THE GAUSS CLASS NUMBER PROBLEM FOR IMAGINARY QUADRATIC FIELDS

Dorian Goldfeld

§1. Introduction

Let $D < 0$ be a fundamental discriminant for an imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$. Such fundamental discriminants D consist of all negative integers that are either $\equiv 1 \pmod 4$ and square–free, or of the form $D = 4m$ with $m \equiv 2$ or $3 \pmod 4$ and square–free. We define

$$h(D) = \#\left\{\frac{\text{group of nonzero fractional ideals } \frac{\mathfrak{a}}{\mathfrak{b}}}{\text{group of principal ideals}\,(\alpha),\ \alpha \in K^\times}\right\},$$

to be the cardinality of the ideal class group of K. In the Disquisitiones Arithmeticae (1801) **[G]**, Gauss showed (using the language of binary quadratic forms) that $h(D)$ is finite. He conjectured that

$$h(D) \longrightarrow \infty \quad \text{as} \quad D \longrightarrow -\infty,$$

a result first proved by Heilbronn **[H]** in 1934. The Disquisitiones also contains tables of binary quadratic forms with small class numbers (actually tables of imaginary quadratic fields of small class number with even discriminant which is a much easier problem to deal with) and Gauss conjectured that his tables were complete. In modern parlance, we can rewrite Gauss' tables (we are including both even and odd discriminants) in the following form.

$h(D)$	1	2	3	4	5
# of fields	9	18	16	54	25
largest $\|D\|$	163	427	907	1555	2683

The problem of finding an effective algorithm to determine all imaginary quadratic fields with a given class number h is known as the Gauss class number h problem. The Gauss class number problem is especially intriguing, because if such an effective algorithm did not exist, then the associated Dirichlet L–function would have to have a real zero, and the generalized Riemann hypothesis would necessarily be false. This problem has a long history (see **[Go2]**) which we do not replicate here, but the first important milestones were obtained by Heegner **[Heg]**, Stark **[St1]**, **[St2]**, and Baker **[B]**, whose work led to the solution of the class number one and two problems. The general Gauss class number problem was finally solved completely by Goldfeld–Gross–Zagier (**[Go1], [Go2], [G-Z]**) in 1985. The key idea of the proof is based on the following theorem (see **[Go 1]** (1976), for an essentially equivalent result) which reduced the problem to a finite amount of computation.

This research is partially supported by the National Science Foundation.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Theorem 1: *Let D be a fundamental discriminant of an imaginary quadratic field. If there exists a modular elliptic curve E (defined over $\mathbb{Q}$) whose associated base change Hasse–Weil L–function $L_{E/\mathbb{Q}(\sqrt{D})}(s)$ has a zero of order ≥ 4 at $s = 1$ then for every $\epsilon > 0$, there exists an effective computable constant $c_\epsilon(E) > 0$, depending only on ϵ, E such that*

$$h(D) > c_\epsilon(E)(\log|D|)^{1-\epsilon}.$$

Note that the L–function of $E/\mathbb{Q}$, $L_E(s)$, always divides $L_{E/\mathbb{Q}(\sqrt{D})}(s)$. If an imaginary quadratic field $Q(\sqrt{D})$ has small class number, then many small primes are inert. It is not hard to show that the existence of an elliptic curve whose associated Hasse–Weil L–function has a triple zero at $s = 1$ is enough to usually guarantee that $L_{E/\mathbb{Q}(\sqrt{D})}(s)$ has a fourth order zero. This idea will be clarified in §3. We also remark, that if $L_{E/\mathbb{Q}(\sqrt{D})}(s)$ had a zero of order $g \geq 4$, then you would get (see **[G1]**) the lower bound

$$h(D) \gg (\log|D|)^{g-3} e^{-21\sqrt{g(\log\log|D|)}}.$$

Actually, **[G1]** also gives a similar result for real quadratic fields ($D > 0$),

$$h(D)\log\epsilon_D \gg (\log|D|)^{g-3} e^{-21\sqrt{g(\log\log|D|)}},$$

where ϵ_D denotes the fundamental unit. In this case, however, it is required that $L_{E/\mathbb{Q}(\sqrt{D})}(s)$ has a zero of order $g \geq 5$ to get a non–trivial lower bound, because $\log\epsilon_D \gg \log D$. The term $e^{-21\sqrt{g(\log\log|D|)}}$ (obtained by estimating a certain product of primes dividing D) is far from optimal, because it simultaneously covers the cases of both real and imaginary quadratic fields. If one considers only imaginary quadratic fields, the term can be easily written as a simple product over primes dividing D. This was done by Oesterlé in 1985 **[O]** who made Theorem 1 explicit. He proved that for $(D,\, 5077) = 1$,

$$h(D) > \frac{1}{55}\log|D| \prod_{p|D,\ p\neq|D|} \left(1 - \frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\right),$$

which allowed one to solve the class number 3 problem. More recently, using the above methods, Arno **[A]** (1992), solved the class number four problem, and subsequently, work with Robinson and Wheeler **[A-R-W]** (1998), and work of Wagner **[Wag]** (1996) gave a solution to Gauss' class number problem for class numbers 5, 6, 7. The most recent advance in this direction is due to Watkins **[Wat]**, who obtained the complete list of all imaginary quadratic fields with class number ≤ 100.

The main aim of this paper is to illustrate the key ideas of the proof of Theorem 1 by giving full details of the proof for the solution of just the class number number one problem. The case of class number one is considerably simpler than the general case, but the proof exemplifies the ideas that work in general. We have not tried to compute or optimize constants, but have focused instead on exposition of the key ideas.

Acknowledgment: The author would like to thank Brian Conrey for several helpful comments.

§2. The Deuring–Heilbronn Phenomenon

Let $\mathbb{Q}(\sqrt{D})$ denote an imaginary quadratic field with class number $h(D) = 1$. If a rational prime p splits completely in $\mathbb{Q}(\sqrt{D})$ then $(p) = \pi \cdot \bar{\pi}$ with $\pi = \left(\frac{m+n\sqrt{D}}{2}\right)$, a principal ideal. It follows that

$$p = \frac{m^2 - n^2 D}{4} \qquad \Longrightarrow \qquad p > \frac{1+|D|}{4}.$$

We have thus shown.

Lemma 2: *Let $\mathbb{Q}(\sqrt{D})$ be an imaginary quadratic field of class number one. Then all primes less than $\frac{1+|D|}{4}$ must be inert.*

Note that Lemma 2 can be used to write down prime producing polynomials **[Ra]**

$$x^2 - x + \frac{|D|+1}{4},$$

(e.g., $x^2 - x + 41$) which takes prime values for $x = 1, 2, \ldots, \frac{|D|-3}{4}$.

Lemma 2 is the simplest example of the more general phenomenon which says that an imaginary quadratic field with small class number has the property that most small rational primes must be inert in that field. It follows that if $h(D) = 1$, then the quadratic character $\chi_D(n) = \left(\frac{D}{n}\right)$ (Kronecker symbol) associated to $\mathbb{Q}(\sqrt{D})$ satisfies $\chi_D(p) = -1$ for most small primes, and thus behaves like the Liouville function. Consequently, we heuristically expect that as $D \to -\infty$ and s fixed with $\Re(s) > \frac{1}{2}$,

$$\begin{aligned} L(s, \chi_D) &= \prod_p \left(1 - \frac{\chi_D(p)}{p^s}\right)^{-1} \\ &\sim \prod_p \left(1 + \frac{1}{p^s}\right)^{-1} = \frac{\zeta(2s)}{\zeta(s)}, \end{aligned}$$

so that analytically the Dirichlet L–function, $L(s, \chi_D)$, associated to $\mathbb{Q}(\sqrt{D})$ behaves like $\frac{\zeta(2s)}{\zeta(s)}$. By $f(s) \sim g(s)$ in a region $s \in \mathcal{R} \subset \mathbb{C}$ we mean that there exists a small $\epsilon > 0$ such that $|f(s) - g(s)| < \epsilon$ in the region $\mathcal{R}$. Here we are appealing to the standard use of approximate functional equations which allow one to replace an L–function by a short (square root of conductor) sum of its early Dirichlet coefficients. This is the basis for the so called zero repelling effects (Deuring–Heilbronn phenomenon) associated to imaginary quadratic fields with small class number. For example, if $h(D) = 1$ and $D \to -\infty$, and D_1 is a fixed discriminant of a quadratic field, then we expect that for $\Re(s) > \frac{1}{2}$,

$$L(s, \chi_{D_1}) L(s, \chi_D \chi_{D_1}) \sim L(2s, \chi_{D_1}),$$

which implies (see **[Da]**) that $L(s, \chi_{D_1})$ has no zeros $\gamma + i\rho$ with $\gamma > \frac{1}{2}$.

§3. Existence of L–functions of Elliptic Curves with Triple Zeroes

Let E be an elliptic curve defined over $\mathbb{Q}$ whose associated Hasse–Weil L–function $L_E(s)$ vanishes at $s = 1$. Let $L_E(s, \chi_d)$ denote the L–function twisted by the quadratic character χ_d of conductor

d, a fundamental discriminant of an imaginary quadratic field. We shall need the Gross–Zagier formula (see **[G–Z]**)

$$\frac{d}{ds}\Big(L_E(s)L_E(s,\chi_d)\Big)_{s=1} = c_E\langle P_d, P_d\rangle, \tag{3.1}$$

where $\langle P_d, P_d\rangle$ is the height pairing of a certain Heegner point P_D and c_E is an explicit constant depending on the elliptic curve E. Gross and Zagier showed that if E is an elliptic curve of conductor 37 and $d = -139$, then the Heegner point is torsion and the height pairing $\langle P_d, P_d\rangle$ vanishes. By (3.1), this gives a construction of an L–function with a triple zero at $s = 1$. Actually, their method is quite general, and many other such examples can be constructed.

Henceforth, we fix E to be the above elliptic curve of conductor $N = 37\cdot 139^2$. Then the Hasse–Weil L–function $L_E(s)$ satisfies the functional equation (see **[Shim]**)

$$\left(\frac{\sqrt{N}}{2\pi}\right)^{1+s}\Gamma(1+s)L_E(1+s) = -\left(\frac{\sqrt{N}}{2\pi}\right)^{1-s}\Gamma(1-s)L_E(1-s),$$

and $L_E(1+s)$ has a MacLaurin expansion of the form

$$L_E(1+s) = c_3s^3 + c_4s^5 \ + \ \{\text{higher odd powers of } s\}.$$

Now, let D, with $|D| > 163$, denote a fundamental discriminant of an imaginary quadratic field with class number one. It is not hard to show that $(D,\ 37\cdot 139) = 1$. Let χ_D denote the quadratic Dirichlet character of conductor D. We define

$$\Lambda_D(s) = \left(\frac{N|D|}{4\pi^2}\right)^s\Gamma(1+s)^2L_E(s)L_E(s,\chi_D). \tag{3.2}$$

Then it can be shown (see **[Shim]**) that $\Lambda_D(s)$ satisfies the functional equation

$$\Lambda_D(1+s) = w\cdot\Lambda_D(1-s), \tag{3.3}$$

with root number $w = \chi_D(-37\cdot 139^2) = \chi_D(-37) = +1$, because the early primes of an imaginary quadratic field $\mathbb{Q}(\sqrt{D})$ with class number one must be inert (Lemma 2). It follows from (3.3) that

$$L_{E/\mathbb{Q}(\sqrt{D})}(s) = L_E(s)L_E(s,\chi_D)$$

has a zero of even order at $s = 1$. Since $L_E(s)$ has a zero of order 3 at $s = 1$, we immediately see that $L_{E/\mathbb{Q}(\sqrt{D})}(s)$ must have a zero of order at least 4 at $s = 1$. This is the main requirement of Theorem 1.

§4. Solution of the Class Number One Problem

Assume D is sufficiently large and the class number $h(D)$ of $\mathbb{Q}(\sqrt{D})$ is one. We will get a contradiction using zero–repelling ideas (Deuring–Heilbronn phenomenon) of section 2. The main idea is to consider the integral I_D defined by:

$$I_D = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Lambda_D(1+s)\,\frac{ds}{s^3},$$

where $\Lambda_D(1+s)$ is given in (3.2).

Lemma 3: *We have $I_D = 0$.*

Proof: If we shift the line of integration to $\Re(s) = -2$, the residue at $s = 0$ is zero because $\Lambda_D(1+s)$ has a fourth order zero at $s = 0$. If immediately follows that

$$\begin{aligned} I_D &= \frac{1}{2\pi i} \int\limits_{-2-i\infty}^{-2+i\infty} \Lambda_D(1+s)\, \frac{ds}{s^3} \\ &= -\frac{1}{2\pi i} \int\limits_{2-i\infty}^{2+i\infty} \Lambda_D(1+s)\, \frac{ds}{s^3} \\ &= -I_D, \end{aligned}$$

after applying the functional equation (3.3) and letting $s \to -s$. Consequently, $I_D = 0$. This completes the proof of Lemma 3.

We will now show that if $h(D) = 1$ and D is sufficiently large then $I_D \neq 0$. The heuristics for obtaining this contradiction are easily seen. We may write the Euler products:

$$L_E(s) = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1}, \tag{4.1}$$

$$L_E(s, \chi_D) = \prod_p \left(1 - \frac{\alpha_p \chi_D(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_p \chi_D(p)}{p^s}\right)^{-1}.$$

The assumption that $h(D) = 1$ implies that $\chi_D(p) = -1$ for all primes $p < \frac{1+|D|}{4}$ (Lemma 2). So we expect that analytically the Euler product $L_E(s)L_E(s, \chi_D)$ should behave like

$$\phi(s) := \prod_p \left(1 - \frac{\alpha_p^2}{p^{2s}}\right)^{-1} \left(1 - \frac{\beta_p^2}{p^{2s}}\right)^{-1},$$

where

$$|\alpha_p|^2 \;=\; |\beta_p|^2 = \alpha_p \beta_p = p,$$

for all but finitely many primes p. Now, if f is the weight two Hecke eigenform associated to E, then we have the symmetric square L–function

$$L\big(s, \mathrm{sym}^2(f)\big) := \prod_p \left(1 - \frac{\alpha_p^2}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p \beta_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p^2}{p^s}\right)^{-1}.$$

Thus $\phi(s)$ is essentially $L\big(2s, \mathrm{sym}^2(f)\big)/\zeta(2s-1)$. It is known that $L\big(s, \mathrm{sym}^2(f)\big)$ is entire which implies that $L\big(2s, \mathrm{sym}^2(f)\big)/\zeta(2s-1)$ vanishes at $s = 1$, a result first proved by **[Ogg]**. This implies that

$$\phi(1) = 0.$$

In fact, $\phi(1)$ has a simple zero at $s = 1$ which seems to contradict the fact that $L_E(s)L_E(s,\chi_D)$ has a fourth order zero. Although it appears that a contradiction could be obtained if $L_E(s)L_E(s,\chi_D)$ had a double zero at $s = 1$, this, unfortunately is not the case. The contradiction is much more subtle and will be shortly clarified.

We now define

$$I_D^* = \frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty} \left(\frac{37\cdot 139^2|D|}{4\pi^2}\right)^{1+s} \Gamma(1+s)^2\phi(1+s)\ \frac{ds}{s^3},$$

which allows us to write

$$0 = I_D \ = \ I_D^* + \ \text{Error}, \tag{4.2}$$

with

$$\text{Error} = \frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty} \left(\frac{37\cdot 139^2|D|}{4\pi^2}\right)^{1+s} \Gamma(1+s)^2\Big[L_E(1+s)L_E(1+s,\chi_D) - \phi(1+s)\Big]\ \frac{ds}{s^3}. \tag{4.3}$$

Lemma 4: *Define Dirichlet coefficients $B_n (n = 1, 2, \dots)$ by the representation*

$$L_E(1+s)L_E(1+s,\chi_D) - \phi(1+s) \ = \ \sum_{n=1}^{\infty} B_n\, n^{-1-s}.$$

We also define Dirichlet coefficients $\nu_D(n)(n = 1, 2, \dots)$ by the representation

$$\zeta(s)L(s,\chi_D) = \sum_{n=1}^{\infty} \nu_D(n)\, n^{-s}.$$

Then $B_n = 0$ for $n < \frac{1+|D|}{4}$. In the other cases, we have

$$|B_n| \ \le \ \begin{cases} 2\nu_D(n)\sqrt{n} & \text{if } \frac{1+|D|}{4} \le n < \left(\frac{1+|D|}{4}\right)^2, \\ 2d_4(n)\cdot\sqrt{n} & \text{if } n \ge \left(\frac{1+|D|}{4}\right)^2, \end{cases}$$

where $d_4(n) = \sum\limits_{d_1d_2d_3d_4=n} 1$.

Proof: The fact that $B_n = 0$ for $n < \frac{1+|D|}{4}$ follows immediately from Lemma 2. The upper bound $|B_n| \le 2d_4(n)\cdot n$ is a consequence of the fact (see (4.1)) that $L_E(1+s)L_E(1+s,\chi_D)$ is an Euler product of degree 4. Thus, the Dirichlet coefficients of $L_E(1+s)L_E(1+s,\chi_D)$ are bounded by the Dirichlet coefficients of the Euler product

$$\prod_p \left(1 - \frac{\sqrt{p}}{p^{1+s}}\right)^{-1} = \sum_{n=1}^{\infty} d_4(n)\sqrt{n}\cdot n^{-1-s}.$$

The extra factor of 2 in the bound for B_n comes from the consideration of the additional Euler product for $\phi(1+s)$.

In the range $\frac{1+|D|}{4} \le n < \left(\frac{1+|D|}{4}\right)^2$, we can only have $B_n \neq 0$ if n is divisible by a prime $q > \frac{1+|D|}{4}$. In this range, it is not possible that q^2 divides n. This implies that $\phi(1+s)$ does not contribute to B_n since $\phi(1+s)$ is a Dirichlet series formed from perfect squares, i.e., of the form $\phi(1+s) = \sum_{k=1}^{\infty} \frac{b(k)}{(k^2)^{1+s}}$. If we let $n = q \cdot m$ then we must have $B_n = a_m \cdot a_q$ where $L_E(s) = \sum_{k=1}^{\infty} a_k \cdot k^{-s}$. Consequently, $|B_n| \le 2|a_m|\sqrt{q}$. It is easy to see that m must be a perfect square because m can only be divisible by primes $< \frac{1+|D|}{4}$. Again, by considering the Euler product (4.1), it follows that in the range $\frac{1+|D|}{4} \le n < \left(\frac{1+|D|}{4}\right)^2$, the coefficients B_n are bounded by $2\nu_D(n)\sqrt{n}$ where $\nu_D(n)\sqrt{n}$ are the Dirichlet coefficients of the Euler product

$$\prod_p \left(1 - \frac{\sqrt{p}}{p^{1+s}}\right)^{-1} \left(1 - \chi_D(p)\frac{\sqrt{p}}{p^{1+s}}\right)^{-1} = \sum_{n=1}^{\infty} \nu_D(n)\sqrt{n} \cdot n^{-1-s}.$$

Clearly,

$$\zeta(s)L(s,\chi_D) = \sum_{n=1}^{\infty} \nu_D(n) \cdot n^{-s}.$$

Lemma 5: *Let $x > 1$. Then*

$$\sum_{x \le n \le 2x} \nu_D(n)\sqrt{n} \ \le\ 4e \cdot x^{\frac{3}{2}} L(1,\chi_D) \ +\ \mathcal{O}\left(|D|^{\frac{3}{2}} x^{-\frac{1}{2}}\right).$$

If we further assume that $|D| > 4$ and $h(D) = 1$, then

$$\sum_{x \le n \le 2x} \nu_D(n)\sqrt{n} \ \le\ 4\pi e \cdot \frac{x^{\frac{3}{2}}}{|D|^{\frac{1}{2}}} \ +\ \mathcal{O}\left(|D|^{\frac{3}{2}} x^{-\frac{1}{2}}\right).$$

Proof: We shall need the well known Mellin transform:

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} x^s \Gamma(s)\, ds = e^{-\frac{1}{x}}.$$

It follows that

$$\sum_{x \le n \le 2x} \nu_D(n)\sqrt{n} \ \le\ \frac{2e}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta\left(s - \frac{1}{2}\right) L(\left(s - \frac{1}{2}, \chi_D\right) \left((2x)^s - x^s\right) \Gamma(s)\, ds$$

$$= \ 2e \sum_{n=1}^{\infty} \nu_D(n)\sqrt{n}\left(e^{-\frac{n}{2x}} - e^{-\frac{n}{x}}\right).$$

Here we have used the fact that $2e\left(e^{-\frac{n}{2x}} - e^{-\frac{n}{x}}\right) > 1$ for $x \le n \le 2x$, and, otherwise, $\nu_D(n) \ge 0$. The above integral can be evaluating by shifting the line of integration to the left to the line $\Re(s) = -\frac{1}{2}$. There is a pole at $s = \frac{3}{2}$ coming from the Riemann zeta function. Consequently

(4.4)
$$\sum_{x \le n \le 2x} \nu_D(n)\sqrt{n} \;\le\; 2eL(1,\chi_D)\Big((2x)^{\frac{3}{2}} - x^{\frac{3}{2}}\Big) + \left| \frac{2e}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \zeta\left(s - \frac{1}{2}\right) L(\left(s - \frac{1}{2}, \chi_D\right) \left((2x)^s - x^s\right) \Gamma(s)\, ds \right|.$$

The functional equation

$$\zeta(s)L(s,\chi_D) = \left(\frac{\sqrt{|D|}}{\pi}\right)^{1-2s} \frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)} \zeta(1-s)L(1-s,\chi_D)$$

together with Stirling's asymptotic formula

$$\lim_{|t|\to\infty} |\Gamma(\sigma+it)| e^{\frac{\pi}{2}|t|} |t|^{\frac{1}{2}-\sigma} = \sqrt{2\pi}$$

imply that the shifted integral in (4.4) converges absolutely and is bounded by $\mathcal{O}\left(|D|^{\frac{3}{2}} x^{-1}\right)$. This completes the first part of the proof of Lemma 5. For the second part, we simply use Dirichlet's class number formula (see **[Da]**), $L(1,\chi_D) = \frac{\pi h(D)}{|D|^{\frac{1}{2}}}$, which holds for $|D| > 4$.

Lemma 6: *For $y > 0$, define*

$$G(y) := \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} y^{s+1} \Gamma(1+s)^2 \, \frac{ds}{s^3}.$$

Then

$$G(y) \;<\; 2y^2 e^{-\frac{1}{\sqrt{y}}}.$$

Proof: Recall the definition of the Gamma function

$$\Gamma(s) = \int_0^\infty e^{-u} u^s \, \frac{du}{u},$$

which satisfies $\Gamma(s+1) = s\Gamma(s)$. It follows that

(4.5)
$$G(y) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} y^{s+1} \int_0^\infty \int_0^\infty e^{-u_1-u_2} (u_1u_2)^s \frac{du_1 du_2}{u_1 u_2} \, \frac{ds}{s}.$$

On the other hand, we have the classical integral

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^s\,\frac{ds}{s} = \begin{cases} 1 & \text{if } x>1,\\ \frac{1}{2} & \text{if x = 1},\\ 0 & \text{if } x<1.\end{cases}$$

If we now apply the above to (4.5) (after interchanging integrals), we obtain

$$G(y) = y \iint\limits_{u_1u_2\,\geq\, y^{-1}} e^{-u_1-u_2}\,\frac{du_1du_2}{u_1u_2}. \tag{4.6}$$

To complete the proof, we use the range of integration, $u_1u_2 \geq y^{-1}$, to show that $\frac{1}{u_1u_2} \leq y$, from which it follows from (4.6) that

$$\begin{aligned} G(y) &\leq y^2 \iint\limits_{u_1u_2\,\geq\, y^{-1}} e^{-u_1-u_2}\,du_1du_2\\ &= y^2\int_0^\infty e^{-\frac{1}{u_2y}-u_2}\,du_2\\ &= y^{\frac{3}{2}}\int_0^\infty e^{-\frac{1}{\sqrt{y}}\left(u_2+\frac{1}{u_2}\right)}\,du_2\\ &\leq 2y^{\frac{3}{2}}\int_1^\infty e^{-\frac{1}{\sqrt{y}}\left(u_2+\frac{1}{u_2}\right)}\,du_2\\ &< 2y^2e^{-\frac{1}{\sqrt{y}}}. \end{aligned}$$

It now follows from (4.3), Lemma 4, and the definition of $G(y)$ given in Lemma 6 that

$$|\text{Error}| \leq \sum_{n\geq\frac{1+|D|}{4}} |B_n|\cdot G\left(\frac{37\cdot 139^2|D|}{4\pi^2n}\right).$$

The bound for $G(y)$ given in Lemma 6 implies that

$$\begin{aligned} |\text{Error}| \leq &\sum_{\frac{1+|D|}{4}\leq n\leq\left(\frac{1+|D|}{4}\right)^2} 4\nu_D(n)\sqrt{n}\cdot\left(\frac{37\cdot 139^2|D|}{4\pi^2n}\right)^2 e^{-\sqrt{\frac{4\pi^2n}{37\cdot 139^2|D|}}}\\ &+\sum_{\left(\frac{1+|D|}{4}\right)^2<n} 4d_4(n)\sqrt{n}\cdot\left(\frac{37\cdot 139^2|D|}{4\pi^2n}\right)^2 e^{-\sqrt{\frac{4\pi^2n}{37\cdot 139^2|D|}}}. \end{aligned}$$

The second sum in the above Error is $\mathcal{O}\left(e^{-c_1\sqrt{|D|}}\right)$ (for some $c_1>0$), so can be ignored. We can, therefore, estimate the Error by breaking it into smaller sums as follows:

$$\begin{aligned} |\text{Error}| \leq 4 &\sum_{k<\log_2\left(\frac{1+|D|}{4}\right)} \frac{37^2\cdot 139^4}{2^{2k-2}\cdot\pi^4} \sum_{\frac{1+|D|}{4}2^{k-1}\leq n\leq\left(\frac{1+|D|}{4}\right)2^k} \nu_D(n)\sqrt{n}\cdot e^{-\sqrt{\frac{4\pi^2n}{37\cdot 139^2|D|}}}\\ &+\mathcal{O}\left(e^{-\sqrt{|D|}}\right). \end{aligned}$$

For each, k, we can apply Lemma 5 to the inner sum over n in the above. It follows that

$$|\text{Error}| \ll |D| \sum_{k<\log\left(\frac{1+|D|}{4}\right)} 2^{-\frac{k}{2}} \ll |D|.$$

It immediately follows that for D sufficiently large, there exists a fixed, effectively computable constant c such that

$$|\text{Error}| \leq c \cdot |D|$$

as $|D| \to \infty$. Combining this bound with (4.2), we have that

$$I_D^* = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left(\frac{37 \cdot 139^2 |D|}{4\pi^2} \right)^{1+s} \Gamma(1+s)^2 \phi(1+s) \, \frac{ds}{s^3} \tag{4.7}$$

satisfies

$$|I_D^*| < c \cdot |D|. \tag{4.8}$$

The integral for I_D^* given in (4.7) can be evaluated by shifting the line of integration to the left. A double pole is encountered at $s = 0$. Actually the term $1/s^3$ contributes a triple pole, but the vanishing of $\phi(1+s)$ at $s = 0$ reduces this to a double pole. Because of the double pole and the known zero–free region for the Riemann zeta function, it is not hard to show that there exists an effectively computable constant $c_1 > 0$ such that

$$|I_D^*| > c_1 D \log D. \tag{4.9}$$

The inequalities (4.8) and (4.9) are contradictory for large D. Consequently, it is not possible that $h(D) = 1$. QED

§5. References

[**A**] S. Arno, *The imaginary quadratic fields of class number 4,* Acta Arith. **60** (1992), 321–334.

[**A-R-W**] S. Arno, M. Robinson, F. Wheeler, *Imaginary quadratic fields with small odd class number,* Acta Arith. **83** (1998), 295–330.

[**B**] A. Baker, *Imaginary quadratic fields with class number 2,* Annals of Math. (2) **94** (1971), 139–152.

[**Da**] H. Davenport, *Multiplicative Number Theory,* Second edition, Revised by H. Montgomery, Grad. Texts in Math. 74, Springer–Verlag (1980).

[**G**] C.F. Gauss, *Disquisitiones Arithmeticae*, Göttengen (1801); English translation by A. Clarke, revised by W. Waterhouse, 1986 Springer–Verlag reprint of the Yale University Press, New Haven, 1966 edition.

[**Go1**] D. Goldfeld *The class number of quadratic fields and the conjectures of Birch and Swinnerton–Dyer,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **3** (1976), 624–663.

[Go2] D. Goldfeld, *Gauss' class number problem for imaginary quadratic fields*, Bull. Amer. Math. Soc. **13** (1985), 23–37.

[G-Z] B. Gross, D.B. Zagier, *Heegner points and derivatives of L–series,* Invent. Math. **84** (1986), 225–320.

[Heg] K. Heegner, *Diophantische Analysis und Modulfunktionen,* Math. Z. **56** (1952), 227–253.

[H] H. Heilbronn, *On the class number in imaginary quadratic fields*, Quarterly J. of Math., **5** (1934), 150–160.

[O] J. Oesterlé, *Le probléme de Gauss sur le nombre de classes,* Enseign. Math. **34** (1988), 43–67.

[Ogg] A. Ogg, *On a convolution of L–series,* Invent. Math. **7** (1969), 297–312.

[Ra] G. Rabinovitch, *Eindeutigkeit der Zerlegung in Primzahlfaktoren in quadratischen Zahlkörpern,* Proc. Fifth Internat. Congress Math. (Cambridge), vol I, (1913), 418–421.

[Shim] G. Shimura, *Introduction to the arithmetic theory of automorphic functions,* Iwanami Shoten – Princeton University Press, (1971).

[St1] H. Stark, *A complete determination of the complex quadratic fields of class–number one,* Mich. Math. J. **14** (1967), 1–27.

[St2] H. Stark, *A transcendence theorem for class–number problems I, II*, Annals of Math. (2) **94** (1971), 153–173; ibid. **96** (1972), 174–209.

[Wag] C. Wagner, *Class number 5, 6 and 7,* Math. Comp. **65** (1996), 785–800.

[Wat] M. Watkins, *Class numbers of imaginary quadratic fields,* to appear.

Dorian Goldfeld, Columbia University Department of Mathematics, New York, NY 10027
E-mail address: goldfeld@columbia.edu