow & \mathcal{CM}^d_n \\ & & \downarrow & & \downarrow \\ & & \mathcal{Z}_m & \xrightarrow{i_m} & \mathcal{M}_n. \end{array}$$

  A point in $\Delta^d_{\mathcal{Z}_m}$ is

  $$(A, A_0, \varphi \in \mathrm{End}^\circ(A), x : A_0 \to A).$$

- K-R hermitian form and char. poly. together define a map

  $$\mathrm{inv} : \ \Delta^d_{\mathcal{Z}_m} \longrightarrow K[T]_{\mathrm{deg}=n} \times K^n,$$

  sending (A, A_0, x, φ) to $a = \mathrm{char}_K(\varphi), b = (b_i)_{0 \le i \le n-1}$ where

  $$b_i = (\varphi^i x, x).$$

Intersection theory (II)

- $$\begin{array}{ccccc} \coprod \Delta^d_{\mathcal{Z}_m}(a,b) & \overset{\sim}{\longrightarrow} & \Delta^d_{\mathcal{Z}_m} & \longrightarrow & \mathcal{CM}^d_n \\ & & \downarrow & & \downarrow \\ & & \mathcal{Z}_m & \xrightarrow{i_m} & \mathcal{M}_n. \end{array}$$

  A point in $\Delta^d_{\mathcal{Z}_m}$ is
  $$(A, A_0, \varphi \in \mathrm{End}^\circ(A), x : A_0 \to A).$$

- K-R hermitian form and char. poly. together define a map
  $$\mathrm{inv} :\ \Delta^d_{\mathcal{Z}_m} \longrightarrow K[T]_{\deg = n} \times K^n,$$
  sending (A, A_0, x, φ) to $a = \mathrm{char}_K(\varphi)$, $b = (b_i)_{0 \le i \le n-1}$ where
  $$b_i = (\varphi^i x, x).$$

Theorem

Let $a \in K[T]_{\deg = n}$ be irreducible, and $b \in K^n$ such that $b_0 \neq 0$.

- *$\Delta^d_{\mathcal{Z}_m}(a, b)$ has support in the supersingular locus above a unique (necessarily inert) place p of $\mathbb{Q}$, and is a proper scheme.*
- *Assume that $p \nmid dD$. Then*

$$\deg {}^{\mathbb{L}}\Delta^d_{\mathcal{Z}_m}(a, b) = \mathrm{Orb}\left((a, b), f_d^{(p)}\right) \cdot \mathrm{Int}_p\left((a, b)\right),$$

where $\mathrm{Int}_p\left((a, b)\right)$ is the intersection number on "local Shimura variety" appearing in AFL (to be recalled below).

Part IV

The Arithmetic Fundamental Lemma conjecture

Relative orbital integrals

Define a family of (weighted) orbital integrals:

$$\mathrm{Orb}\left(\gamma, \mathbf{1}_{\mathfrak{gl}_n(O_F)}, s\right) = \int_{\mathrm{GL}_{n-1}(F)} \mathbf{1}_{\mathfrak{gl}_n(O_F)}(h^{-1}\gamma h) \left|\det(h)\right|^s (-1)^{\mathrm{val}(\det(h))}\, dh.$$

This can be viewed as a generating series of lattice counting of O_F-lattices $\Lambda^\flat$:

$$\left\{\Lambda^\flat \subset F^{n-1} \middle| \Lambda = \Lambda^\flat \oplus O_F \cdot e_n \text{ is stable under } \gamma.\right\}$$

The condition can be restated as ("local CM condition")

$$O_F[\gamma] \subset \mathrm{End}(\Lambda).$$

Jacquet–Rallis FL

Theorem (Yun–Gordan (large p), Beuzart–Plessis (all odd p))

Let $\gamma \in \mathfrak{gl}_n(F)$ *match an element* $g \in \mathrm{G}(F)$*, regular semisimple. Then*

$$\pm\mathrm{Orb}\left(\gamma, \mathbf{1}_{\mathfrak{gl}_n(O_F)}, s=0\right) = \mathrm{Orb}(g, \mathbf{1}_{\mathrm{Aut}(\Lambda)}).$$

Remark

1. (Xiao) J-R FL $\Longrightarrow$ Langlands–Shelstad FL for unitary groups (Theorem of Laumon–Ngo).
2. (Xiao, in progress) J-R FL $\Longrightarrow$ weighted FL for unitary groups.

Unitary Rapoport–Zink space

- F'/F : an unramified quadratic extension of p-adic fields.
- $\mathbb{X}_n$: n-dim'l *Hermitian supersingular formal* $O_{F'}$*-modules of signature* $(1, n-1)$ (unique up to isogeny).
- $\mathcal{N}_n$: the unitary Rapoport–Zink formal moduli space over $\mathrm{Spf}(O_{\breve{F}})$ (parameterizing "deformations" of $\mathbb{X}_n$).
- The group $\mathrm{Aut}^0(\mathbb{X}_n)$ is a unitary group in n-variable and acts on $\mathcal{N}_n$.
- The $\mathcal{N}_n$'s are non-archimedean analogs of Hermitian symmetric domains. They have a "skeleton" given by a union of Deligne–Lusztig varieties for unitary groups over finite fields.

Local intersection numbers

- A natural closed embedding $\delta : \mathcal{N}_{n-1} \to \mathcal{N}_n$, and its graph

$$\Delta : \mathcal{N}_{n-1} \longrightarrow \mathcal{N}_{n-1,n} = \mathcal{N}_{n-1} \times_{\mathrm{Spf} O_{\breve{F}}} \mathcal{N}_n.$$

  Denote by $\Delta_{\mathcal{N}_{n-1}}$ the image of Δ.
- The group $\mathrm{G}(F) := \mathrm{Aut}^0(\mathbb{X}_{n-1}) \times \mathrm{Aut}^0(\mathbb{X}_n)$ acts on $\mathcal{N}_{n-1,n}$. For (nice) $g \in \mathrm{G}(F)$, we define the intersection number

$$\begin{aligned}\mathrm{Int}(g) &= \left(\Delta_{\mathcal{N}_{n-1}}, g \cdot \Delta_{\mathcal{N}_{n-1}}\right)_{\mathcal{N}_{n-1,n}} \\ &:= \chi\left(\mathcal{N}_{n-1,n}, \mathcal{O}_{\Delta_{\mathcal{N}_{n-1}}} \otimes^{\mathbb{L}} \mathcal{O}_{g \cdot \Delta_{\mathcal{N}_{n-1}}}\right).\end{aligned}$$

The arithmetic fundamental lemma (AFL) conjecture

Then the local version of the global “arithmetic intersection conjecture" is

Conjecture (Z. ’12)

Let $\gamma \in \mathfrak{gl}_n(F)$ *match an element* $g \in \mathrm{G}(F)$, *strongly regular semisimple. Then*

$$\pm \frac{d}{ds}\Big|_{s=0} \mathrm{Orb}\left(\gamma, \mathbf{1}_{\mathfrak{gl}_n(O_F)}, s\right) = -\mathrm{Int}(g) \cdot \log q.$$

The status

Theorem (Z. '19)

The AFL conjecture holds when $F = \mathbb{Q}_p$ and $p > n$.

Remark

1. The case $n = 3$, Z '12 (A simplified proof when $p \geq 5$ is given by Mihatsch.)
2. Rapoport–Terstiege–Z. '13: $p \geq \frac{n}{2} + 1$, and *minuscule* elements $g \in \mathrm{G}(F)$. (A simplified proof is given by Li –Zhu.)
3. He–Li–Zhu, 2018: *minuscule* case but no restriction on p.

Thank you!

The arithmetic fundamental lemma for the diagonal cycles

Wei Zhang

Massachusetts Institute of Technology

The first JNT Biennial, Cetraro, 2019