A new proof of the Jacquet-Rallis fundamental lemma

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Motivation : Gan-Gross-Prasad conjecture for unitary groups

- E/F quad ext of number fields, $W \subset V$ Hermitian spaces $/E$ of dim n, $n+1$, $G = U(W) \times U(V) \hookleftarrow H = U(W)$ (diagonally) and $\pi = \pi_n \boxtimes \pi_{n+1} \hookrightarrow \mathcal{A}_{\text{cusp}}(G(F)\backslash G(\mathbb{A}))$ an irred cuspidal automorphic representation.
- Automorphic period : $\phi \in \pi \mapsto \mathcal{P}_H(\phi) = \int_{H(F)\backslash H(\mathbb{A})} \phi(h)dh$.
- Let $\pi_E = \pi_{n,E} \boxtimes \pi_{n+1,E}$ be the quadratic base-change to $\mathrm{GL}_{n,E} \times \mathrm{GL}_{n+1,E}$ (Mok, Kaletha-Minguez-Shin-White) and $L(s, \pi_{n,E} \times \pi_{n+1,E})$ the corresponding Rankin-Selberg L-function.

Conjecture (Gan-Gross-Prasad)

Assume that π_E is generic. We have

$$L(\frac{1}{2}, \pi_{n,E} \times \pi_{n+1,E}) \neq 0 \Leftrightarrow \exists \pi' \text{“in the same L-packet as } \pi\text{" st } \mathcal{P}_H \mid_{\pi'} \neq 0.$$

- Refined version (Ichino-Ikeda, N. Harris) : $|\mathcal{P}_H(\phi)|^2 \sim L(\frac{1}{2}, \pi_{n,E} \times \pi_{n+1,E})$.
- There has been a lot of progress on these conjectures recently : W. Zhang, H. Xue, B.-P. (using Relative Trace Formulas)/ Jiang-L. Zhang, Ginzburg-Jiang-Rallis (automorphic descent)/ Grobner-Lin (by rationality results for special values of L-functions).

Jacquet-Rallis approach trough comparison of RTFs

- Jacquet and Rallis have proposed to attack these conjectures through a comparison of Relative Trace Formulas (RTF).
- RTF are analytic tools introduced by Jacquet that relate automorphic periods to more geometric distributions known as (relative) orbital integrals. Roughly, RTFs are associated to triples $H_1 \hookrightarrow G' \hookleftarrow H_2$ with G' reductive and can be thought as distributions on double coset spaces $H_1 \backslash G' / H_2$.

Jacquet-Rallis (simple) Relative Trace Formulas

- RTF for $H\backslash G/H$: For 'nice' test fns $f \in C_c^\infty(G(\mathbb{A}))$

$$\sum_{\delta \in H(F)\backslash G_{rs}(F)/H(F)} O(\delta, f) = \sum_{\phi \in \mathcal{A}_{\text{cusp}}(G)} \mathcal{P}_H(R(f)\phi)\overline{\mathcal{P}_H(\phi)}$$

  where the right sum runs over an ONB, $R(f)$ is the right convolution by f, $G_{rs} \subset G$ is the open subset of *regular semi-simple* elts (i.e. trivial stabilizer and closed orbit) for the $H \times H$-action and

$$O(\delta, f) = \int_{H(\mathbb{A})\times H(\mathbb{A})} f(h_1 \delta h_2) dh_1 dh_2$$

  are *relative orbital integrals*.

- Set $H_1 = \mathrm{GL}_{n,E} \hookrightarrow G' = \mathrm{GL}_{n,E} \times \mathrm{GL}_{n+1,E} \hookleftarrow H_2 = \mathrm{GL}_{n,F} \times \mathrm{GL}_{n+1,F}$ with $\eta : H_2(F)\backslash H_2(\mathbb{A}) \to \{\pm 1\}$.
- RTF for $H_1\backslash G'/(H_2, \eta)$: For 'nice' test fns $f' \in C_c^\infty(G'(\mathbb{A}))$

$$\sum_{\gamma \in H_1(F)\backslash G'_{rs}(F)/H_2(F)} O(\gamma, f') = \sum_{\phi \in \mathcal{A}_{\text{cusp}}(G')} \mathcal{P}_{H_1}(R(f')\phi)\overline{\mathcal{P}_{H_2,\eta}(\phi)}$$

  where this time

$$O(\gamma, f') = \int_{H_1(\mathbb{A})\times H_2(\mathbb{A})} f'(h_1 \gamma h_2)\eta(h_2) dh_1 dh_2.$$

Comparison

$$\sum_{\delta \in H(F)\backslash G_{rs}(F)/H(F)} O(\delta, f) = \sum_{\phi \in \mathcal{A}_{\text{cusp}}(G)} \mathcal{P}_H(R(f)\phi)\overline{\mathcal{P}_H(\phi)}$$

$$\sum_{\gamma \in H_1(F)\backslash G'_{rs}(F)/H_2(F)} O(\gamma, f') = \sum_{\phi \in \mathcal{A}_{\text{cusp}}(G')} \mathcal{P}_{H_1}(R(f')\phi)\overline{\mathcal{P}_{H_2,\eta}(\phi)}$$

- Goal : compare these RTFs. Why ? $\mathcal{P}_{H_1}$ is a Rankin-Selberg period giving $L(\frac{1}{2}, \pi_{n,E} \times \pi_{n+1,E})$ whereas $\mathcal{P}_{H_2,\eta}$ if the Flicker-Rallis period detecting image of base-change (i.e. autom repns of the form π_E) $\rightsquigarrow$ from the GGP conj we expect spectral sides to “match".
- We will deduce this from a “matching" of geometric sides i.e. we need to produce sufficiently many test fns (f, f') st the LHS are equal. As in the paradigm of endoscopy we do this “orbit by orbit".
- Correspondence of orbits : $H(k)\backslash G_{rs}(k)/H(k) \hookrightarrow H_1(k)\backslash G'_{rs}(k)/H_2(k)$, $\delta \leftrightarrow \gamma$, for $k = F$ or F_v.
- Orbital integrals are local : if $f = \prod_v f_v$, $f' = \prod_v f'_v$ then $O(\delta, f) = \prod_v O(\delta, f_v)$, $O(\gamma, f') = \prod_v O(\gamma, f'_v)$ (products of local orbital integrals) $\rightsquigarrow$ we look for a local matching of functions.

Local matching and fundamental lemma

- v a place of F. We say that $f_v \in C_c^\infty(G_v)$ and $f'_v \in C_c^\infty(G'_v)$ *match* (Not : $f_v \leftrightarrow f'_v$) if

$$O(\delta, f_v) = \Omega_v(\gamma) O(\gamma, f'_v) \text{ for } G_{v,rs} \ni \delta \leftrightarrow \gamma \in G'_{v,rs}$$

$$O(\gamma, f'_v) = 0 \text{ if } \gamma \text{ is not in the image of the corresp.}$$

  and where $(\Omega_v)_v$ are *explicit* transfer factors st $\prod_v \Omega_v(\gamma) = 1$ for $\gamma \in G'_{rs}(F)$.
- For the global comparison to be effective we need :
  - Smooth Transfer : $\forall f_v \in C_c^\infty(G_v)$, $\exists f'_v \in C_c^\infty(G'_v)$ st $f_v \leftrightarrow f'_v$ + a converse (W. Zhang p-adic case, H. Xue Arch case) ;
  - Fundamental Lemma (FL) : $\mathbf{1}_{G(\mathcal{O}_v)} \leftrightarrow \mathbf{1}_{G'(\mathcal{O}_v)}$ for a.a. v (Z. Yun in positive char, transferred to char zero (and $reschar(v) \gg 1$) by J. Gordon).

Theorem

The Jacquet-Rallis FL is true at v provided $reschar(v) \neq 2$ and "everything is unramified at v".

- Not a new result but the proof is substantially different (Harmonic analysis vs geometry of Hitchin fibrations+Model theory) ;
- J. Xiao : this FL implies the "usual" endoscopic fundamental lemma for unitary groups (theorem of Laumon-Ngô).

Statement of the FL for Lie algebras

- E/F unramified quad ext of p-adic fields, $\mathcal{O} = \mathcal{O}_F$. We consider the actions

$$U_n = \{g \in \mathrm{GL}_{n,E} \mid {}^T\overline{g}g = I_n\} \curvearrowright \mathfrak{h}_{n+1} = \{X \in \mathfrak{gl}_{n+1,E} \mid X = {}^T\overline{X}\}$$
$$\text{and } \mathrm{GL}_{n,F} \curvearrowright \mathfrak{gl}_{n+1,F}$$

  by conjugation where $U_n \hookrightarrow U_{n+1}$ or $\mathrm{GL}_n \hookrightarrow \mathrm{GL}_{n+1}$ by $g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}$.

- $X \in \mathfrak{h}_{n+1}$ or $\mathfrak{gl}_{n+1}$ is regular semi-simple (for this action) iff the X-orbits of

  $e_{n+1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ and $e^*_{n+1} = (0, \ldots, 0, 1)$ generate $E^{n+1} = M_{n+1,1}(E)$ and $E_{n+1} = M_{1,n+1}(E)$ resp.

- Orbital integrals :

$$X \in \mathfrak{h}^{rs}_{n+1}, f \in C^\infty_c(\mathfrak{h}_{n+1}) \rightsquigarrow O(X,f) = \int_{U_n} f(gXg^{-1})dg,$$

$$Y \in \mathfrak{gl}^{rs}_{n+1}, f' \in C^\infty_c(\mathfrak{gl}_{n+1}) \rightsquigarrow O(Y,f') = \omega(Y) \int_{\mathrm{GL}_n} f'(gYg^{-1})\eta_{E/F}(\det g)dg$$

  where $\omega(Y) = \pm 1$ is a transfer factor and $\eta_{E/F} : F^\times \to F^\times/N(E^\times) = \{\pm 1\}$.

- Correspondence of orbits : $X \in \mathfrak{h}_{n+1}^{rs} \leftrightarrow Y \in \mathfrak{gl}_{n+1}^{rs}$ if they are $\mathrm{GL}_n(E)$ conjugate inside $\mathfrak{gl}_{n+1}(E) \rightsquigarrow \mathfrak{h}_{n+1}^{rs}/U_n \hookrightarrow \mathfrak{gl}_{n+1}^{rs}/\mathrm{GL}_n$.
- Matching of functions : $C_c^\infty(\mathfrak{h}_{n+1}) \ni f \leftrightarrow f' \in C_c^\infty(\mathfrak{gl}_{n+1})$ if

$$O(Y, f') = O(X, f) \text{ whenever } Y \leftrightarrow X,$$

$$O(Y, f') = 0 \text{ o/w (i.e. if } Y \text{ isn't in the image of the correp.).}$$

- Jacquet-Rallis fundamental lemma (JRFL) for Lie algebras : $\mathbf{1}_{\mathfrak{h}_{n+1}(\mathcal{O})} \leftrightarrow \mathbf{1}_{\mathfrak{gl}_{n+1}(\mathcal{O})}$.

Proposition (Z. Yun)

When $p \neq 2$, JRFL for Lie algebras implies the original JRFL (for groups).

Theorem

JRFL for Lie algebras is true for any p.

The proof

- We need to show that

$$(\star) \qquad O(Y, \mathbf{1}_{\mathfrak{gl}_{n+1}(\mathcal{O})}) = O(X, \mathbf{1}_{\mathfrak{h}_{n+1}(\mathcal{O})}) \text{ for } Y \leftrightarrow X,$$

$$(\star\star) \qquad O(Y, \mathbf{1}_{\mathfrak{gl}_{n+1}(\mathcal{O})}) = 0 \text{ otherwise.}$$

- The proof is by induction on n and a certain quad form q will play a crucial role :

$$X = \begin{pmatrix} X' & b \\ c & \lambda \end{pmatrix} \in \mathfrak{h}_{n+1} \text{ or } \mathfrak{gl}_{n+1} \mapsto q(X) = cb \in F$$

(here $X' \in \mathfrak{h}_n$ or $\mathfrak{gl}_n$, $b \in E^n$, $c \in E_n$ and $\lambda \in F$).

Important remarks :

- If $X \in \mathfrak{h}_{n+1}^{rs} \leftrightarrow Y \in \mathfrak{gl}_{n+1}^{rs}$ then $q(X) = q(Y)$.
- Orbital integrals are locally constant on rs locus $\Rightarrow$ just need to show $(\star)$ and $(\star\star)$ when $q(Y) \neq 0$.

The induction hypothesis will only be used for the next lemma.

Lemma

$(\star)$ and $(\star\star)$ hold when $|q(Y)| \geqslant 1$.

Proof of the lemma

- If $|q(Y)| > 1$: as q is invt and takes integral values on integral points

$$O(Y, \mathbf{1}_{\mathfrak{gl}_{n+1}(\mathcal{O})}) = O(X, \mathbf{1}_{\mathfrak{h}_{n+1}(\mathcal{O})}) = 0.$$

- If $|q(Y)| = 1 (= |q(X)|)$: Up to conjugacy (by U_n or GL_n), we may assume

$$X = \begin{pmatrix} X' & \alpha e_n \\ \overline{\alpha} e_n^* & \lambda \end{pmatrix}, \quad Y = \begin{pmatrix} Y' & \beta e_n \\ e_n^* & \lambda \end{pmatrix}$$

  where $\alpha \in \mathcal{O}_E^\times$, $q(Y) = \beta = N(\alpha) \in \mathcal{O}^\times$ and $X' \in \mathfrak{h}_n^{rs} \leftrightarrow Y' \in \mathfrak{gl}_n^{rs}$.

- Therefore, by the induction hypothesis, we just need to show

$$(A) \qquad O(Y, \mathbf{1}_{\mathfrak{gl}_{n+1}(\mathcal{O})}) = O(Y', \mathbf{1}_{\mathfrak{gl}_n(\mathcal{O})}),$$

$$(B) \qquad O(X, \mathbf{1}_{\mathfrak{h}_{n+1}(\mathcal{O})}) = O(X', \mathbf{1}_{\mathfrak{h}_n(\mathcal{O})}).$$

- Both (A) and (B) are easy to check using that GL_{n-1} (resp. U_{n-1}) is the stabilizer of (e_n^*, e_n) (resp. of e_n) and that $\mathrm{GL}_n(\mathcal{O})$ (resp. $U_n(\mathcal{O})$) acts transitively on vectors in $\mathcal{O}_n \oplus \mathcal{O}^n$ of pairing 1 (resp. vectors in $\mathcal{O}_E^n$ of Hermitian norm 1) ■

Partial Fourier Transform and Weil representation

- Fix $\psi : F \to \mathbb{C}^\times$ unramified and let $\mathcal{F} \curvearrowright C_c^\infty(\mathfrak{h}_{n+1}$ or $\mathfrak{gl}_{n+1})$ be defined by

$$(\mathcal{F} f)\begin{pmatrix} X' & b \\ {}^T\overline{b} & \lambda \end{pmatrix} = \int_{E^n} f\begin{pmatrix} X' & c \\ {}^T\overline{c} & \lambda \end{pmatrix} \psi(\mathrm{Trace}_{E/F}({}^T\overline{c}b))dc,$$

$$(\mathcal{F} f')\begin{pmatrix} Y' & b \\ c & \lambda \end{pmatrix} = \int_{F^n \oplus F_n} f'\begin{pmatrix} Y' & b' \\ c' & \lambda \end{pmatrix} \psi(c'b + cb')db'dc'.$$

- Note that $\mathcal{F}\,\mathbf{1}_{\mathfrak{h}_{n+1}(\mathcal{O})} = \mathbf{1}_{\mathfrak{h}_{n+1}(\mathcal{O})}$ and $\mathcal{F}\,\mathbf{1}_{\mathfrak{gl}_{n+1}(\mathcal{O})} = \mathbf{1}_{\mathfrak{gl}_{n+1}(\mathcal{O})}$.

Theorem (W.Zhang)

If $C_c^\infty(\mathfrak{h}_{n+1}) \ni f \leftrightarrow f' \in C_c^\infty(\mathfrak{gl}_{n+1})$ then $\mathcal{F} f \leftrightarrow \mathcal{F} f'$.

- Weil representation : $\exists$ repn $SL_2(F) \curvearrowright C_c^\infty(\mathfrak{gl}_{n+1}$ or $\mathfrak{h}_{n+1})$ characterized by
  - $\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} f(X) = \psi(tq(X))f(X)$ for $t \in F$,
  - $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} f = \mathcal{F} f$.

Descent of Weil representations to orbital integrals

- Set
$$\mathrm{Orb}(\mathfrak{h}) = \left\{ \mathrm{Orb}(f) : X \in \mathfrak{h}_{n+1}^{rs} \mapsto O(X,f) \mid f \in C_c^\infty(\mathfrak{h}_{n+1}) \right\},$$
$$\mathrm{Orb}(\mathfrak{gl}) = \left\{ \mathrm{Orb}(f') : Y \in \mathfrak{gl}_{n+1}^{rs} \mapsto O(Y,f') \mid f' \in C_c^\infty(\mathfrak{gl}_{n+1}) \right\}.$$
- Both $\mathrm{Orb}(\mathfrak{h})$ and $\mathrm{Orb}(\mathfrak{gl})$ can be seen as spaces of functions on $\mathcal{A} = \mathfrak{gl}_{n+1}^{rs} / \mathrm{GL}_n \hookleftarrow \mathfrak{h}_{n+1}^{rs} / U_n$ (via extension by zero).

Proposition

The Weil representations descent to repns of $\mathrm{SL}_2(F)$ on $\mathrm{Orb}(\mathfrak{h})$ and $\mathrm{Orb}(\mathfrak{gl})$ coinciding on the intersection.

Proof :

- This is clear for the action of $\begin{pmatrix} 1 & F \\ & 1 \end{pmatrix}$ as q is invt and $X \leftrightarrow Y \Rightarrow q(X) = q(Y)$;
- For $w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ this follows from Zhang's result;
- Now, $\mathrm{SL}_2(F) = \langle w, \begin{pmatrix} 1 & F \\ & 1 \end{pmatrix} \rangle$. ■

End of the proof

- Set $\Phi = \mathrm{Orb}(\mathbf{1}_{\mathfrak{gl}_{n+1}(\mathcal{O})}) - \mathrm{Orb}(\mathbf{1}_{\mathfrak{h}_{n+1}(\mathcal{O})})$. We want : $\Phi(a) = 0$ for every $a \in \mathcal{A}$ st $q(a) \neq 0$.
- By the (first) lemma, for every $t \in \mathfrak{p}_F^{-1}$ and $a \in \mathcal{A}$ we have

$$\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \Phi(a) = \psi(tq(a))\Phi(a) = \begin{cases} \Phi(a) & \text{if } |q(a)| < 1, \\ 0 & \text{otherwise.} \end{cases} = \Phi(a)$$

(as ψ is unramified). Hence, Φ is fixed by $\begin{pmatrix} 1 & \mathfrak{p}_F^{-1} \\ & 1 \end{pmatrix}$.

- On the other hand, for $w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ we have

$$w\Phi = \mathrm{Orb}(\mathcal{F}\mathbf{1}_{\mathfrak{gl}_{n+1}(\mathcal{O})}) - \mathrm{Orb}(\mathcal{F}\mathbf{1}_{\mathfrak{h}_{n+1}(\mathcal{O})}) = \mathrm{Orb}(\mathbf{1}_{\mathfrak{gl}_{n+1}(\mathcal{O})}) - \mathrm{Orb}(\mathbf{1}_{\mathfrak{h}_{n+1}(\mathcal{O})}) = \Phi$$

i.e. Φ is fixed by w.

- Since $\mathrm{SL}_2(F) = \langle w, \begin{pmatrix} 1 & \mathfrak{p}_F^{-1} \\ & 1 \end{pmatrix} \rangle$, Φ is fixed by the whole action of $\mathrm{SL}_2(F)$.
- In particular,

$$\Phi(a) = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \Phi(a) = \psi(tq(a))\Phi(a)$$

for every $a \in \mathcal{A}$ and $t \in F$ and this implies $\Phi(a) = 0$ when $q(a) \neq 0$.

Thank you !