Moments for L-functions for $GL_r \times GL_{r-1}$

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We exhibit elementary kernels Pé which produce sums of integral moments for cuspforms f on GL_r by

$$\int_{Z_{\mathbb{A}} GL_r(k)\backslash GL_r(\mathbb{A})} \text{Pé} \cdot |f|^2 = \sum_{F \text{ on } GL_{r-1}} \int_{\text{Re}(s)=\frac{1}{2}} \frac{|L(s, f \otimes F)|^2}{\langle F, F\rangle} M(s)\, ds \;+\; (\text{continuous part})$$

over number fields k, with certain weights $M(s)$. Here F runs over an orthogonal basis for cuspforms on GL_{r-1}. There are further continuous-spectrum terms analogous to the discrete-spectrum sum over cuspforms. The kernel (Poincaré series) Pé admits a spectral decomposition, surprisingly consisting of only three parts: a leading term, a sum arising from cuspforms on GL_2, and a continuous part from GL_2. That is, no cuspforms on GL_ℓ with $2 < \ell \le r$ contribute. This spectral decomposition makes possible the meromorphic continuation of Pé in auxiliary parameters.

Moments of level-one holomorphic elliptic modular forms were treated in [Good 1983] and [Good 1986], the latter using an idea that is a precursor of part of the present approach. Level-one waveforms over $\mathbb{Q}$ appear in [Diaconu-Goldfeld 2006a], over $\mathbb{Q}(i)$ in [Diaconu-Goldfeld 2006b]. Arbitrary level, groundfield, and ∞-type for GL_2 are in [Diaconu-Garrett 2006] and [Diaconu-Garrett 2008].

We do have in mind application not only to cuspforms, but also to truncated Eisenstein series (with cuspidal data) or wave packets of Eisenstein series, giving a non-trivial application of harmonic analysis on larger groups GL_r to L-functions attached to smaller groups, for example, on GL_1, giving high integral moments of $\zeta_k(s)$.

For context, we review the [Diaconu-Goldfeld 2006a] treatment of spherical waveforms f for $GL_2(\mathbb{Q})$. In that case, the sum of moments is a single term

$$\int_{Z_{\mathbb{A}} GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})} \text{Pé}(g)\, |f(g)|^2\, dg = \frac{1}{2\pi i} \int_{\text{Re}(s)=\frac{1}{2}} L(s'+s, f) \cdot \overline{L}(s, f) \cdot \Gamma(s, s', s'', f_\infty)\, ds$$

where $\Gamma(s, s', s'', f_\infty)$ is a ratios of products of gammas, with arguments depending upon the archimedean data attached to f. Here the Poincaré series $\text{Pé}(g) = \text{Pé}(g, s', s'')$ has a *spectral expansion*

$$\text{Pé}(s', s'') \;=\; \frac{\pi^{\frac{1-s''}{2}}\Gamma(\frac{s''-1}{2})}{\pi^{-\frac{s''}{2}}\Gamma(\frac{s''}{2})} \cdot E_{1+s'} \;+\; \tfrac{1}{2} \sum_{F \text{ on } GL_2} \frac{L(\frac{1}{2}+s', \overline{F})}{\langle F, F\rangle} \cdot \mathcal{G}(\tfrac{1}{2} - it_F, s', s'') \cdot F$$

$$+\frac{1}{4\pi i} \int_{\text{Re}(s)=\frac{1}{2}} \frac{\zeta(s'+s)\,\zeta(s'+1-s)}{\xi(2-2s)}\, \mathcal{G}(1-s, s', s'') \cdot E_s\, ds \qquad (\text{for } \text{Re}(s') \gg \tfrac{1}{2},\ \text{Re}(s'') \gg 0)$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, where $\mathcal{G}$ is essentially a product of gamma function values [1]

$$\mathcal{G}(s,s',s'') = \pi^{-(s'+\frac{s''}{2})} \frac{\Gamma(\frac{s'+1-s}{2})\,\Gamma(\frac{s'+s}{2})\,\Gamma(\frac{s'-s+s''}{2})\,\Gamma(\frac{s'+s-1+s''}{2})}{\Gamma(s'+\frac{s''}{2})}$$

and F is summed over (an orthogonal basis for) spherical (at finite primes) cuspforms on GL_2 with Laplacian eigenvalues $\frac{1}{4}+t_F^2$, and E_s is the usual spherical Eisenstein series

$$E_s\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = |y|^s + \frac{\xi(2-2s)}{\xi(2s)}\,|y|^{1-s} + \ldots$$

It is not obvious, but the continuous part (the *integral* of Eisenstein series) cancels the pole at $s'=1$ of the leading term, and when evaluated at $s'=0$ is [2]

$$\text{Pé}(g,0,s'') \;=\; (\text{holomorphic at } s'\!=\!0) \;+\; \frac{1}{2}\sum_{F \text{ on } GL_2} \frac{L(\frac{1}{2},\overline{F})}{\langle F,F\rangle}\cdot \mathcal{G}(\tfrac{1}{2}-it_F,0,s'')\cdot F$$

$$+\frac{1}{4\pi i}\int_{\text{Re}\,(s)=\frac{1}{2}} \frac{\zeta(s)\,\zeta(1-s)}{\xi(2-2s)}\,\mathcal{G}(1-s,0,s'')\cdot E_s\,ds$$

In this spectral expansion, the coefficient in front of a cuspform F includes $\mathcal{G}$ evaluated at $s'=0$ and $s=\frac{1}{2}\pm it_F$, namely

$$\mathcal{G}(\tfrac{1}{2}-it_F,0,s'') = \pi^{-\frac{s''}{2}} \frac{\Gamma(\frac{\frac{1}{2}-it_F}{2})\,\Gamma(\frac{\frac{1}{2}+it_F}{2})\,\Gamma(\frac{s''-\frac{1}{2}-it_F}{2})\,\Gamma(\frac{s''-\frac{1}{2}+it_F}{2})}{\Gamma(\frac{s''}{2})}$$

The gamma function has poles at $0,-1,-2,\ldots$, so this coefficient has poles at $s''=\frac{1}{2}\pm it_F$, $-\frac{3}{2}\pm it_F$, ... Over $\mathbb{Q}$, among spherical cuspforms (or for any fixed level) these values have no accumulation point. [3] The continuous part of the spectral side at $s'=0$ is

$$\frac{1}{4\pi i}\int_{\text{Re}\,(s)=\frac{1}{2}} \frac{\xi(s)\,\xi(1-s)}{\xi(2-2s)}\,\frac{\Gamma(\frac{s''-s}{2})\,\Gamma(\frac{s''-1+s}{2})}{\Gamma(\frac{s''}{2})}\cdot E_s\,ds$$

with gamma factors grouped with corresponding zeta functions, to form the completed L-functions ξ. Thus, the evident pole of the leading term at $s''=1$ can be exploited, using the obvious continuation to $\text{Re}\,(s'')>1/2$.

Further, a subtle contour-shifting argument [4] shows that the continuous part of this spectral decomposition has a meromorphic continuation to $\mathbb{C}$ with poles at $\rho/2$ for zeros ρ of ζ, in addition to the obvious poles from the gamma functions.

[1] This from [Diaconu-Goldfeld 2006a], the result of a direct computation with the simplest useful choice of archimedean data in the Poincaré series.

[2] This evaluation of the meromorphic continuation, from [Diaconu-Goldfeld 2006a], is not trivial. Note that the leading term (after continuation) is reminiscent of the Kronecker limit formula. See [Asai 1977].

[3] The discreteness of the parameters t_F as F ranges over cuspforms of a fixed level follows from the compactness of test-function operators on cuspforms, from [Gelfand-PS 1963]. In particular, at this point we do *not* need any sort of *Weyl's law*, and, thus, do not need trace formula computations yet.

[4] See [Diaconu-Goldfeld 2006a]. A contour shifting argument is also necessary to meromorphically continue the continuous part of the spectral decomposition to $s'=0$ in the first place.

Already for GL_2, over general groundfields k, infinitely many Hecke characters enter [5] both the spectral decomposition of the Poincaré series and the moment expression. This naturally complicates isolation of literal moments, and complicates analysis of poles via the spectral expansion. Suppressing constants, the moment expansion is a sum of twists by χ's

$$\int_{Z_{\mathbb{A}}GL_2(k)\backslash GL_2(\mathbb{A})} \text{Pé} \cdot |f|^2 = \sum_\chi \int_{\text{Re}\,(s)=\frac{1}{2}} L(s'+s, f\otimes\chi)\cdot L(1-s,\overline{f}\otimes\overline{\chi})\cdot M_\chi(s)\, ds$$

And, suppressing constants, the spectral expansion is

$$\text{Pé} \;=\; (\infty-\text{part})\cdot E_{1+s'} \;+\; \sum_{F \text{ on } GL_2} (\infty-\text{part})\cdot \frac{L(\frac{1}{2}+s',\overline{F})}{\langle F,F\rangle}\cdot F$$

$$+\;\; \sum_\chi \int_{\text{Re}\,(s)=\frac{1}{2}} \frac{L(s'+s,\overline{\chi})\,L(s'+1-s,\chi)}{L(2-2s,\overline{\chi}^2)}\,\mathcal{G}_\chi(s)\cdot E_{s,\chi}\, ds$$

In the simplest case beyond GL_2, take f a spherical cuspform [6] on GL_3 over [7] $\mathbb{Q}$. We construct a weight function $\Gamma(s,s',s'',f_\infty,F_\infty)$ depending upon complex parameters s, s', and s'', *and* upon the *archimedean* data for both f and cuspforms F on GL_2, such that $\Gamma(s,s',s'',f_\infty,F_\infty)$ has explicit asymptotic behavior, and such that the *moment expansion* is

$$\int_{Z_{\mathbb{A}}GL_3(\mathbb{Q})\backslash GL_3(\mathbb{A})} \text{Pé}(s',s'')\cdot|f|^2\, dg = \sum_{F \text{ on } GL_2} \frac{1}{2\pi i}\int_{\text{Re}\,(s)=\frac{1}{2}} \frac{|L(s,f\otimes F)|^2}{\langle F,F\rangle}\cdot\Gamma(s,0,s'',f_\infty,F_\infty)\, ds$$

$$+\frac{1}{4\pi i}\frac{1}{2\pi i}\sum_{k\in\mathbb{Z}} \int_{\text{Re}\,(s_1)=\frac{1}{2}}\int_{\text{Re}\,(s_2)=\frac{1}{2}} \frac{|L(s_1,f\otimes E^{(k)}_{1-s_2})|^2}{|\xi(1-2it_2)|^2}\cdot\Gamma(s_1,0,s'',f_\infty,E^{(k)}_{1-s_2,\infty})\, ds_1\, ds_2$$

where F runs over (an orthogonal basis for) all level-one cuspforms on GL_2, with *no* restriction on the right K_∞-type, and $E^{(k)}_s$ is the usual level-one Eisenstein series of K_∞-type k. Here and throughout, for $\text{Re}\,(s)=1/2$, write $1-s$ in place of $\overline{s}$, to maintain holomorphy in complex-conjugated parameters. In this vein, over $\mathbb{Q}$, it is reasonable to put

$$L(s_1,f\otimes\overline{E}^{(k)}_{s_2}) = L(s_1,f\otimes E^{(k)}_{1-s_2}) = \frac{L(s_1+\frac{1}{2}-s_2,f)\cdot L(s_1-\frac{1}{2}+s_2,f)}{\zeta(2-2s_2)} \qquad \text{(finite-prime parts only)}$$

since the natural normalization of the Eisenstein series $E^{(k)}_{s_2}$ on GL_2 contributes the denominator $\zeta(2-2s_2)$. Meromorphic continuation in s' and evaluation at $s'=0$ gives the desired specialization of the *moment expansion*. There is also a meromorphic continuation in the parameter s'' in the archimedean data.

More generally, for a cuspform f on GL_r with $r\geq 3$, whether over $\mathbb{Q}$ or over a numberfield, the *moment expansion* includes an infinite sum [8] of $|L(s,f\otimes F)|^2/\langle F,F\rangle$ over an orthogonal basis for cuspforms F

[5] See [Sarnak 1985], [Diaconu-Goldfeld 2006b] for $\mathbb{Q}(i)$, and [Diaconu-Garrett 2006] for number fields.

[6] For our purposes, a *cuspform* generates an irreducible unitary representation of the adele group, so has a central character, and the representation factors over primes. This factorization follows from the admissibility of irreducible unitaries of reductive groups over archimedean and non-archimedean local fields (the former due to Harish-Chandra in the 1950s, the latter reduced to the supercuspidal case by Harish-Chandra, and completed by J. Bernstein in 1972).

[7] The assumption of groundfield $\mathbb{Q}$ achieves the minor simplification that for $GL_2(\mathbb{Q})$ there is a single (family of) Eisenstein series participating in the spectral decomposition.

[8] In fact, it is non-trivial to prove (after Selberg) that there are infinitely-many spherical waveforms for $SL_2(\mathbb{Z})$.

on GL_{r-1}, as well as *integrals* of products of L-functions $L(s, f \otimes F)$ for F ranging over cuspforms on $GL_{r_1} \times \ldots \times GL_{r_\ell}$ for all partitions $(r_1, \ldots, r_\ell)$ of r. Correspondingly, the natural normalization of the cuspidal-data Eisenstein series gives products of convolution L-functions $L(*, F_i \otimes F_j)$ in the denominators of these terms, as well as factors $\langle F_i, F_i\rangle^{1/2} \cdot \langle F_j, F_j\rangle^{1/2}$. [9]

Generally, the spectral expansion for GL_r is an induced-up version of that for GL_2. Suppressing constants, using groundfield $\mathbb{Q}$ to skirt Hecke characters,

$$\text{Pé} \;=\; (\infty-\text{part})\cdot E^{r-1,1}_{s'+1} \;+\; \sum_{F \text{ on } GL_2} (\infty-\text{part})\cdot \frac{L(\frac{rs'+r-2}{2}+\frac{1}{2}, \overline{F})}{\langle F, F\rangle}\cdot E^{r-2,2}_{\frac{s'+1}{2}, F}$$

$$+\int_{\mathrm{Re}\,(s)=\frac{1}{2}} (\infty-\text{part})\cdot \frac{\zeta(\frac{rs'+r-2}{2}+\frac{1}{2}-s)\cdot\zeta(\frac{rs'+r-2}{2}+\frac{1}{2}+s)}{\zeta(2-2s)} \;\cdot\; E^{r-2,1,1}_{s'+1,\, s-\frac{s'+1}{2},\, -s-\frac{s'+1}{2}}\; ds$$

where the Eisenstein series are normalized naively. The continuous part has a pole that cancels the obvious pole of the leading term at $s' = 0$.

Again over $\mathbb{Q}$, the *most-continuous* part of the moment expansion for GL_n is of the form

$$\int_{\mathrm{Re}\,(s)=\frac{1}{2}} \int_{t\in\Lambda} |L(s, f\otimes E^{\min}_{\frac{1}{2}+it})|^2\, M_t(s)\, ds\, dt = \int\int_{\Lambda} \left|\frac{\Pi_{1\le \ell\le n-1}\, L(s+it_\ell, f)}{\Pi_{1\le j<\ell<n}\, \zeta(1-it_j+it_\ell)}\right|^2 M_t(s)\, ds\, dt$$

where

$$\Lambda = \{t \in \mathbb{R}^{n-1} \;:\; t_1+\ldots+t_{n-1} = 0\}$$

and where M is a weight function depending upon f and F. More generally, let $n-1 = m\cdot b$. For F on GL_m, let

$$F^{\Delta} = F\otimes\ldots\otimes F$$

on $GL_m \times \ldots \times GL_m$. Inside the moment expansion we have (recall Langlands-Shahidi)

$$\int_{\mathrm{Re}\,(s)=\frac{1}{2}} \int_{\Lambda} |L(s, f\otimes E_{F^{\Delta},\, \frac{1}{2}+it})|^2\, M_{F,t}(s)\, ds\, dt = \int\int \left|\frac{\Pi_{1\le\ell\le b}\, L(s+it_\ell, f\otimes F)}{\Pi_{1\le j<\ell\le b}\, L(1-it_j+it_\ell, F\otimes F^{\vee})}\right|^2 M\, ds\, dt$$

If we replace the cuspform f on $GL_n(\mathbb{Q})$ by a (truncated) minimal-parabolic Eisenstein series E_α with $\alpha \in \mathbb{C}^{n-1}$, the most-continuous part of the moment expansion contains a term

$$\int\int_{\Lambda} \left|\frac{\Pi_{1\le\mu\le n,\, 1\le\ell\le n-1}\, \zeta(\alpha_\mu+s+it_\ell)}{\Pi_{1\le j<\ell<n}\, |\zeta(1-it_j+it_\ell)}\right|^2 ds\, dt$$

Taking $\alpha = 0 \in \mathbb{C}^{n-1}$ gives

$$\int\int_{\Lambda} \left|\frac{\Pi_{1\le\ell\le n-1}\;\zeta(s+it_\ell)^n}{\Pi_{1\le j<\ell<n}\, \zeta(1-it_j+it_\ell)}\right|^2 M\, ds\, dt$$

For example, for GL_3, where $\Lambda = \{(t, -t)\} \approx \mathbb{R}$,

$$\int\int_{\mathbb{R}} \left|\frac{\zeta(s+it)^3\cdot\zeta(s-it)^3}{\zeta(1-2it)}\right|^2 M\; ds\, dt$$

and for GL_4

$$\int_{(s)}\int_{\Lambda} \left|\frac{\zeta(s+it_1)^4\cdot\zeta(s+it_2)^4\cdot\zeta(s+it_3)^4}{\zeta(1-it_1+it_2)\,\zeta(1-it_1+it_3)\,\zeta(1-it_2+it_3)}\right|^2 M\, ds\, dt$$

[9] The identification of these denominators in the natural normalization of the Eisenstein series is part of the refinement in [Shahidi 1978] and [Shahidi 1983] of [Langlands 1976]'s treatment of L-functions arising in *constant terms* of these Eisenstein series. Here we need to keep track of constants, due to the average over F.

1. *The moment expansion*

Let $G = GL_r$ over a number field k. Let P be the standard maximal proper parabolic

$$P = P^{r-1,1} = \{\begin{pmatrix} (r-1)\text{-by-}(r-1) & * \\ 0 & 1\text{-by-}1 \end{pmatrix}\}$$

Let

$$U = \{\begin{pmatrix} 1_{r-1} & * \\ 0 & 1 \end{pmatrix}\} \qquad H = \{\begin{pmatrix} (r-1)\text{-by-}(r-1) & 0 \\ 0 & 1 \end{pmatrix}\}$$

and

$$N = \{\text{upper triangular unipotent elements in } H\}$$
$$= (\text{unipotent radical of standard minimal parabolic in } H)$$

Let Z be the center of G. Let K_v be the standard maximal compact in the k_v-valued points G_v of G. Thus, for $v < \infty$, $K_v = GL_r(\mathfrak{o}_v)$. For $v \approx \mathbb{R}$, take $K_v = O_r(\mathbb{R})$. For $v \approx \mathbb{C}$ take $K_v = U(r)$.

The standard choice of non-degenerate character on $N_k U_k \backslash N_{\mathbb{A}} U_{\mathbb{A}}$ is

$$\psi(n \cdot u) = \psi_0(n_{12} + n_{23} + \ldots + n_{r-2,r-1}) \cdot \psi_0(u_{r-1,r})$$

where ψ_0 is a fixed non-trivial character on $\mathbb{A}/k$. A cuspform [10] f has a Fourier expansion [12] along NU

$$f(g) = \sum_{\xi \in N_k \backslash H_k} W_f(\xi g) \quad \text{where} \quad W_f(g) = \int_{N_k U_k \backslash N_{\mathbb{A}} U_{\mathbb{A}}} \overline{\psi}(nu)\, f(nug)\, dn\, du$$

The (Whittaker) function $W_f(g)$ factors over primes. [13]

[1.1] Poincaré series

For $s' \in \mathbb{C}$, let

$$\varphi = \bigotimes_v \varphi_v$$

where for v finite

$$\varphi_v(g) = \begin{cases} \left|(\det A)/d^{r-1}\right|_v^{s'} & (\text{for } g = mk \text{ with } m = \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \text{ in } Z_v H_v \text{ and } k \in K_v) \\ 0 & (\text{otherwise}) \end{cases}$$

[10] A *cuspform* f satisfies the Gelfand-Fomin-Graev condition [11] $\int_{N_k \backslash N_{\mathbb{A}}} f(ng)\, dn = 0$ for all unipotent radicals N of (proper, rational) parabolics, generates an irreducible representation locally everywhere (hence, has a central character ω), and is in $L^2(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}, \omega)$.

[12] Fourier-Whittaker expansions for $GL(n)$ with $n > 2$ are due to [Shalika 1974] and [Piatetski-Shapiro 1975], independently.

[13] Uniqueness of Whittaker models, by [Shalika 1974] at archimedean places, [Gelfand-Kazhdan 1975] at non-archimedean, implies the factorization over primes. The local factors are ambiguous up to constants, naturally. For *cuspforms* f at primes v where f is *spherical*, the spherical Whittaker function W_v° with the same local data as f, normalized by $W_v^\circ(1) = 1$, is *the* standard choice. However, even at good primes, for natural normalization in *construction* of Eisenstein series presents us most naturally with a local Whittaker function which is an image under the intertwining operator from principal series to the Whittaker model. This contributes an extra factor which is a product of L-functions, as studied at length in [Langlands 1971], [Shahidi 1978], [Shahidi 1983], and elsehwere. Luckily, our subsequent convolution L-functions will lack archimedean factors, so avoid the most acute concern about choices of data at archimedean places, thus skirting issues addressed in [Jacquet-Shalika 1990], [Cogdell-PS 2003].

and for v archimedean require right K_v-invariance and left equivariance

$$\varphi_v(mg) = \left|\frac{\det A}{d^{r-1}}\right|_v^{s'} \cdot \varphi_v(g) \qquad \text{(for } g \in G_v\text{, for } m = \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \in Z_v H_v\text{)}$$

Thus, for $v|\infty$, the further data determining φ_v consists of its values on U_v. The simplest useful choice is [14]

$$\varphi_v \begin{pmatrix} 1_{r-1} & x \\ 0 & 1 \end{pmatrix} = (1 + |x_1|^2 + \ldots + |x_{r-1}|^2)^{-s''/2} \qquad \text{(where } x = \begin{pmatrix} x_1 \\ \vdots \\ x_{r-1} \end{pmatrix}\text{, and } s'' \in \mathbb{C}\text{)}$$

and where the norm $|x_1|^2 + \ldots + |x_{r-1}|^2$ is normalized to be invariant under K_v. [15] Thus, φ is left $Z_{\mathbb{A}} H_k$-invariant. We attach to φ a **Poincaré series**

$$\text{Pé}(g) = \sum_{\gamma \in Z_k H_k \backslash G_k} \varphi(\gamma g)$$

[1.2] Two obvious unwindings

Integrate the norm-squared $|f|^2$ of a cuspform f against Pé. The typical first unwinding is

$$\int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \text{Pé}(g)\, |f(g)|^2\, dg = \int_{Z_{\mathbb{A}} H_k \backslash G_{\mathbb{A}}} \varphi(g)\, |f(g)|^2\, dg$$

Next, express f in its Fourier-Whittaker expansion, and unwind further:

$$\int_{Z_{\mathbb{A}} H_k \backslash G_{\mathbb{A}}} \varphi(g) \sum_{\xi \in N_k \backslash H_k} W_f(\xi g)\, \overline{f}(g)\, dg = \int_{Z_{\mathbb{A}} N_k \backslash G_{\mathbb{A}}} \varphi(g)\, W_f(g)\, \overline{f}(g)\, dg$$

[1.3] Iwasawa decomposition, simplification of integral

Suppose for simplicity that f is right $K_{\mathbb{A}}$-invariant, so we can use an Iwasawa decomposition $G = (HZ)UK$ (everywhere locally) to rewrite the whole integral as

$$\int_{N_k \backslash H_{\mathbb{A}} \times U_{\mathbb{A}}} \varphi(hu)\, W_f(hu)\, \overline{f}(hu)\, dh\, du$$

[1.4] Spectral decomposition on GL_{r-1}

Use a spectral decomposition [16] for $F \in L^2(H_k \backslash H_{\mathbb{A}})$, inexplicitly

$$F = \int_{(\eta)} \langle F, \eta \rangle \cdot \eta\, d\eta$$

[14] Over $\mathbb{Q}$ and $\mathbb{Q}(i)$, in effect this is the choice of archimedean data in [Good 1983], [Diaconu-Goldfeld 2006a], [Diaconu-Goldfeld 2006b].

[15] Thus, for this purpose, to be K_v-invariant at *real* places v, we use the standard norm. At complex places we *also* use the standard norm, *not* the norm compatible with the product formula. That is, for this purpose, for $v \approx \mathbb{C}$ the norm is *not* $|z|_v = z \cdot \bar{z}$, but is $|z| = \sqrt{z \cdot \bar{z}}$.

[16] Let $H^1 = \{h \in H_{\mathbb{A}} : |\det h| = 1\}$. Then $H_k \backslash H_{\mathbb{A}} = H_k \backslash H^1 \times (0, +\infty)$ is the relevant decomposition. The most obvious *continuous* part coming from $(0, +\infty)$ will eventually give integrals over vertical lines. Still $L^2(H_k \backslash H^1)$ has both continuous and discrete parts in its decomposition. Since some of the necessary functions η for such a spectral decomposition are not literally in L^2 (both Eisenstein series and ordinary exponentials in Fourier transforms and Fourier inversion), the inner integral is not at all symmetric. Further, that integral, (as well as the outer) only make literal sense for F in a suitable (dense) subspace (e.g., *pseudo-Eisenstein series* and Schwartz functions in the corresponding circumstances), and the mappings indicated must be defined by isometric extension. See [Langlands 1976] and [Moeglin-Waldspurger 1995].

where each η generates an irreducible representation of $H_{\mathbb{A}}$.

[1.5] Expand $\overline{f}(hu)$ Since $\overline{f}$ is left H_k-invariant, it decomposes along $H_k \backslash H_{\mathbb{A}}$ as

$$\overline{f}(hu) = \int_{(\eta)} \eta(h) \int_{H_k \backslash H_{\mathbb{A}}} \overline{\eta}(m)\, \overline{f}(mu)\, dm\, d\eta$$

Unwind the Fourier-Whittaker expansion of $\overline{f}$

$$\overline{f}(hu) = \int_{(\eta)} \eta(h) \int_{H_k \backslash H_{\mathbb{A}}} \overline{\eta}(m) \sum_{\xi \in N_k \backslash H_k} \overline{W}_f(\xi m u)\, dm\, d\eta$$

$$= \int_{(\eta)} \eta(h) \int_{N_k \backslash H_{\mathbb{A}}} \overline{\eta}(m)\, \overline{W}_f(mu)\, dm\, d\eta$$

Then the whole integral is

$$\int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \text{Pé}(g)\, |f(g)|^2\, dg$$

$$= \int_{(\eta)} \int_{N_k \backslash H_{\mathbb{A}} \times U_{\mathbb{A}}} \varphi(hu)\, \eta(h)\, W_f(hu) \int_{N_k \backslash H_{\mathbb{A}}} \overline{W}_f(mu)\, \overline{\eta}(m)\, dm\, dh\, du\ d\eta$$

[1.6] Decoupling at finite primes The part of the integrand that depends upon $u \in U$ is

$$\int_{U_{\mathbb{A}}} \varphi(hu)\, W_f(hu)\, \overline{W}_f(mu)\, du = \varphi(h)\, W_f(h)\, \overline{W}_f(m) \cdot \int_{U_{\mathbb{A}}} \varphi(u)\, \psi(huh^{-1})\, \overline{\psi}(mum^{-1})\, du$$

The latter integrand visibly factors over primes.

[1.6.1] **Lemma:** Let v be a finite prime. For $h, m \in H_v$ such that $W_{f,v}(h) \neq 0$ and $\overline{W}_{f,v}(m) \neq 0$,

$$\int_{U_v} \varphi_v(h)\, \psi_v(huh^{-1})\, \overline{\psi}_v(mum^{-1})\, du = \int_{U_v \cap K_v} 1\, du$$

Proof: At a finite place v, $\varphi_v(u) \neq 0$ if and only if $u \in U_v \cap K_v$, and for such u

$$\psi_v(huh^{-1}) \cdot W_{f,v}(h) = W_{f,v}(huh^{-1} \cdot h) = W_{f,v}(hu) = W_{f,v}(h) \cdot 1$$

by the right K_v-invariance. Thus, for $W_{f,v}(h) \neq 0$, $\psi_v(huh^{-1}) = 1$, and similarly for $\psi_v(mum^{-1})$. Thus, the finite-prime part of the integral over U_v is just the integral of 1 over $U_v \cap K_v$, as indicated. ///

[1.7] Archimedean kernel The archimedean part of the integral does not necessarily decouple. Thus, with subscripts ∞ denoting the infinite-adele part of various objects, for $h, m \in H_\infty$, define

$$\mathcal{K}(h,m) = \int_{U_\infty} \varphi_\infty(u)\, \psi_\infty(huh^{-1})\, \overline{\psi}_\infty(mum^{-1})\, du$$

The whole integral is

$$\int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \text{Pé}(g)\, |f(g)|^2\, dg = \int_{(\eta)} \int_{N_k \backslash H_{\mathbb{A}}} \int_{N_k \backslash H_{\mathbb{A}}} \mathcal{K}(h,m)\, \varphi(h) \Big(W_f(h)\, \eta(h)\Big) \Big(\overline{W}_f(m)\, \overline{\eta}(m)\Big)\, dm\, dh\, d\eta$$

[1.8] Fourier expansion of η Normalize the volume of $N_k\backslash N_{\mathbb{A}}$ to 1. Thus, for a left N_k-invariant function F on $H_{\mathbb{A}}$

$$\int_{N_k\backslash H_{\mathbb{A}}} F(h)\,dh = \int_{N_{\mathbb{A}}\backslash H_{\mathbb{A}}} \int_{N_k\backslash N_{\mathbb{A}}} F(nh)\,dn\,dh$$

Using the left $N_{\mathbb{A}}$-equivariance of W by ψ, and the left $N_{\mathbb{A}}$-invariance of φ,

$$\int_{N_k\backslash N_{\mathbb{A}}} \varphi(nh)\,\eta(nh)\,W_f(nh)\,dn = \varphi(h)\,W_f(h) \int_{N_k\backslash N_{\mathbb{A}}} \psi(n)\,\eta(nh)\,dn = \varphi(h)\,W_f(h)\,W_\eta(h)$$

where

$$W_\eta(h) = \int_{N_k\backslash N_{\mathbb{A}}} \psi(n)\,\eta(nh)\,dn$$

(The integral is not against $\overline{\psi}(n)$, but $\psi(n)$.) That is, the integral over $N_k\backslash H_{\mathbb{A}}$ is equal to an integral against (up to an alteration of the character) the Whittaker function W_η of η, which factors [17] over primes. The whole integral is

$$\int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} \text{Pé}(g)\,|f(g)|^2\,dg = \int_{(\eta)} \int_{N_{\mathbb{A}}\backslash H_{\mathbb{A}}} \int_{N_{\mathbb{A}}\backslash H_{\mathbb{A}}} \mathcal{K}(h,m)\,\varphi(h) \Big(W_f(h)\,W_\eta(h)\Big) \Big(\overline{W}_f(m)\,\overline{W}_\eta(m)\Big)\,dm\,dh\,d\eta$$

And the η^{th} part is a product of two Euler products. It is evident that for f right K_{fin}-invariant only right $(K_{\text{fin}} \cap H_{\text{fin}})$-invariant η's will appear, due to the decoupling. However, at archimedean places v right K_v-invariance of f does *not* allows us to restrict our attention to right $(K_v \cap H_v)$-invariant η.

[1.9] Appearance of the parameter s In fact, as usual,

$$H_k\backslash H_{\mathbb{A}} \approx GL_{r-1}(k)\backslash GL_{r-1}(\mathbb{A}) \approx \mathbb{R}^+ \times H_k\backslash H^1$$

where $\mathbb{R}^+$ is positive real numbers, and

$$H^1 = \{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \;:\; a \in GL_{r-1}(\mathbb{A}),\; |\det a| = 1\}$$

The quotient $H_k\backslash H^1$ has finite volume. Thus, the spectral decomposition uses functions

$$\eta \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \;=\; |\det a|^s \cdot F(a) \quad \text{with} \quad F \in L^2(H_k\backslash H^1),\;\; s \in i\mathbb{R}$$

The real part of the parameter s will necessarily be shifted in the subsequent discussion. Thus, the functions η above are of the form $|\det|^s \otimes F$, and the Whittaker function W_η of $\eta = |\det|^s \otimes F$ is

$$W_\eta \begin{pmatrix} a & \\ & 1 \end{pmatrix} = |\det a|^s \cdot W_F(a)$$

where W_F is the Whittaker function of F, normalized here by

$$W_F(g) = \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}(n)\,F(ng)\,dn$$

[17] When η is either a cuspform (in a strong sense) or is an Eisenstein series attached to suitable (e.g., cuspidal, in a strong sense) data, this Whittaker function W_η factors over primes. The usual normalization for the *spherical* Whittaker function W_v° at finite places v is $W_v^\circ(1) = 1$. This is the normalization we take for *cuspforms*, but this is incompatible with a standard normalization of Eisenstein series $E(g) = \sum_\gamma \varphi(\gamma g)$ by requiring $\varphi(1) = 1$. This produces a normalizing denominator which (as in simple cases of Langlands-Shahidi) is a product of convolution L-functions.

where N is the unipotent radical of the standard minimal parabolic in GL_{r-1}.

[1.10] Non-archimedean local factors

In terms of s and F, the non-archimedean local factors are[18]

$$\int_{N_v \backslash H_v} |\det a|^{s+s'} \, W_{f,v}\begin{pmatrix} a & \\ & 1 \end{pmatrix} W_{F,v}(a)\, da = \frac{L_v(s+s'+\frac{1}{2}, f \otimes F)}{\langle F,F\rangle^{1/2}} \qquad (\text{for } \mathrm{Re}(s+s') \gg 0)$$

The second Euler product is the complex conjugate of this, but lacking the shift by s', namely, the *complex conjugate* of

$$\int_{N_v \backslash H_v} |\det a'|^{s} \, W_{f,v}\begin{pmatrix} a' & \\ & 1 \end{pmatrix} W_{F,v}(a')\, da' = \frac{L_v(s+\frac{1}{2}, f \otimes F)}{\langle F,F\rangle^{1/2}} \qquad (\text{for } \mathrm{Re}(s+s') \gg 0 \text{ and } \mathrm{Re}(s) \gg 0)$$

When $\eta = |\det|^s \otimes F$ is not cuspidal, but, instead, is an Eisenstein series with cuspidal data, it still does generate an irreducible representation of $G_{\mathbb{A}}$. At a place v where η generates a spherical representation, the Euler product expansion of degree $r \cdot (r-1)$ falls apart into smaller factors, *and* has a *denominator* arising from the (natural) normalization of the cuspidal-data Eisenstein series entering. Discussion of these terms and their normalizations is postponed.

[1.11] Replace $\overline{s}$ by $1-s$ on $\mathrm{Re}(s) = 1/2$

The global integrals for the L-functions $L(s'+s+\frac{1}{2}, f \otimes F)$ and $L(s+\frac{1}{2}, f \otimes F)$ only converge for $\mathrm{Re}(s'+s) \gg 0$ and $\mathrm{Re}(s) \gg 0$, so we will need to meromorphically continue. To this end, it is most convenient for the whole integral to be *holomorphic* in s, rather than having both s and $\overline{s}$ appear.

To these ends, first absorb the $1/2$ into s by replacing s by $s + \frac{1}{2}$, so we have

$$L(s'+s, f\otimes F) \cdot \overline{L(s, f \otimes F)}$$

and want to eventually move to the line $\mathrm{Re}(s) = 1/2$. To avoid the anti-holomorphy in the second factor, since $\overline{s} = 1-s$ on the line $\mathrm{Re}(s) = 1/2$, we can rewrite this as

$$L(s'+s, f\otimes F) \cdot L(1-s, \overline{f} \otimes \overline{F}) \qquad (\text{for } \mathrm{Re}(1-s) \gg 0 \text{ and } \mathrm{Re}(s'+s) \gg 0)$$

[1.12] The vertical integral(s)

Keep in mind that we have absorbed a $1/2$ into s, and have replaced $\overline{s}$ by $1-s$. The archimedean part of the whole integral is the function $\Gamma_{\varphi_\infty}(s, s', f, F)$ defined by

$$\Gamma_{\varphi_\infty}(s,s',f,F) =$$

$$\int_{N_\infty \backslash H_\infty} \int_{N_\infty \backslash H_\infty} \mathcal{K}(h,m)\, |\det a|^{s'+s-\frac{1}{2}} \, |\det a'|^{\frac{1}{2}-s} \left(W_{f,\infty}\begin{pmatrix} a & \\ & 1 \end{pmatrix} W_{F,\infty}(a) \right)$$

$$\times \quad \left(\overline{W}_{f,\infty}\begin{pmatrix} a' & \\ & 1 \end{pmatrix} \overline{W}_{F,\infty}(a') \right) da\, da' \qquad \left(\text{with } h = \begin{pmatrix} a & \\ & 1 \end{pmatrix} \text{ and } m = \begin{pmatrix} a' & \\ & 1 \end{pmatrix}\right)$$

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[18] The L-functions $L(s, f \otimes F)$ attached to *cuspforms* f and F by these *zeta integrals* are Euler products with local factor *equal* to that of the L-functions $L(s, \pi_f \times \pi_F)$ attached to the corresponding (irreducible cuspidal automorphic) representations at all finite primes at which f and F are spherical. At other finite places the local factors of $L(s, f \otimes F)$ may be polynomial (in q_v^{-s} and q_v^{s}) multiples of the corresponding local factor of $L(s, \pi_f \times \pi_F)$. At archimedean places, $L_v(s, f\otimes F)/L_v(s, \pi_f \times \pi_F)$ is entire, etc. Last, but not least, there are global constant factors sometimes denoted $\rho_f(1)$ which for newforms f for $GL(2)$ are the ratios of the first Fourier coefficient of f to the Petersson norm squared of f. See [Hoffstein-Lockhart 1994], [Bernstein-Reznikoff 1999], and [Sarnak 1985], [Sarnak 1992]. Here the latter global normalizing constant is accommodated by a division by $\langle F,F\rangle^{1/2}$, to make $F/\langle F,F\rangle^{1/2}$ run through an orthonormal basis when F runs through an orthogonal basis.

since $\varphi_\infty(h) = |\det a|^{s'}$. Note that this depends only upon the archimedean data attached to f and F. Thus, so far, the whole is

$$\int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \text{Pé}(g)\, |f(g)|^2 \, dg$$

$$= \sum_{F \text{ on } GL(r-1)} \int_{\text{Re}(s)=\frac{1}{2}} \Gamma_{\varphi_\infty}(s, s', f, F)\, \frac{L(s'+s, f\otimes F)\, L(1-s, \overline{f}\otimes\overline{F})}{\langle F, F\rangle}\, dt$$

$$+ \text{ (continuous part)} \qquad\qquad (\text{with } \text{Re}(s') \gg 0)$$

Again, we want to meromorphically continue to $s' = 0$.

[1.12.1] **Remark:** With or without detailed knowledge of the *residual* part of L^2 (meaning that consisting of square-integrable iterated residues of cuspidal-data Eisenstein series), automorphic forms in the residual spectrum not admitting Whittaker models do not enter in this expansion.

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2. *Spectral expansion: reduction to* GL_2

The Poincaré series admits a spectral expansion in terms of Eisenstein series, cuspforms, and L-functions, preparing for its meromorphic continuation. This section reduces the general spectral expansion to the case $r = 2$.

[2.1] Poisson summation Form the Poincaré series in two stages to allow application of Poisson summation, namely

$$\text{Pé}(g) = \sum_{Z_k H_k \backslash G_k} \varphi(\gamma g) = \sum_{Z_k H_k U_k \backslash G_k} \sum_{\beta \in U_k} \varphi(\beta\gamma g) = \sum_{Z_k H_k U_k \backslash G_k} \sum_{\psi \in (U_k \backslash U_{\mathbb{A}})^\frown} \widehat{\varphi}_{\gamma g}(\psi)$$

where

$$\widehat{\varphi}_g(\psi) = \int_{U_{\mathbb{A}}} \overline{\psi}(u)\, \varphi(ug)\, du \qquad\qquad (\text{for } g \in G_{\mathbb{A}})$$

[2.2] The leading term The inner summand for $\psi = 1$ gives a vector from which an extremely degenerate Eisenstein series[19] for the $(r-1, 1)$ parabolic $P^{r-1,1} = ZHU$ is formed by the outer sum. That is,

$$g \to \int_{U_{\mathbb{A}}} \varphi(ug)\, du$$

is left equivariant by a character on $P^{r-1,1}_{\mathbb{A}}$, and is left invariant by $P^{r-1,1}_k$, namely,

$$\int_{U_{\mathbb{A}}} \varphi(upg)\, du = \int_{U_{\mathbb{A}}} \varphi(p \cdot p^{-1}up \cdot g)\, du = \delta_{P^{r-1,1}}(m) \cdot \int_{U_{\mathbb{A}}} \varphi(m \cdot u \cdot g)\, du$$

$$= \left|\frac{\det A}{d^{r-1}}\right|^{s'+1} \int_{U_{\mathbb{A}}} \varphi(ug)\, du \qquad\qquad \left(\text{where } p = \begin{pmatrix} A & * \\ 0 & d \end{pmatrix},\ A \in GL_{r-1},\ d \in GL_1\right)$$

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[19] These degenerate (spherical) Eisenstein series $E_s^{r-1,1}$ are readily understood via Poisson summation, imitating Riemann *et alia*. There is a simple pole at $s = 1$, with constant residue, and no other poles in $\text{Re}(s) \geq 1/r$. That first pole will be cancelled (when s' goes to 0) by the continuous part of the spectral decomposition.

The normalization[20] is explicated by setting $g = 1$:

$$\int_{U_{\mathbb{A}}} \varphi(u)\, du = \int_{U_\infty} \varphi_\infty \cdot \int_{U_{\rm fin}} \varphi_{\rm fin} = \int_{U_\infty} \varphi_\infty \cdot \text{meas}\,(U_{\rm fin} \cap K_{\rm fin}) = \int_{U_\infty} \varphi_\infty$$

A natural normalization would have been that this value be 1, so the Eisenstein series here implicitly includes the archimedean integral and finite-prime measure constant as factors:

$$\int_{U_\infty} \varphi_\infty \quad \cdot \quad E^{r-1,1}_{s'+1}(g) \;=\; \sum_{\gamma \in P_k^{r-1,1}\backslash G_k} \Big(\int_{U_{\mathbb{A}}} \varphi(u\gamma g)\, du \Big)$$

As advance warning: the pole at $s' = 0$ of this leading term will be *cancelled* by a contribution from the continuous part of the spectral decomposition, below.

[2.3] Main terms: appearance of $\mathfrak{Q}$ from GL_2

The group H_k is transitive on non-trivial characters on $U_k\backslash U_{\mathbb{A}}$. As usual, for fixed non-trivial character ψ_0 on $k\backslash A$, let

$$\psi^\xi(u) = \psi_0(\xi \cdot u_{r-1,r}) \qquad\qquad (\text{for } \xi \in k^\times)$$

The spectral expansion of Pé with the obvious leading term removed, is

$$\sum_{\gamma \in P_k^{r-1,1}\backslash G_k} \;\sum_{\alpha \in P_k^{r-2,1}\backslash H_k} \left(\sum_{\xi \in k^\times} \widehat{\varphi}_{\alpha\gamma g}(\psi^\xi) \right)$$

where $P^{r-2,1}$ is the parabolic subgroup of $H \approx GL_{r-1}$. Let

$$U' \;=\; \{ \begin{pmatrix} 1_{r-2} & & * \\ & 1 & \\ & & 1 \end{pmatrix} \} \qquad U'' \;=\; \{ \begin{pmatrix} 1_{r-2} & & \\ & 1 & * \\ & & 1 \end{pmatrix} \}$$

Let

$$\Theta \;=\; \{ \begin{pmatrix} 1_{r-2} & & \\ & * & * \\ & * & * \end{pmatrix} \}$$

Then the expansion of the Poincaré series with leading term removed is

$$\sum_{\gamma \in P_k^{r-2,1,1}\backslash G_k} \left(\sum_{\xi \in k^\times} \int_{U''_{\mathbb{A}}} \overline{\psi}^\xi(u'') \int_{U'_{\mathbb{A}}} \varphi(u'u''\gamma g)\, du'\, du'' \right)$$

$$= \sum_{\gamma \in P_k^{r-2,2}\backslash G_k} \;\sum_{\alpha \in P^{1,1}\backslash \Theta_k} \left(\sum_{\xi \in k^\times} \int_{U''_{\mathbb{A}}} \overline{\psi}^\xi(u'') \int_{U'_{\mathbb{A}}} \varphi(u'u''\alpha\gamma g)\, du'\, du'' \right)$$

Letting

$$\widetilde{\varphi}(g) = \int_{U'_{\mathbb{A}}} \varphi(u'g)\, du'$$

[20] Recall an integration-theory trick. Given a unimodular topological group G, use a topological group decomposition $G = PK$ with K *compact open* as follows. To integrate a right K-invariant function f on G, with the total mass of K normalized to 1, we have $\int_G f = \int_P f$ with a *left* Haar measure on P normalized so that the measure (in P) of $K \cap P$ is 1.

the expansion becomes

$$\sum_{\gamma \in P_k^{r-2,2}\backslash G_k} \sum_{\alpha \in P^{1,1}\backslash \Theta_k} \sum_{\xi \in k^\times} \int_{U''_{\mathbb{A}}} \overline{\psi}^{\xi}(u'')\, \widetilde{\varphi}(u''\alpha\gamma g)\, du''$$

We claim the equivariance

$$\widetilde{\varphi}(pg) = |\det A|^{s'+1} \cdot |a|^{s'} \cdot |d|^{-(r-1)s'-(r-2)} \cdot \widetilde{\varphi}(g) \qquad \left(\text{for } p = \begin{pmatrix} A & * & * \\ & a & \\ & & d \end{pmatrix} \in G_{\mathbb{A}}, \text{ with } A \in GL_{r-2}\right)$$

This is verified by changing variables in the defining integral: let $x \in \mathbb{A}^{r-1}$ and compute

$$\begin{pmatrix} 1_{r-2} & & x \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} A & b & c \\ & a & \\ & & d \end{pmatrix} = \begin{pmatrix} A & b & c+xd \\ & a & \\ & & d \end{pmatrix} = \begin{pmatrix} A & b & c \\ & a & \\ & & d \end{pmatrix} \begin{pmatrix} 1_{r-2} & & A^{-1}xd \\ & 1 & \\ & & 1 \end{pmatrix}$$

Thus, $|\det A|^{s'} \cdot |a|^{s'} \cdot |d|^{-(r-1)s'}$ comes out of the definition of φ, and another $|\det A| \cdot |d|^{2-r}$ from the change-of-measure in the change of variables replacing x by Ax/d in the integral defining $\widetilde{\varphi}$ from φ. Note that

$$|a|^{s'} \cdot |d|^{-(r-1)s'-(r-2)} = \left|\det \begin{pmatrix} a & \\ & d \end{pmatrix}\right|^{-\frac{(r-2)}{2}\cdot(s'+1)} \cdot |a/d|^{\frac{rs'+(r-2)}{2}}$$

Thus, letting

$$\Phi(g) = \sum_{\alpha \in P_k^{1,1}\backslash \Theta_k} \sum_{\xi \in k^\times} \int_{U''_{\mathbb{A}}} \overline{\psi}^{\xi}(u'')\, \widetilde{\varphi}(u''\alpha g)\, du''$$

we can write

$$\text{Pé}(g) \;-\; \sum_{\gamma \in P_k^{r-1,1}\backslash G_k} \int_{U_{\mathbb{A}}} \varphi(u\gamma g)\, du = \sum_{\gamma \in P_k^{r-2,2}\backslash G_k} \Phi(\gamma g)$$

This is not an Eisenstein series for $P^{r-2,2}$ in the strictest sense. An expression in terms of genuine Eisenstein series is helpful in understanding meromorphic continuations.

Define a GL_2 kernel $\varphi^{(2)}$ for a Poincaré series as follows. We require right invariance by the maximal compact subgroups locally everywhere, and left equivariance

$$\varphi^{(2)}\left(\begin{pmatrix} a & \\ & d \end{pmatrix} \cdot D\right) \;=\; |a/d|^{s} \cdot \varphi^{(2)}(D)$$

Then the the archimedean data $\varphi^{(2)}_\infty$ is completely specified by

$$\varphi^{(2)}_\infty \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \;=\; \widetilde{\varphi} \begin{pmatrix} 1_{r-2} & & \\ & 1 & x \\ & & 1 \end{pmatrix} \qquad (\text{with } \widetilde{\varphi} \text{ as above})$$

Then put

$$\varphi^*(s, \widetilde{\varphi}, D) \;=\; \sum_{\xi \in k^\times} \int_{U_{\mathbb{A}}} \overline{\psi}^{\xi}(u)\, \varphi^{(2)}(s, uD)\, du \qquad (\text{with } U \text{ now the unipotent radical of } P^{1,1} \text{ in } GL_2)$$

The corresponding GL_2 Poincaré series with leading term removed is

$$\mathfrak{Q}(s, D) = \sum_{\alpha \in P_k^{1,1}\backslash GL_2(k)} \varphi^*(s, \alpha D)$$

Thus, for

$$g = \begin{pmatrix} A & * \\ & D \end{pmatrix} \qquad \text{(with } A \in GL_{r-2}(\mathbb{A}) \text{ and } D \in GL_2(\mathbb{A}))$$

the inner integral

$$g \to \int_{U''_{\mathbb{A}}} \overline{\psi}(u'')\, \widetilde{\varphi}(u''g)\, du''$$

is expressible in terms of the kernel φ^* for $\mathfrak{Q}$, namely,

$$\sum_{\xi \in k^\times} \int_{U''_{\mathbb{A}}} \overline{\psi}^\xi(u'')\, \widetilde{\varphi}(u''g)\, du'' \;=\; |\det A|^{s'+1} \cdot |\det D|^{-\frac{(r-2)}{2}\cdot(s'+1)} \cdot \varphi^*\left(\frac{rs'+r-2}{2}, D\right)$$

Thus,

$$\sum_{\alpha \in P_k^{1,1}\backslash \Theta_k} \sum_{\xi \in k^\times} \int_{U''_{\mathbb{A}}} \overline{\psi}^\xi(u'')\, \widetilde{\varphi}(u''\alpha g)\, du'' \;=\; |\det A|^{s'+1} \cdot |\det D|^{-\frac{(r-2)}{2}\cdot(s'+1)} \cdot \mathfrak{Q}\left(\frac{rs'+r-2}{2}, D\right)$$

Thus, to obtain a (not necessarily L^2) spectral decomposition of the Poincaré series Pé (with main term removed) we first determine the (L^2) spectral decomposition of $\mathfrak{Q}$ for $r = 2$, and then form $P^{r-2,2}$ Eisenstein series from the spectral fragments.

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3. *Spectral expansion for* GL_2

The spectral expansion of the Poincaré series for GL_2 is easy, because the Poincaré series (with leading term removed) is essentially a kernel for the linear map $f \to L(s' + \frac{1}{2}, f)$.

[3.1] Recall the set-up Take $r = 2$ and $G = GL_2$, and $\varphi = \bigotimes_v \varphi_v$ with φ_v constructed above. Namely, at a finite prime v

$$\varphi_v(\begin{pmatrix} a & * \\ & d \end{pmatrix} \cdot \theta) = |a/d|_v^{s'} \qquad \text{(for } \theta \in K_v)$$

and at infinite v we have at least the left equivariance

$$\varphi_v(\begin{pmatrix} a & * \\ & d \end{pmatrix} \cdot g) = |a/d|_v^{s'} \cdot \varphi_v(g)$$

Let

$$H = \{\begin{pmatrix} * & \\ & 1 \end{pmatrix}\} \qquad U = \{\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}\} \qquad P = \{\begin{pmatrix} * & * \\ & * \end{pmatrix}\}$$

Form a P_k-invariant function

$$\Phi(g) = \sum_{\xi \in k^\times} \int_{U_k\backslash U_{\mathbb{A}}} \overline{\psi}_0(\xi \cdot u_{1,2})\, \varphi(ug)\, du$$

and the Poincaré series

$$\mathfrak{Q}(g) = \sum_{P_k\backslash G_k} \Phi(\gamma g)$$

For GL_2, the $L^2(G_k\backslash G^1)$ decomposition [21] has a *cuspidal* part, a *continuous* part, and a *residual* part. Since $\mathfrak{Q}$ is right $K_{\mathbb{A}}$-invariant and has trivial central character, all the spectral components will have these attributes.

[21] We grant that suitable parameter choices put Pé (with a leading term, an Eisenstein series, removed) in L^2, allowing such a decomposition, which persists by analytic continuation even when $\mathfrak{Q}$ is not in L^2. See estimates over $\mathbb{Q}$ and $\mathbb{Q}(i)$ for GL_2, but with arbitrary level, in [Zhang 2005] and [Zhang 2006]. The general ideas of the spectral decomposition for GL_2, at least over $\mathbb{Q}$, have been understood for a long time. See [Selberg 1956], [Roelcke 1956], [Godement 1966a], [Godement 1966b].

[3.2] Cuspidal part Computation of $\langle \text{Pé}, F\rangle$ for F a cuspform can be done directly. This pairing can *also* be computed as for the pairing against Eisenstein series below, but the cuspform case allows an illuminating direct computation. Let F be a spherical cuspform[22] with trivial central character, and Fourier-Whittaker expansion

$$F(g) = \sum_{\gamma \in H_k} W_F(\gamma g) \qquad\qquad (\text{where } W_F(g) = \textstyle\int_{U_k\backslash U_{\mathbb{A}}} \overline{\psi}(u)\, F(ug)\, du)$$

Unwinding the Poincaré series gives[23]

$$\langle \text{Pé}, F\rangle = \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \text{Pé} \cdot \overline{F} = \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \mathfrak{Q} \cdot \overline{F} = \int_{Z_{\mathbb{A}} H_k \backslash G_{\mathbb{A}}} \varphi \cdot \overline{F}$$

Since F is a cuspform on GL_2, its Fourier-Whittaker expansion is $F(g) = \sum_{\gamma \in H_k} W_F(\gamma g)$. Substituting the Fourier expansion of F into the integral unwinds further to

$$\int_{Z_{\mathbb{A}} \backslash G_{\mathbb{A}}} \varphi(g)\, \overline{W}_F(g)\, dg = \prod_{v<\infty} \int_{Z_v \backslash G_v} \varphi_v(g)\, \overline{W}_{F,v}(g)\, dg \cdot \prod_{v|\infty} \int_{Z_v \backslash G_v} \varphi_v(g)\, \overline{W}_{F,v}(g)\, dg$$

At finite primes v, the right K_v-invariance implies (via an Iwasawa decomposition) that the v^{th} integral is

$$\int_{H_v \times U_v} \varphi_v(hu)\, \overline{W}_{F,v}(hu)\, dh\, du = \int_{H_v} \varphi_v(h)\, \overline{W}_{F,v}(h)\, dh = \int_{k_v^\times} |a|_v^{s'}\, \overline{W}_{F,v} \begin{pmatrix} a & \\ & 1 \end{pmatrix} da$$

since φ_v is supported on $H_v K_v$. We can adjust F by a scalar (without changing $F/\langle F, F\rangle^{1/2}$ so that the non-archimedean Whittaker functions for the cuspform F give non-archimedean integrals over H_v which are exactly the local L-factors $L_v(s' + \frac{1}{2}, \overline{F})$. The archimedean integrals are not the usual gamma factors, but still do depend only upon the local data of F. Thus,

$$\langle \text{Pé}, F\rangle = \Big(\int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot W_{\overline{F},\infty}\Big) \cdot L(s' + \tfrac{1}{2}, \overline{F}) \qquad\qquad (F \text{ a suitable cuspform})$$

[3.3] Poisson summation To isolate the *leading term* of the Poincaré series, as well as to determine the continuous part of the spectral decomposition, recall the rewritten form of the Poincaré series via Poisson summation:

$$\text{Pé}(g) = \sum_{\gamma \in Z_k H_k \backslash G_k} \varphi(\gamma g) = \sum_{\gamma \in Z_k H_k U_k \backslash G_k} \sum_{\beta \in U_k} \varphi(\beta\gamma g) = \sum_{\gamma \in Z_k H_k U_k \backslash G_k} \sum_{\psi \in (U_k \backslash U_{\mathbb{A}})^\frown} \widehat{\varphi}_{\gamma g}(\psi)$$

where

$$\widehat{\varphi}_g(\psi) = \int_{U_{\mathbb{A}}} \overline{\psi}(u)\, \varphi(ug)\, du \qquad\qquad (\text{for } g \in G_{\mathbb{A}})$$

[3.4] The leading term The inner summand for trivial ψ gives a vector from which an Eisenstein series is formed by the outer sum. That is,

$$g \to \int_{U_{\mathbb{A}}} \varphi(ug)\, du$$

[22] As earlier, for us a *cuspform* generates an irreducible unitary representation of $G_{\mathbb{A}}$.

[23] In this context, the pairing $\langle , \rangle$ can be taken to be an integral over $Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}$, since the integrand has trivial central character.

is left equivariant by a character on $P_{\mathbb{A}}$, and is left invariant by P_k, namely,

$$\int_{U_{\mathbb{A}}} \varphi(upg)\,du = \int_{U_{\mathbb{A}}} \varphi(p\cdot p^{-1}up\cdot g)\,du = \delta_P(m)\cdot\int_{U_{\mathbb{A}}} \varphi(m\cdot u\cdot g)\,du$$

$$= |a/d|^{s'+1}\int_{U_{\mathbb{A}}} \varphi(ug)\,du \qquad (\text{where } p = \begin{pmatrix} a & * \\ 0 & d\end{pmatrix})$$

The normalization is understood by setting $g = 1$:

$$\int_{U_{\mathbb{A}}} \varphi(u)\,du = \int_{U_\infty} \varphi_\infty \cdot \int_{U_{\text{fin}}} \varphi_{\text{fin}} = \int_{U_\infty} \varphi_\infty$$

The natural normalization would have been that this value be 1, so this Eisenstein series includes the archimedean integral and finite-prime measure constant as factors:

$$(\text{leading term of Pé}) \;=\; \Big(\int_{U_\infty} \varphi_\infty\Big) \;\cdot\; E_{s'+1}$$

[3.5] Continuous part

First, let

$$\mathfrak{Q} = \text{Pé} - (\text{leading term}) = \text{Pé} - (\text{factor})\cdot E_{s'+1} = \sum_{\gamma\in Z_k H_k U_k\backslash G_k}\;\sum_{\psi\neq 1} \widehat{\varphi}_{\gamma g}(\psi)$$

be the Poincaré series with its leading term removed. Let

$$\kappa = \text{meas}\,(\mathbb{J}^1/k^\times)$$

The residue of the zeta function of k at $z = 1$ is

$$\text{Res}_{z=1}\,\zeta_k(z) \;=\; \frac{\kappa}{|D_k|^{1/2}} \qquad (\text{where } D_k \text{ is the discriminant of } k)$$

The continuous part of the Poincaré series is [24]

$$\frac{1}{4\pi i\kappa}\sum_\chi \int_{\text{Re}\,(s)=\frac{1}{2}} \langle \mathfrak{Q}, E_{s,\chi}\rangle \cdot E_{s,\chi}\;ds$$

Computation of the pairing of $\mathfrak{Q}$ against Eisenstein series is best approached indirectly, unlike the pairings against cuspforms. [25] For GL_2, the simple transitivity of H_k on non-trivial characters allows $\mathfrak{Q}$ to be rewritten, with any fixed choice of non-trivial ψ, as

$$\mathfrak{Q}(g) = \sum_{\gamma\in Z_k H_k U_k\backslash G_k}\;\sum_{\beta\in H_k} \widehat{\varphi}_{\beta\gamma g}(\psi) = \sum_{\gamma\in Z_k U_k\backslash G_k} \widehat{\varphi}_{\gamma g}(\psi)$$

[24] This is the usual spectral inversion formula, as in many sources. For example, see [Godement 1966a] for ground field $\mathbb{Q}$. The volume constant κ necessary in the general case is inconspicuous when $k = \mathbb{Q}$, because in that case its value is 1. The $4\pi i$ is really $2\cdot 2\pi i$, where the extra factor of 2 reflects the fact that we indicate an integral over the entire vertical line, rather than half.

[25] To evaluate the integral of the Poinaré series (with leading term removed) against an Eisenstein series, the computation here is more efficient than the approach which would unwind the Eisenstein series and then compute a Mellin transform of the constant term of the Poincaré series.

Integrate $\mathfrak{Q}$ against an Eisenstein series[26] $E_{s,\chi}$ with χ an unramified Hecke character χ.

$$\langle \mathfrak{Q}, E_{s,\chi} \rangle = \int_{Z_\mathbb{A} G_k \backslash G_\mathbb{A}} \mathfrak{Q}(g)\, \overline{E}_{s,\chi}(g)\, dg$$

In the latter integral, unwind $\mathfrak{Q}$ to obtain

$$\int_{Z_\mathbb{A} U_k \backslash G_\mathbb{A}} \widehat{\varphi}_g(\psi)\, \overline{E}_{s,\chi}(g)\, dg = \int_{Z_\mathbb{A} U_k \backslash G_\mathbb{A}} \int_{U_\mathbb{A}} \overline{\psi}(u)\, \varphi(ug)\, \overline{E}_{s,\chi}(g)\, du\, dg$$

$$= \int_{Z_\mathbb{A} U_\mathbb{A} \backslash G_\mathbb{A}} \int_{U_\mathbb{A}} \overline{\psi}(u)\, \varphi(ug)\, \overline{W}^E_{s,\chi}(g)\, du\, dg = \int_{Z_\mathbb{A} U_\mathbb{A} \backslash G_\mathbb{A}} \int_{U_\mathbb{A}} \varphi(ug)\, \overline{W}^E_{s,\chi}(ug)\, du\, dg$$

$$= \int_{Z_\mathbb{A} \backslash G_\mathbb{A}} \varphi(g)\, \overline{W}^E_{s,\chi}(g)\, dg \qquad \left(\text{where } W^E_{s,\chi}(g) = \int_{U_k \backslash U_\mathbb{A}} \overline{\psi}(u)\, E_{s,\chi}(ug)\, du\right)$$

The Whittaker function of the Eisenstein series does factor over primes, into local factors depending only upon the local data at v

$$W^E_{s,\chi} = \bigotimes_v W^E_{s,\chi,v}$$

Thus,

$$\langle \mathfrak{Q}, E_{s,\chi} \rangle = \prod_v \int_{Z_v \backslash G_v} \varphi_v(g)\, \overline{W}^E_{s,\chi,v}(g)\, dg$$

At finite v, using an Iwasawa decomposition and the vanishing of φ_v off $H_v U_v$, as in the integration against cuspforms, the local factor is

$$\int_{k_v^\times} |a|_v^{s'}\, \overline{W}^E_{s,\chi,v} \begin{pmatrix} a & \\ & 1 \end{pmatrix} da$$

However, for Eisenstein series, the natural normalization of the Whittaker functions differs from that used for cuspforms, instead presenting the local Whittaker functions as images under intertwining operators. Specifically, define the *normalized spherical vector* for data s, χ_v

$$\varphi_v^\circ(p\theta) = |a/d|_v^s \cdot \chi_v(a/d) \qquad \left(\text{for } p = \begin{pmatrix} a & * \\ & d \end{pmatrix} \text{ in } P_v \text{ and } \theta \text{ in } K_v\right)$$

The corresponding (spherical) local Whittaker function for Eisenstein series is the integral[27]

$$W^E_{s,\chi,v}(g) = \int_{U_v} \overline{\psi}(u)\, \varphi_v^\circ(w_\circ ug)\, du$$

where $w_\circ$ is the longest Weyl element. The Mellin transform of the Eisenstein-series normalization W_v^E compares to the Mellin transform of the usual normalization W_v° of the Whittaker function (with the same data s, χ_v) as follows. Let $\mathfrak{o}_v^*$ be the local inverse different at v, and let $\mathfrak{d}_v \in k_v^\times$ be such that

$$(\mathfrak{o}_v^*)^{-1} = \mathfrak{d}_v \cdot \mathfrak{o}_v$$

[26] As with the pairing against cuspforms, because the integrand has trivial central character, we can take the pairing integral to be an integral over $Z_\mathbb{A} G_k \backslash G_\mathbb{A}$.

[27] This integral only converges nicely for $\mathrm{Re}(s) \gg 0$, but admits a meromorphic continuation in s by various means. For example, the relatively elementary algebraic form of Bernstein's *continuation principle* applies, since the dimension of the space of intertwining operators from the principal series to the Whittaker space is one-dimensional.

Let $\mathfrak{d}$ be the idele with v^{th} component $\mathfrak{d}_v$ at finite places v and component 1 at archimedean places. Let $\nu(x) = |x|$. Then for finite v the v^{th} local integral is [28]

$$\int_{k_v^\times} |a|_v^{s'}\, \overline{W}^E_{s,\chi,v} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} da = |\mathfrak{d}|^{1/2} \cdot \frac{L_v(s'+\frac{1}{2}, \nu^{\overline{s}}\overline{\chi}_v) \cdot L_v(s'+\frac{1}{2}, \nu^{1-\overline{s}}\chi_v)}{\overline{L}_v(2s, \chi_v^2)} \cdot |\mathfrak{d}_v|^{-(s'+1-\overline{s})}\, \overline{\chi}(\mathfrak{d}_v)$$

$$= |\mathfrak{d}|^{1/2} \cdot \frac{L_v(s'+\overline{s}, \overline{\chi}_v) \cdot L_v(s'+1-\overline{s}, \chi_v)}{\overline{L}_v(2s, \chi_v^2)} \cdot |\mathfrak{d}_v|^{-(s'+1-\overline{s})}\, \chi(\mathfrak{d}_v)$$

Then

$$\langle \mathfrak{Q}, E_{s,\chi} \rangle \;=\; \Big(\int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot \overline{W}^E_{s,\chi,\infty} \Big) \cdot |\mathfrak{d}|^{1/2} \cdot \frac{L(s'+\overline{s}, \overline{\chi}) \cdot L(s'+1-\overline{s}, \chi)}{\overline{L}(2s, \chi^2)} \cdot |\mathfrak{d}|^{-(s'+1-\overline{s})}\, \chi(\mathfrak{d})$$

[3.6] Maintaining holomorphy The integral of $\langle \mathfrak{Q}, E_{s,\chi}\rangle \cdot E_s$ along the vertical line $\mathrm{Re}(s) = 1/2$ is anti-holomorphic in the first s and holomorphic in the second s, inconvenient when we want to move the line of integration. To maintain holomorphy, as earlier, since $\overline{s} = 1 - s$ on $\mathrm{Re}(s) = \frac{1}{2}$, we rewrite

$$\langle \mathfrak{Q}, E_{s,\chi} \rangle = \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \text{Pé} \cdot E_{1-s,\overline{\chi}}$$

The latter expression is holomorphic in s and has the expected Euler product expansion for $\mathrm{Re}(1-s) \gg 0$. With this adjustment,

$$\langle \mathfrak{Q}, E_{s,\chi} \rangle \;=\; \Big(\int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot W^E_{1-s,\overline{\chi},\infty} \Big) \cdot |\mathfrak{d}|^{1/2} \cdot \frac{L(s'+1-s, \overline{\chi}) \cdot L(s'+s, \chi)}{L(2-2s, \overline{\chi}^2)} \cdot |\mathfrak{d}|^{-(s'+s)}\, \chi(\mathfrak{d})$$

Thus, with the contour at $\mathrm{Re}(s) = \mathrm{Re}(1-s) = 1/2$ and with $\mathrm{Re}(s') \gg 0$,

$$\text{Pé} \;=\; \int_{U_\infty} \varphi_\infty \cdot E_{s'+1} + \sum_F \Big(\int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot W_{\overline{F},\infty} \Big) \cdot L(s'+\tfrac{1}{2}, \overline{F}) \cdot \frac{F}{\langle F, F \rangle}$$

$$+ \sum_\chi \frac{\chi(\mathfrak{d})}{4\pi i \kappa} \int_{\mathrm{Re}(s)=\frac{1}{2}} \Big(\int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot W^E_{1-s,\overline{\chi},\infty} \Big) \frac{L(s'+1-s, \overline{\chi}) \cdot L(s'+s, \chi)}{L(2-2s, \overline{\chi}^2)} \cdot |\mathfrak{d}|^{-(s'+s-1/2)} \cdot E_{s,\chi}\, ds$$

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4. *Continuation and holomorphy at $s' = 0$ for GL_2*

The meromorphic continuation and non-obvious cancellation [29] in the spectral decomposition of the Poincaré series for GL_2 are reviewed in detail here, over an arbitrary number field, proving that the spectral expression for the Poincaré series is holomorphic at $s' = 0$.

To obtain the desired expression for integral moments, we must set $s' = 0$, which on the spectral side requires that the meromorphically continued Poincaré series be *holomorphic* at $s' = 0$. The (obvious) leading Eisenstein series $E_{s'+1}$ (without the archimedean factor) has a *pole* at $s'+1 = 1$, that is, at $s' = 0$, but we will see now that this is cancelled by part of the *continuous* spectrum of the Poincaré series. This cancellation is independent of the choice of archimedean data φ_∞.

[28] This elementary comparison is reproduced in an appendix.

[29] This cancellation argument was given in detail over $\mathbb{Q}$ in [Diaconu-Goldfeld 2006a].

[4.1] The leading term From above, the leading term Eisenstein series, including the extra leading archimedean factor (and measure constant from finite primes) is

$$\Big(\int_{U_\infty} \varphi_\infty\Big) \cdot E_{s'+1}(g) = \sum_{\gamma \in P_k\backslash G_k} \int_{U_\mathbb{A}} \varphi(u\gamma g)\, du$$

[4.2] Two residues of the continuous part The only fragment of the continuous part of the spectral decomposition of $\mathfrak{Q}$ with poles to the right of the line $\mathrm{Re}(s) = 1/2$ is that with *trivial* Hecke character χ. Replacing (as above) $\overline{s}$ by $1 - s$ to maintain holomorphy in the integral [30] for $\langle \mathfrak{Q}, E_s\rangle$, the term with relevant poles is [31]

$$\frac{1}{4\pi i\kappa} \int_{\mathrm{Re}(s)=\frac{1}{2}} \langle \mathfrak{Q}, E_s\rangle \cdot E_s\, ds$$

$$= \frac{1}{4\pi i\kappa} \int_{\mathrm{Re}(s)=\frac{1}{2}} \Big(\int_{Z_\infty\backslash G_\infty} \varphi_\infty \cdot W^E_{1-s,\overline{\chi},\infty}\Big) \frac{\zeta_k(s'+1-s)\cdot \zeta_k(s'+s)}{\zeta_k(2-2s)} |\mathfrak{d}|^{-(s'+s-1/2)}\, E_s\, ds$$

Aiming to analytically continue to $s' = 0$, in the integral first take $\mathrm{Re}(s') = 1/2 + \varepsilon$, and move the contour from $\mathrm{Re}(s) = 1/2$ to $\mathrm{Re}(s) = 1/2 - 2\varepsilon$. This picks up the residue of the integrand due to the pole of $\zeta_k(s'+s)$ at $s' + s = 1$, that is, at $s = 1 - s'$. This contributes

$$\frac{1}{4\pi i\kappa} \cdot 2\pi i \cdot \left(\int_{Z_\infty\backslash G_\infty} \varphi_\infty \cdot W^E_{s',\infty}\right) \cdot \frac{\zeta_k(2s')\cdot \mathrm{Res}_{z=1}\zeta_k(z)}{\zeta_k(2s')} |\mathfrak{d}|^{-1/2} \cdot E_{1-s'}$$

$$= \frac{1}{2} \cdot \left(\int_{Z_\infty\backslash G_\infty} \varphi_\infty \cdot W^E_{s',\infty}\right) \cdot |\mathfrak{d}|^{1/2} \cdot |\mathfrak{d}|^{-1/2} \cdot E_{1-s'} = \frac{1}{2} \cdot \left(\int_{Z_\infty\backslash G_\infty} \varphi_\infty \cdot W^E_{s',\infty}\right) \cdot E_{1-s'}$$

The vertical integral on $\mathrm{Re}(s) = 1/2 - 2\varepsilon$ is holomorphic in s' in (at least) the strip

$$\tfrac{1}{2} - \varepsilon \le \mathrm{Re}(s') \le \tfrac{1}{2} + \varepsilon \qquad\qquad (\text{while } \mathrm{Re}(s) = \tfrac{1}{2} - 2\varepsilon)$$

Move s' to $\mathrm{Re}(s') = 1/2 - \varepsilon$, and then move the vertical integral from the contour $\sigma = 1/2 - 2\varepsilon$ back to the contour $\sigma = 1/2$. This picks up (-1) times the residue at the pole of $\zeta_k(s' + 1 - s)$ at 1, that is, at $s = s'$, with another sign due to the sign on s inside this zeta function. Thus, we pick up the residue, namely

$$\frac{1}{4\pi i\kappa} \cdot 2\pi i \cdot \left(\int_{Z_\infty\backslash G_\infty} \varphi_\infty \cdot W_{1-s',\infty}\right) \cdot \frac{\mathrm{Res}_{z=1}\zeta_k(z)\cdot \zeta_k(2s')}{\zeta_k(2-2s')} \cdot |\mathfrak{d}|^{-2s'+\frac{1}{2}} \cdot E_{1-(1-s')}$$

$$= \frac{1}{2} \cdot \left(\int_{Z_\infty\backslash G_\infty} \varphi_\infty \cdot W_{1-s',\infty}\right) \cdot \frac{\zeta_k(2s')}{\zeta_k(2-2s')} \cdot |\mathfrak{d}|^{-2s'+1} \cdot E_{s'}$$

[4.3] Use of functional equation of E_s The latter expression can be simplified by using the functional equation of $E_{s'}$. Specifically, letting ζ_∞ be the gamma factor for ζ_k,

$$\zeta_\infty(2s')\cdot \zeta_k(2s')\cdot E_{s'} = |\mathfrak{d}|^{2s'-1} \cdot \zeta_\infty(2-2s')\cdot \zeta_k(2-2s')\cdot E_{1-s'}$$

[30] The pairing $\langle,\rangle$ can be taken to be an integration over $Z_\mathbb{A} G_k\backslash G_\mathbb{A}$, since the central character is trivial.

[31] Again, κ is the natural volume of $\mathbb{J}^1/k^\times$, and is invisible in the case of $k = \mathbb{Q}$ because in that case $\kappa = 1$. And $\kappa/D_k^{1/2}$ is the residue of ζ_k at 1, again invisible for groundfield $\mathbb{Q}$.

In the situation at hand, this gives

$$\frac{\zeta_k(2s')}{\zeta_k(2-2s')} \cdot E_{s'} = |\mathfrak{d}|^{2s'-1} \cdot \frac{\zeta_\infty(2-2s')}{\zeta_\infty(2s')} \cdot E_{1-s'}$$

Thus, the powers of $|\mathfrak{d}|$ cancel, and the second residue can be rewritten as

$$\frac{1}{2} \cdot \left(\int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot W_{1-s',\infty} \right) \cdot \frac{\zeta_\infty(2-2s')}{\zeta_\infty(2s')} \cdot E_{1-s'}$$

These two contributions from the continuous part are

$$\tfrac{1}{2} \cdot \left(\int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot \left[W_{s',\infty} + \frac{\zeta_\infty(2-2s')}{\zeta_\infty(2s')} \cdot W_{1-s',\infty} \right] \right) \cdot E_{1-s'}$$

[4.4] Use of functional equation of archimedean Whittaker functions The archimedean-place (and finite-prime) Whittaker functions are normalized by presenting the Whittaker functions by the usual intertwining integral (and analytically continuing), as follows. Let

$$\eta_s(um\theta) = |a/d|_v^s \qquad \text{(for } u \in U,\ m = \begin{pmatrix} a & \\ & d \end{pmatrix},\ \theta \in K_v)$$

The normalization of the Whittaker function is

$$W^E_{s,v}(g) = \int_{U_v} \overline{\psi}(u)\, \eta_s(w_\circ ug)\, du \qquad \text{(for } \mathrm{Re}(s) \gg 0, \text{ fixed non-trivial } \psi)$$

Further, for v archimedean

$$W^E_{s,v} \begin{pmatrix} a & \\ & 1 \end{pmatrix} = \int_{k_v} \overline{\psi}_0(x) \left| \frac{a}{aa^\iota + xx^\iota} \right|_v^s dx = |a|_v^{1-s} \int_{k_v} \overline{\psi}_0(ax) \frac{1}{|1+xx^\iota|_v^s} dx$$

by replacing x by ax, where ι is complex conjugation for $v \approx \mathbb{C}$ and is the identity map for $v \approx \mathbb{R}$. The usual computation shows that

$$W^E_{s,\mathbb{R}} \begin{pmatrix} a & \\ & 1 \end{pmatrix} = \frac{|a|^{1/2}}{\pi^{-s}\Gamma(s)} \int_0^\infty e^{-\pi(t+\frac{1}{t})|a|}\, t^{s-\frac{1}{2}} \frac{dt}{t} = \frac{1}{\zeta_\mathbb{R}(2s)} \times (\text{invariant under } s \leftrightarrow 1-s)$$

and, similarly,[32]

$$W^E_{s,\mathbb{C}} \begin{pmatrix} a & \\ & 1 \end{pmatrix} = \frac{|a|}{(2\pi)^{-2s}\Gamma(2s)} \int_0^\infty e^{-2\pi(t+\frac{1}{t})|a|}\, t^{2s-1} \frac{dt}{t} = \frac{1}{\zeta_\mathbb{C}(2s)} \times (\text{invariant under } s \leftrightarrow 1-s)$$

Thus, in both cases,

$$\frac{\zeta_v(2-2s)}{\zeta_v(2s)} \cdot W^E_{1-s,v} \begin{pmatrix} a & \\ & 1 \end{pmatrix} = W^E_{s,v} \begin{pmatrix} a & \\ & 1 \end{pmatrix} \qquad \text{(for } v \text{ archimedean)}$$

[32] In the displayed identity for $v \approx \mathbb{C}$, an unadorned absolute value is *not* $|x|_\mathbb{C} = xx^\iota$, but, rather, the standard $|x| = \sqrt{xx^\iota}$. The appropriate measure is *double* the usual.

and

$$W^E_{1-s} = \frac{\zeta_\infty(2s)}{\zeta_\infty(2-2s)} \cdot W^E_s$$

Thus, in fact, the second of the two residues from the continuous part is identical to the first. Thus, these two residues combine to give

$$\Big(\int_{Z_\infty\backslash G_\infty} \varphi_\infty \cdot W_{s',\infty}\Big) \cdot E_{1-s'}$$

Thus, to prove that the leading term's pole at $s' = 0$ is cancelled by the poles of these residues at $s' = 0$, we must show that

$$\Big(\int_{U_\infty} \varphi_\infty\Big) \cdot E_{1+s'} + \Big(\int_{Z_\infty\backslash G_\infty} \varphi_\infty \cdot W_{s',\infty}\Big) \cdot E_{1-s'} \;=\; (\text{holomorphic at } s' = 0)$$

[4.5] Invocation of Fourier inversion Thus, to prove cancellation of poles at $s' = 0$ it suffices to prove a *local* fact, namely, that

$$\int_{U_v} \varphi_v = \Big(\int_{Z_v\backslash G_v} \varphi_v \cdot W^E_{s',v}\Big)\Big|_{s'=0} \qquad (\text{for all archimedean } v)$$

The Whittaker function $W^E_{s,v}$ can be presented in terms of a Fourier transform

$$W^E_{s,v}\begin{pmatrix} a & \\ & 1\end{pmatrix} = |a|^{1-s} \cdot \widehat{\Psi}_{s,v}(a) \qquad (\text{where } \Psi(x) = \Psi_s(x) = |1 + xx^\iota|_v^{-s},\ v \text{ archimedean})$$

Let

$$\Phi(x) = \varphi_v \begin{pmatrix} 1 & x \\ & 1\end{pmatrix}$$

Using the right K_v-invariance and an Iwasawa decomposition, and noting that Φ and Ψ are *even*,

$$\int_{Z_v\backslash G_v} \varphi_v \cdot W^E_{s',v} = \int_{k_v^\times}\int_{k_v} |a|^{s'}\, \Phi(x) \cdot |a|^{1-s'}\, \widehat{\Psi}(a)\, \psi(ax)\, dx\, da$$

$$= \int_{k_v^\times} \widehat{\Phi}(a)\, \widehat{\Psi}(a)\, |a|\, da = \int_{k_v^\times} \Phi(a)\, \Psi(a)\, |a|\, da$$

by Fourier inversion, since $|a|da$ is an *additive* Haar measure. Writing the latter expression out, it is

$$\int_{k_v} \varphi_v \begin{pmatrix} 1 & a \\ & 1\end{pmatrix} \cdot \frac{1}{|1+aa^\iota|_v^{s'}}\, |a|\, da$$

The equality

$$\int_{Z_v\backslash G_v} \varphi_v \cdot W^E_{s',v} = \int_{k_v^\times} \varphi_v \begin{pmatrix} 1 & a \\ & 1\end{pmatrix} \cdot \frac{1}{|1+aa^\iota|_v^{s'}}\, |a|\, da$$

holds at first only for $\mathrm{Re}(s') \gg 0$, but then extends by analytic continuation for φ_v sufficiently integrable on N_v. Thus, we can evaluate the latter integral at $s' = 0$, obtaining

$$\int_{Z_v\backslash G_v} \varphi_v \cdot W^E_{0,v} = \int_{k_v^\times} \varphi_v \begin{pmatrix} 1 & a \\ & 1\end{pmatrix} |a|\, da$$

At archimedean places v, the normalization $|a|\,da$ for an additive Haar measure, in terms of a multiplicative Haar measure da, is correct. This proves that, indeed,

$$\Big(\int_{Z_v\backslash G_v} \varphi_v \cdot W^E_{s',v}\Big)\Big|_{s'=0} \;=\; \int_{U_v} \varphi_v \qquad \text{(for all archimedean } v)$$

which yields the desired cancellation.

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5. *Spectral expansion for GL_r*

The spectral decomposition of the Poincaré series for $r = 2$ yields that for $r > 2$ by inducing. For $r > 2$, nothing remains after all the non-L^2 terms are removed. The non-L^2 terms are induced from the genuinely L^2 spectral expansion of the Poincaré series on GL_2.

Before carrying out the spectral expansion for $r = 2$, we had found that

$$\text{Pé}(g) \;=\; \Big(\int_{U_\infty}\varphi_\infty\Big)\cdot E^{r+1,1}_{s'+1}(g) \;+\; \sum_{\gamma\in P_k^{r-2,2}\backslash G_k} \Phi(\gamma g)$$

where

$$\Phi\begin{pmatrix} A & * \\ & D\end{pmatrix} = |\det A|^{s'+1}\cdot|\det D|^{-(r-2)\cdot\frac{s'+1}{2}}\cdot \mathfrak{Q}(\frac{rs'+r-2}{2},\widetilde{\varphi},D) \qquad \text{(with } A\in GL_{r-2} \text{ and } D\in GL_2)$$

with $\mathfrak{Q}$ from GL_2, and

$$\widetilde{\varphi}(g) = \int_{U'_k\backslash U'_{\mathbb{A}}} \varphi(u'g)\,du'$$

Thus, formation of Pé (with its leading term removed) amounts to forming an Eisenstein series from Φ, with analytical properties explicated by expressing Φ as a superposition of vectors generating irreducibles.

[5.1] Decomposition into irreducibles

From the decomposition of $\mathfrak{Q}$ on GL_2 for $\mathrm{Re}(s')\gg 0$

$$\Phi\begin{pmatrix} A & * \\ 0 & D\end{pmatrix} \;=\; |\det A|^{2\cdot\frac{s'+1}{2}}\cdot|\det D|^{-(r-2)\cdot\frac{s'+1}{2}}\cdot\mathfrak{Q}(\frac{rs'+r-2}{2},\widetilde{\varphi},D)$$

$$= |\det A|^{2\cdot\frac{s'+1}{2}}\cdot|\det D|^{-(r-2)\cdot\frac{s'+1}{2}}\sum_F\Big(\int_{PGL_2(k_\infty)}\widetilde{\varphi}_\infty\cdot W_{\overline{F},\infty}\Big)\cdot L(\frac{rs'+r-2}{2}+\tfrac12,\overline{F})\cdot\frac{F}{\langle F,F\rangle}$$

$$+\;|\det A|^{2\cdot\frac{s'+1}{2}}\cdot|\det D|^{-(r-2)\cdot\frac{s'+1}{2}}\sum_\chi\frac{\chi(\mathfrak{d})}{4\pi i\kappa}$$

$$\int_{\mathrm{Re}(s)=\frac12}\Big(\int_{PGL_2(k_\infty)}\widetilde{\varphi}_\infty\cdot W^E_{1-s,\overline{\chi},\infty}\Big)\cdot\frac{L(\frac{rs'+r-2}{2}+1-s,\overline{\chi})\cdot L(\frac{rs'+r-2}{2}+s,\chi)}{L(2-2s,\overline{\chi}^2)}\cdot|\mathfrak{d}|^{-(\frac{rs'+r-2}{2}+s-1/2)}\cdot E_{s,\chi}(D)\,ds$$

[5.2] Least continuous part of Poincaré series

Let

$$\Phi_{\frac{s'+1}{2},F}(\begin{pmatrix} A & * \\ 0 & D\end{pmatrix}\cdot\theta) = |\det A|^{2\cdot\frac{s'+1}{2}}\cdot|\det D|^{-(r-2)\cdot\frac{s'+1}{2}}\,F(D) \qquad (\text{for } \theta\in K_{\mathbb{A}})$$

and define a half-degenerate Eisenstein series[33]

$$E^{r-2,2}_{\frac{s'+1}{2},F}(g) = \sum_{\gamma \in P_k^{r-2,2}\backslash G_k} \Phi_{\frac{s'+1}{2},F}(\gamma g)$$

Then the *most-cuspidal* (or *least continuous*) part of the Poincaré series is

$$\sum_F \left(\int_{PGL_2(k_\infty)} \widetilde{\varphi} \cdot W_{\overline{F},\infty} \right) \cdot \frac{L(\frac{rs'+r-2}{2} + \frac{1}{2}, \overline{F})}{\langle F, F \rangle} \cdot E^{r-2,2}_{\frac{s'+1}{2},F} \qquad \text{(cuspforms } F \text{ on } GL_2)$$

The *summands* of this expression have relatively well understood meromorphic continuations. As discussed in an appendix, the half-degenerate Eisenstein series $E^{r-2,2}_{s,F}$ has *no poles* in $\mathrm{Re}(s) \geq 1/2$. With $s = (s'+1)/2$ this assures absence of poles in $\mathrm{Re}(s') \geq 0$.

[5.3] Continuous part of the Poincare series

The Eisenstein series integral part of $\mathfrak{Q}$ on GL_2 gives degenerate Eisenstein series attached to the $(r-2,1,1)$-parabolic in GL_r. This arises from a similar consideration as for GL_2 *cuspforms*, but for GL_2 *Eisenstein series*, as follows.

Let $E_{s,\chi}$ be the usual Eisenstein series for GL_2, and let

$$\Phi_{\frac{s'+1}{2},E_{s,\chi}}(\begin{pmatrix} A & * \\ 0 & D \end{pmatrix} \cdot \theta) = |\det A|^{2\cdot\frac{s'+1}{2}} \cdot |\det D|^{-(r-2)\cdot\frac{s'+1}{2}} E_{s,\chi}(D) \qquad (\text{for } \theta \in K_{\mathbb{A}})$$

and define an Eisenstein series

$$E^{r-2,2}_{\frac{s'+1}{2},E_{s,\chi}}(g) = \sum_{\gamma \in P_k^{r-2,2}\backslash G_k} \Phi_{\frac{s'+1}{2},E_{s,\chi}}(\gamma g)$$

For given $s \in \mathbb{C}$, an easy variant of Godement's criterion[34] proves convergence for sufficiently large $\mathrm{Re}(s')$.

Then, ignoring the issue of interchange of sums and integrals, the Poincaré series has *most continuous* part

$$\sum_\chi \frac{\chi(\mathfrak{d})}{4\pi i\kappa} \int_{\mathrm{Re}(s)=\frac{1}{2}} \left(\int_{PGL_2(k_\infty)} \widetilde{\varphi}_\infty \cdot W^E_{1-s,\overline{\chi},\infty} \right)$$
$$\times \frac{L(\frac{rs'+r-2}{2} + 1 - s, \overline{\chi}) \cdot L(\frac{rs'+r-2}{2} + s, \chi)}{L(2-2s, \overline{\chi}^2)} \cdot |\mathfrak{d}|^{-(\frac{rs'+r-2}{2}+s-1/2)} \cdot E^{r-2,2}_{\frac{s'+1}{2},E_{s,\chi}} \, ds$$

That is, it is the *analytically continued* E_s on the line $\mathrm{Re}(s) = \frac{1}{2}$ that enters. However, as usual, for $\mathrm{Re}(s) \gg 0$ and $\mathrm{Re}(s') \gg 0$ this iterated formation of Eisenstein series is equal to a single-step Eisenstein series. The equality persists after analytic continuation.

Thus, let

$$\Phi_{s_1,s_2,s_3,\chi}(\begin{pmatrix} A & * & * \\ 0 & m_2 & * \\ 0 & 0 & m_3 \end{pmatrix} \cdot \theta)$$
$$= |\det A|^{s_1} \cdot |m_2|^{s_2}\chi(m_2) \cdot |m_3|^{s_3}\overline{\chi}(m_3) \qquad (\text{for } \theta \in K_{\mathbb{A}} \text{ and } A \in GL_{r-2})$$

[33] Visibly, this Eisenstein series is a $P^{r-2,2}$ Eisenstein series *degenerate* on the (upper-left) Levi component GL_{r-2} for $r > 3$, while *cuspidal* on the (lower-right) Levi component GL_2. Basic analytic properties of such Eisenstein series are accessible by relatively elementary methods, as noted in the appendix on half-degenerate Eisenstein series.

[34] A neo-classical version of Godement's criterion is in [Borel 1966].

and

$$E^{r-2,1,1}_{s_1,s_2,s_3,\chi} = \sum_{\gamma \in P_k^{r-2,1,1}\backslash G_k} \Phi_{s_1,s_2,s_3,\chi}(\gamma g)$$

Then, ignoring the issue of interchange of sums and integrals, the Poincaré series has *most continuous* part

$$\sum_\chi \frac{\chi(\mathfrak{d})}{4\pi i \kappa} \int_{\mathrm{Re}\,(s)=\frac{1}{2}} \Big(\int_{PGL_2(k_\infty)} \widetilde{\varphi}_\infty \cdot W^E_{1-s,\overline{\chi},\infty} \Big)$$

$$\times \; \frac{L(\frac{rs'+r-2}{2}+1-s,\overline{\chi}) \cdot L(\frac{rs'+r-2}{2}+s,\chi)}{L(2-2s,\overline{\chi}^2)} \cdot |\mathfrak{d}|^{-(\frac{rs'+r-2}{2}+s-1/2)} \cdot E^{r-2,1,1}_{2\cdot\frac{s'+1}{2},\; s-(r-2)\cdot\frac{s'+1}{2},\; -s-(r-2)\cdot\frac{s'+1}{2},\; \chi}\; ds$$

[5.4] Comment It is remarkable that there are no further terms in the spectral expansion of Pé, beyond the main term, the cuspidal GL_2 part induced up to GL_r, and the continuous GL_2 part induced up to GL_r.

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6. *Continuation and cancellation for* GL_r

Take $r > 2$. In meromorphically continuing the Poincaré series to $s' = 0$ the leading term

$$\Big(\int_{U_\infty} \varphi_\infty \Big) \cdot E^{r+1,1}_{s'+1}(g)$$

seems to create an obstruction, having a pole at $s' = 0$ (coming from the pole of the Eisenstein series). As with GL_2, it is not obvious, but this pole is *cancelled* by poles coming from the trivial-χ integral of Eisenstein series

$$\frac{1}{4\pi i\kappa} \int_{\mathrm{Re}\,(s)=\frac{1}{2}} \Big(\int_{PGL_2(k_\infty)} \widetilde{\varphi}_\infty (\frac{rs'+r-2}{2}) \cdot W^E_{1-s,\infty} \Big)$$

$$\times \; \frac{\zeta(\frac{rs'+r-2}{2}+1-s) \cdot \zeta(\frac{rs'+r-2}{2}+s)}{\zeta(2-2s)} \cdot |\mathfrak{d}|^{-(\frac{rs'+r-2}{2}+s-1/2)} \cdot E^{r-2,1,1}_{2\cdot\frac{s'+1}{2},\; s-(r-2)\cdot\frac{s'+1}{2},\; -s-(r-2)\cdot\frac{s'+1}{2}}\; ds$$

Unlike the GL_2 case, for $r > 2$ the relevant poles of this integral are due to the Eisenstein series, not to the zeta functions.

[6.1] Residue of leading term First, recall from the appendix that the leading term has residue at $s' = 0$ given explicitly by the residue of the Eisenstein series, namely [35]

$$\mathrm{Res}_{s'=0} \Big(\int_{U_\infty} \varphi_\infty \Big) \cdot E^{r+1,1}_{s'+1}(g) = \Big(\int_{U_\infty} \varphi_\infty \Big) \cdot \mathrm{Res}_{s'=0} E^{r+1,1}_{s'+1}(g) = \Big(\int_{U_\infty} \varphi_\infty \Big) \cdot \frac{\kappa \cdot |\mathfrak{d}|^{r/2}}{r \cdot \xi(r)}$$

[6.2] The initial situation The spectral decomposition initially holds for $\mathrm{Re}\,(s') \gg 0$ while the integral

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[35] As usual, the function $\xi(s)$ is the zeta function of the underlying number field with gamma factors added, but *not* attempting to compensate for the *conductor* or *epsilon factor* (the latter being absent here). Thus, the functional equation is

$$\xi(s) = |\mathfrak{d}|^{1/2} \cdot |\mathfrak{d}|^{-(1-s)} \cdot \xi(1-s) = |\mathfrak{d}|^{s-\frac{1}{2}} \cdot \xi(1-s)$$

where $\mathfrak{d}$ is a finite idele generating the local different everywhere. The first $|\mathfrak{d}|^{1/2}$ is the product of standard measures of local integers at ramified primes, and $|\mathfrak{d}|^{-(1-s)}$ arises because the Fourier transform of the characteristic function of the local integers is the characteristic function of the local inverse different.

is along $\mathrm{Re}(s) = 1/2$. For brevity, let the three complex arguments to the Eisenstein series be denoted

$$s_1 = 2 \cdot \frac{s'+1}{2} \qquad s_2 = s - (r-2)\frac{s'+1}{2} \qquad s_3 = -s - (r-2)\frac{s'+1}{2}$$

For $\mathrm{Re}(s') \gg 0$, certainly $\mathrm{Re}(s_1 - s_2) > r-1$ and $\mathrm{Re}(s_1 - s_3) > r$, although $\mathrm{Re}(s_2 - s_3) = 1$, since $\mathrm{Re}(s) = 1/2$. Thus, for convergence, the Eisenstein series $E^{r-2,1,1}_{s_1,s_2,s_3}$ must be understood as formed by inducing up along $P^{r-2,2}$ the meromorphic continuation of $E^{1,1}_{s_2,s_3}$. Godement's criterion[36] yields convergence.

[6.3] Move s' to the left edge Aiming to move s' toward 0, first move s' as far to the left as possible without violating either of the two conditions

$$\begin{cases} \mathrm{Re}(s_1 - s_2) > r-1 & \text{(first condition)} \\ \mathrm{Re}(s_1 - s_3) > r & \text{(second condition)} \end{cases}$$

Let $\sigma = \mathrm{Re}(s)$ and $\sigma' = \mathrm{Re}(s')$. Expanded, the first condition is

$$2 \cdot \frac{\sigma'+1}{2} - \left(\sigma - (r-2) \cdot \frac{\sigma'+1}{2}\right) > r-1$$

which simplifies to

$$r \cdot \frac{\sigma'+1}{2} > \sigma + r - 1$$

and then

$$\frac{\sigma'+1}{2} > \frac{\sigma}{r} + 1 - \frac{1}{r}$$

and finally

$$\sigma' > 1 - \frac{2(1-\sigma)}{r} \qquad \text{(first condition, simplified)}$$

Similarly, the second condition becomes

$$\sigma' > 1 - \frac{2\sigma}{r} \qquad \text{(second condition, simplified)}$$

For $\sigma = \mathrm{Re}(s) = 1/2$, which is where we start, these two conditions are equivalent, namely

$$\sigma' > 1 - \frac{1}{r} \qquad \text{(either condition, at } \sigma = \tfrac{1}{2}\text{)}$$

Thus, move s' close to the left edge of the region defined by these conditions: for small $\varepsilon > 0$ move $\sigma' = \mathrm{Re}(s')$ to $\sigma' = 1 - \frac{1}{r} + \varepsilon$.

[6.4] First contour shift and residue Now solve the two inequalities for σ in terms of σ'. These are

$$\begin{cases} \sigma < \frac{r}{2} \cdot \sigma' - \frac{r-2}{2} & \text{(first condition, rearranged)} \\ \sigma > -\frac{r}{2} \cdot \sigma' + \frac{r}{2} & \text{(second condition, rearranged)} \end{cases}$$

With $\sigma' = 1 - \frac{1}{r} + \varepsilon$, move σ to the left, from $\frac{1}{2}$ to $\frac{1}{2} - \frac{r+1}{2}\varepsilon$. This causes the *second* condition (but not the first) to be violated:

$$\begin{cases} \frac{1}{2} - \frac{r+1}{2} \cdot \varepsilon < \frac{1}{2} + \frac{r}{2} \cdot \varepsilon = \frac{r}{2} \cdot (1 - \frac{1}{r} + \varepsilon) - \frac{r-2}{2} & \text{(first condition holds)} \\ \frac{1}{2} - \frac{r+1}{2} \cdot \varepsilon \not> \frac{1}{2} - \frac{r}{2} \cdot \varepsilon = -\frac{r}{2} \cdot (1 - \frac{1}{r} + \varepsilon) + \frac{r}{2} & \text{(second condition fails)} \end{cases}$$

[36] A neo-classical version of Godement's criterion appears in [Borel 1966].

Thus, we pick up $2\pi i$ times the residue of the pole of the Eisenstein series at

$$s = -\frac{r}{2} \cdot s' + \frac{r}{2} = \frac{r}{2} \cdot (1 - s')$$

The corresponding residue of the $P^{r-2,1,1}$ Eisenstein series is (from the appendix)

$$\frac{\kappa \cdot |\mathfrak{d}|^{\frac{r-1}{2}}}{\xi(r-1)} \cdot \frac{\xi(s_2 - s_3 - 1)}{\xi(s_2 - s_3)} \cdot E^{r-1,1}_{s_1-1,s_2-1}$$

$$= \frac{\kappa \cdot |\mathfrak{d}|^{\frac{r-1}{2}}}{\xi(r-1)} \cdot \frac{\xi(r(1-s') - 1)}{\xi(r(1-s'))} \cdot E^{r-1,1}_{s',-(r-1)s'}$$

With $s = r(1-s')/2$, the zeta functions in the integral of Eisenstein series become

$$\frac{\zeta(\frac{rs'+r-2}{2} + 1 - \frac{r}{2}(1-s')) \cdot \zeta(\frac{rs'+r-2}{2} + \frac{r}{2}(1-s'))}{\zeta(2 - r(1-s'))} = \frac{\zeta(rs') \cdot \zeta(r-1)}{\zeta(2 - r + rs')}$$

That is, apart from the $2\pi i$ and archimedean factor, we have picked up

$$(2\pi i) \cdot \frac{\zeta(rs') \cdot \zeta(r-1)}{\zeta(2-r+rs')} \cdot \frac{\kappa \cdot |\mathfrak{d}|^{\frac{r-1}{2}}}{\xi(r-1)} \cdot \frac{\xi(r(1-s') - 1)}{\xi(r(1-s'))} \cdot E^{r-1,1}_{s'}$$

From the appendix,

$$\mathrm{Re}_{s'=0}\, \xi(rs') \cdot E^{r-1,1}_{s'} = -\frac{\kappa}{r}$$

The only other finite-prime zeta which misbehaves at $s' = 0$ is the $\zeta(2 - r + rs')$ in the denominator. From the functional equation

$$\zeta(s) = \frac{\xi(s)}{\zeta_\infty(s)} = \frac{|\mathfrak{d}|^{s-\frac{1}{2}} \cdot \xi(1-s)}{\zeta_\infty(s)}$$

we have

$$\frac{1}{\zeta(2-r+rs')} = |\mathfrak{d}|^{\frac{1}{2}-(2-r+rs')} \cdot \frac{\zeta_\infty(2-r+rs')}{\xi(1-(2-r+rs'))} = |\mathfrak{d}|^{\frac{1}{2}-2+r-rs'} \cdot \frac{\zeta_\infty(2-r+rs')}{\xi(r-1-rs')}$$

Thus, the iterated residue at $s' = 0$ of this residue (without $2\pi i$ and without the archimedean factor) is

$$\mathrm{Res}_{s'=0} \left(\frac{\zeta_\infty(2-r+rs')}{\zeta_\infty(rs')} \right) \cdot \frac{\zeta(r-1) \cdot \kappa \cdot |\mathfrak{d}|^{\frac{3r}{2}-2} \cdot \xi(r-1)}{\xi(r-1) \cdot \xi(r-1) \cdot \xi(r)} \cdot \left(-\frac{\kappa}{r} \right)$$

$$= -\,\mathrm{Res}_{s'=0} \left(\frac{\zeta_\infty(2-r+rs')}{\zeta_\infty(rs')} \right) \cdot \frac{\kappa \cdot |\mathfrak{d}|^{\frac{3r}{2}-2}}{\zeta_\infty(r-1) \cdot \xi(r)} \cdot \left(\frac{\kappa}{r} \right)$$

[6.5] The archimedean factor

Take

$$s = \frac{r}{2} \cdot (1 - s')$$

and use the functional equation for the GL_2 archimedean Whittaker function, namely

$$W^E_{1-s} = \frac{\zeta_\infty(2s)}{\zeta_\infty(2-2s)} \cdot W^E_s$$

Then the archimedean integral is

$$\int_{PGL_2(k_\infty)} \widetilde{\varphi}_\infty(\frac{rs'+r-2}{2}) \cdot \frac{\zeta_\infty(r \cdot (1-s'))}{\zeta_\infty(2 - r \cdot (1-s'))} \cdot W^E_{\frac{r}{2}(1-s'),\infty}$$

Continuing this discussion shows that the continuous-spectrum part of the spectral decomposition cancels the pole of the singular term, giving the meromorphic continuation to $s' = 0$.

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7. Appendix: half-degenerate Eisenstein series

Take $q > 1$, and let f be a cuspform on $GL_q(\mathbb{A})$, in the strong sense that f is in $L^2(GL_q(k)\backslash GL_q(\mathbb{A})^1)$, and f meets the Gelfand-Fomin-Graev conditions

$$\int_{N_k\backslash N_{\mathbb{A}}} f(ng)\,dn = 0 \qquad \text{(for almost all } g\text{)}$$

and f generates an irreducible representation of $GL_q(k_\nu)$ locally at all places ν of k. For a Schwartz function Φ on $\mathbb{A}^{q\times r}$ and Hecke character χ, let

$$\varphi(g) = \varphi_{\chi,f,\Phi}(g) = \chi(\det g)^q \int_{GL_q(\mathbb{A})} f(h^{-1})\,\chi(\det h)^r\,\Phi(h\cdot[0_{q\times(r-q)}\ 1_q]\cdot g)\,dh$$

This function φ has the same central character as f. It is left invariant by the adele points of the unipotent radical

$$N = \{\begin{pmatrix} 1_{r-q} & * \\ & 1_r \end{pmatrix}\} \qquad \text{(unipotent radical of } P = P^{r-q,q}\text{)}$$

The function φ is left invariant under the *k-rational* points M_k of the standard Levi component of P,

$$M = \{\begin{pmatrix} a & \\ & d \end{pmatrix} : a \in GL_{r-q},\ d \in GL_r\}$$

To understand the normalization, observe that

$$\xi(\chi^r, f, \Phi(0,*)) = \varphi(1) = \int_{GL_q(\mathbb{A})} f(h^{-1})\,\chi(\det h)^r\,\Phi(h\cdot[0_{q\times(r-q)}\ 1_q])\,dh$$

is a zeta integral as in [Godement-Jacquet 1972] for the standard L-function attached to the cuspform f (or perhaps a contragredient). Thus, the Eisenstein series formed from φ includes this zeta integral as a factor, so write

$$\xi(\chi^r, f, \Phi(0,*))\ \cdot\ E^P_{\chi,f,\Phi}(g)\ =\ \sum_{\gamma\in P_k\backslash GL_r(k)} \varphi(\gamma\, g) \qquad \text{(convergent for } \mathrm{Re}(\chi) >> 0\text{)}$$

Now prove the meromorphic continuation via Poisson summation:

$$\xi(\chi^r, f, \Phi(0,*))\cdot E^P_{\chi,f,\Phi}(g)$$

$$= \chi(\det g)^q \sum_{\gamma\in P_k\backslash GL_r(k)} \int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r} \sum_{\alpha\in GL_q(k)} \Phi(h^{-1}\cdot[0\ \alpha]\cdot g)\,dh$$

$$= \chi(\det g)^q \int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r} \sum_{y\in k^{q\times r},\ \text{full rank}} \Phi(h^{-1}\cdot y\cdot g)\,dh$$

The Gelfand-Fomin-Graev condition on f will compensate for the otherwise-irksome full-rank constraint. Anticipating that we can drop the rank condition suggests that we define

$$\Theta_\Phi(h,g) = \sum_{y\in k^{q\times r}} \Phi(h^{-1}\cdot y\cdot g)$$

As in [Godement-Jacquet 1972], the non-full-rank terms integrate to 0: [37]

[7.0.1] Proposition: For f a cuspform, less-than-full-rank terms integrate to 0, that is,

$$\int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r} \sum_{y\in k^{q\times r},\ \text{rank} < q} \Phi(h^{-1}\cdot y\cdot g)\,dh = 0$$

Proof: Since this is asserted for arbitrary Schwartz functions Φ, we can take $g = 1$. By linear algebra, given $y_0 \in k^{q\times r}$ of rank ℓ, there is $\alpha \in GL_q(k)$ such that

$$\alpha\cdot y_0 = \begin{pmatrix} y_{\ell\times r} \\ 0_{(q-\ell)\times r}\end{pmatrix} \qquad \text{(with } \ell\text{-by-}r \text{ block } y_{\ell\times r} \text{ of rank } \ell)$$

Thus, without loss of generality fix y_0 of the latter shape. Let Y be the orbit of y_0 under left multiplication by the rational points of the parabolic

$$P^{\ell,q-\ell} = \{\begin{pmatrix} \ell\text{-by-}\ell & * \\ 0 & (q-\ell)\text{-by-}(q-\ell)\end{pmatrix}\} \subset GL_q$$

This is some set of matrices of the same shape as y_0. Then the subsum over $GL_q(k)\cdot y_0$ is

$$\int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r} \sum_{y\in GL_q(k)\cdot y_0} \Phi(h^{-1}\cdot y)\,dh = \int_{P_k^{\ell,q-\ell}\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r} \sum_{y\in Y} \Phi(h^{-1}\cdot y)\,dh$$

Let N and M be the unipotent radical and standard Levi component of $P^{\ell,q-\ell}$,

$$N = \begin{pmatrix} 1_\ell & * \\ 0 & 1_{q-\ell}\end{pmatrix} \qquad M = \begin{pmatrix} \ell\text{-by-}\ell & 0 \\ 0 & (q-\ell)\text{-by-}(q-\ell)\end{pmatrix}$$

Then the integral can be rewritten as an iterated integral

$$\int_{N_k M_k\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r} \sum_{y\in Y} \Phi(h^{-1}\cdot y)\,dh$$

$$= \int_{N_{\mathbb{A}} M_k\backslash GL_q(\mathbb{A})} \sum_{y\in Y} \int_{N_k\backslash N_{\mathbb{A}}} f(nh)\,\chi(\det nh)^{-r}\,\Phi((nh)^{-1}\cdot y)\,dn\,dh$$

$$= \int_{N_{\mathbb{A}} M_k\backslash GL_q(\mathbb{A})} \sum_{y\in Y} \chi(\det h)^{-r}\,\Phi(h^{-1}\cdot y)\left(\int_{N_k\backslash N_{\mathbb{A}}} f(nh)\,dn\right) dh$$

[37] There are issues of convergence. First, for $\mathrm{Re}(\chi)$ sufficiently large, the integral

$$\chi(\det g)^q \sum_{\gamma\in P_k\backslash GL_r(k)} \int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r}\,\Theta(h,g)\,dh$$

is absolutely convergent. Also, we have the integrals analogous to integrals over $k^\times\backslash\mathbb{J}^+$. That is, let $GL_q^+ = \{h \in GL_q(\mathbb{A}) : |\det h| \geq 1\}$. Then, for *arbitrary* χ, using the fact that cuspforms f are of rapid decay in Siegel sets,

$$\chi(\det g)^q \sum_{\gamma\in P_k\backslash GL_r(k)} \int_{GL_q(k)\backslash GL_q^+} f(h)\,\chi(\det h)^{-r} \sum_{y\in k^{q\times r},\ \text{full rank}} \Phi(h^{-1}\cdot y\cdot g)\,dh$$

is absolutely convergent.

since all fragments but $f(nh)$ in the integrand are left invariant by $N_{\mathbb{A}}$. But the inner integral of $f(nh)$ is 0, by the Gelfand-Fomin-Graev condition, so the whole is 0. ///

Let ι denote the transpose-inverse involution(s). Poisson summation gives

$$\Theta_\Phi(h,g) = \sum_{y\in k^{q\times r}} \Phi(h^{-1}\cdot y\cdot g)$$

$$= |\det(h^{-1})^\iota|^r \,|\det g^\iota|^q \sum_{y\in k^{q\times r}} \widehat{\Phi}((h^\iota)^{-1}\cdot y\cdot g^\iota) \;=\; |\det(h^{-1})^\iota|^r \,|\det g^\iota|^q \,\Theta_{\widehat{\Phi}}(h^\iota, g^\iota)$$

As with Θ_Φ, the not-full-rank summands in $\Theta_{\widehat{\Phi}}$ integrate to 0 against cuspforms. Thus, letting

$$GL_q^+ = \{h\in GL_q(\mathbb{A}) \;:\; |\det h|\geq 1\} \qquad\qquad GL_q^- = \{h\in GL_q(\mathbb{A}) \;:\; |\det h|\leq 1\}$$

$$\xi(\chi^r, f, \Phi(0,*))\cdot E^P_{\chi,f,\Phi}(g) = \chi(\det g)^q \int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r}\,\Theta_\Phi(h,g)\,dh$$

$$= \chi(\det g)^q \int_{GL_q(k)\backslash GL_q^+} f(h)\,\chi(\det h)^{-r}\,\Theta_\Phi(h,g)\,dh + \chi(\det g)^q \int_{GL_q(k)\backslash GL_q^-} f(h)\,\chi(\det h)^{-r}\,\Theta_\Phi(h,g)\,dh$$

$$= \quad \chi(\det g)^q \int_{GL_q(k)\backslash GL_q^+} f(h)\,\chi(\det h)^{-r}\,\Theta_\Phi(h,g)\,dh$$

$$+\, \chi(\det g)^q \int_{GL_q(k)\backslash GL_q^-} |\det(h^{-1})^\iota|^r \,|\det g^\iota|^q\, f(h)\,\chi(\det h)^{-r}\,\Theta_{\widehat{\Phi}}(h^\iota, g^\iota)\,dh$$

By replacing h by h^ι in the second integral, convert it to an integral over $GL_q(k)\backslash GL_q^+$, and the whole is

$$\xi(\chi^r, f, \Phi(0,*))\cdot E^P_{\chi,f,\Phi}(g) \;=\; \chi(\det g)^q \int_{GL_q(k)\backslash GL_q^+} f(h)\,\chi(\det h)^{-r}\,\Theta_\Phi(h,g)\,dh$$

$$+\, \nu\chi^{-1}(\det g^\iota)^q \int_{GL_q(k)\backslash GL_q^+} f(h^\iota)\;\; \nu\chi^{-1}(\det h^\iota)^{-r}\,\Theta_{\widehat{\Phi}}(h, g^\iota)\,dh$$

Since $f\circ\iota$ is a cuspform, the second integral is entire in χ. Thus, we have proven

$$\xi(\chi^r, f, \Phi(0,*))\;\cdot E^P_{\chi,f,\Phi} \quad \text{is entire}$$

[7.1] Remark Except for the extreme case $q = r-1$, these Eisenstein series are degenerate, so occur only as (iterated) *residues* of cuspidal-data Eisenstein series. Assessing poles of residues is less effective in the present special circumstances than the above argument.

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8. *Appendix: Eisenstein series for* GL_2

We compute Mellin transforms of the Whittaker functions attached to Eisenstein series for GL_2 (with trivial central character, spherical, over a number field k). The natural normalization of Eisenstein series adds a further local L-factor to the Mellin transform (as well as a measure constant at ramified primes), by contrast to the usual Mellin transform of the usual normalization W_v° of the spherical Whittaker function. And at the end we give a precise function equation.

[8.1] The standard Eisenstein series Let $G = GL_2$ and

$$P = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\} \quad N = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\} \quad M = \{\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}\}$$

and let Z be the center. An absolutely unramified Hecke character can be decomposed as $\chi(\alpha) \cdot |\alpha|^s$ corresponding to the usual decomposition

$$\mathbb{J}/k^\times \approx \mathbb{J}^1/k^\times \times (0, +\infty)$$

The standard normalization of the spherical function (in the tensor product of principal series) is $\varphi = \bigotimes_v \varphi_v$ with

$$\varphi_v(\begin{pmatrix} a & * \\ 0 & 1 \end{pmatrix} \cdot \theta) = |a/d|_v^s \cdot \chi_v(a/d) \qquad (\theta \text{ in the standard maximal compact of } GL_2(k_v))$$

Thus, $\varphi_v(1) = 1$ for all v. The standard Eisenstein series attached to data s, χ is

$$E(g) = E_{s,\chi}(g) = \sum_{P_k \backslash G_k} \varphi(\gamma g)$$

[8.2] The Eisenstein Whittaker functions

Let ψ be the standard non-trivial character on $\mathbb{A}/k$. Let $\mathfrak{o}_v$ be the local integers at a finite place v, let p be the rational prime lying under v. The dual lattice to $\mathfrak{o}_v$ with respect to trace is

$$\mathfrak{o}_v^* = \{\alpha \in k_v \ : \ \mathrm{tr}_{Q_p}^{k_v}(\alpha\beta) \in \mathbb{Z}_p \ : \ \text{ for all } \beta \in \mathfrak{o}_v\}$$

This dual lattice $\mathfrak{o}_v^*$ is exactly the kernel of the standard ψ. Let

$$W^E(g) = \int_{N_k \backslash N_\mathbb{A}} \overline{\psi}(n)\, E(ng)\, dn$$

where the measure is normalized so that the total measure of $N_k \backslash N_\mathbb{A}$ is 1. As usual, this unwinds to

$$\int_{N_k \backslash N_\mathbb{A}} \overline{\psi}(n)\, \varphi(ng)\, dn + \int_{N_\mathbb{A}} \overline{\psi}(n)\, \varphi(w_\circ ng)\, dn$$

As ψ is non-trivial, the first of the two integrals is 0. The second integral has an Euler product, with v^{th} factor

$$W_v^E(g) = \int_{N_v} \overline{\psi}(n)\, \varphi_v(w_\circ ng)\, dn$$

[8.3] Archimedean local computations

At archimedean places, for present application we do not need the Mellin transform of the Whittaker functions. Instead, we need to present the archimedean-place Eisenstein-Whittaker functions in a form that emphasizes local functional equations. As above, let

$$\varphi_v(\begin{pmatrix} a & * \\ 0 & 1 \end{pmatrix} \cdot \theta) = |a/d|_v^s \cdot \chi_v(a/d) \qquad (\theta \text{ in the standard maximal compact of } GL_2(k_v))$$

and

$$W_v^E(g) = \int_{N_v} \overline{\psi}(n)\, \varphi_v(w_\circ ng)\, dn$$

By the right K_v invariance, Z_v-invariance, and left N_v equivariance, to understand W_v^E it suffices to take g of the form

$$g = \begin{pmatrix} y & \\ & 1 \end{pmatrix}$$

For v complex and $x \in k_v$, let $x^\sigma = \overline{x}$, and for v real let σ be the identity map. For archimedean v, the standard Iwasawa decomposition is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} y & \\ & 1 \end{pmatrix} = \begin{pmatrix} -y/r & x^\sigma/r \\ 0 & r \end{pmatrix} \cdot \begin{pmatrix} x^\sigma/r & -y^\sigma/r \\ y/r & x/r \end{pmatrix} \qquad \text{(where } r = \sqrt{xx^\sigma + yy^\sigma}\text{)}$$

Thus,

$$\varphi_v \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y & \\ & 1 \end{pmatrix} = \varphi_v \begin{pmatrix} -y/r & * \\ 0 & r \end{pmatrix} = \left| \frac{y}{r^2} \right|_v^s = \left| \frac{y}{xx^\sigma + yy^\sigma} \right|_v^s$$

The Fourier-Whittaker transform is

$$\int_{k_v} \overline{\psi}(x) \left| \frac{y}{xx^\sigma + yy^\sigma} \right|_v^s dx = |y|_v \int_{k_v} \overline{\psi}(xy) \left| \frac{y}{yy^\sigma(xx^\sigma + 1)} \right|_v^s dx = |y|_v^{1-s} \int_{k_v} \overline{\psi}(xy) \left| \frac{1}{xx^\sigma + 1} \right|_v^s dx$$

For v real, the following computation is familiar, but the normalizations are important, so we recall it. Using the identity

$$u^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-tu}\, t^s \,\frac{dt}{t} \qquad \text{(for } u > 0 \text{ and } \mathrm{Re}(s) > 0\text{)}$$

rewrite the Fourier-Whittaker transform as

$$\frac{|y|^{1-s}}{\Gamma(s)} \int_0^\infty \int_{\mathbb{R}} \overline{\psi}(xy)\, e^{-t(1+x^2)}\, t^s \, dx \, \frac{dt}{t} = \frac{|y|^{1-s}\sqrt{\pi}}{\Gamma(s)} \int_0^\infty \int_{\mathbb{R}} \overline{\psi}(xy\frac{\sqrt{\pi}}{\sqrt{t}})\, e^{-t}\, e^{-\pi x^2}\, t^{s-\frac{1}{2}}\, dx \,\frac{dt}{t}$$

$$= \frac{|y|^{1-s}\sqrt{\pi}}{\Gamma(s)} \int_0^\infty e^{-t}\, e^{-\pi y^2 \cdot \frac{\pi}{t}}\, t^{s-\frac{1}{2}} \,\frac{dt}{t} \qquad \text{(Fourier transform after replacing } x \text{ by } x \cdot \tfrac{\sqrt{\pi}}{\sqrt{t}}\text{)}$$

Replacing t by $t \cdot \pi y$ turns the whole into

$$(\pi y)^{s-\frac{1}{2}} \cdot \frac{|y|^{1-s}\sqrt{\pi}}{\Gamma(s)} \int_0^\infty e^{-\pi(t+\frac{1}{t})|y|}\, t^{s-\frac{1}{2}} \,\frac{dt}{t} = \frac{\sqrt{y}}{\pi^{-s}\Gamma(s)} \int_0^\infty e^{-\pi(t+\frac{1}{t})|y|}\, t^{s-\frac{1}{2}} \,\frac{dt}{t} \qquad (v \text{ real})$$

For v complex, the computation is less familiar, but follows the same course. Keep in mind that the complex norm fitting into the product formula is the square of the usual:

$$|y|_{\mathbb{C}} = |y\overline{y}|_{\mathbb{R}} = |y|^2$$

The character is

$$\psi_{\mathbb{C}}(xy) = \psi_{\mathbb{R}}(\mathrm{tr}_{\,\mathbb{C}/\mathbb{R}}(xy))$$

In particular, for $x, y \in \mathbb{C}$, the trace $\mathrm{tr}_{\,\mathbb{C}/\mathbb{R}}(xy)$ is *double* the standard $\mathbb{R}$-valued pairing on $\mathbb{C} \approx \mathbb{R}^2$. This 2 appears explicitly in the Gaussian after a Fourier transform. And the measure on $\mathbb{C}$ is twice the usual Lebesgue measure on $\mathbb{R}^2$. We write this factor explicitly from the outset. The same trick with the gamma function rewrites the Fourier-Whittaker transform as

$$\frac{2|y|_{\mathbb{C}}^{1-s}}{\Gamma(2s)} \int_0^\infty \int_{\mathbb{C}} \overline{\psi}(xy)\, e^{-t(1+xx^\sigma)}\, t^s \, dx \,\frac{dt}{t} = \frac{2|y|^{2-2s}\pi}{\Gamma(2s)} \int_0^\infty \int_{\mathbb{R}} \overline{\psi}(xy\frac{\sqrt{\pi}}{\sqrt{t}})\, e^{-t}\, e^{-\pi xx^\sigma}\, t^{2s-1}\, dx \,\frac{dt}{t}$$

$$= \frac{2|y|^{2-2s}\pi}{\Gamma(2s)} \int_0^\infty e^{-t}\, e^{-\pi 4yy^\sigma \cdot \frac{\pi}{t}}\, t^{2s-1} \,\frac{dt}{t} \qquad \text{(Fourier transform after replacing } x \text{ by } x \cdot \tfrac{\sqrt{\pi}}{\sqrt{t}}\text{)}$$

Again, the $4 = 2^2$ in $4yy^\sigma$ appears because of the normalization of the pairing. Replacing t by $t \cdot 2\pi y$ turns the whole into

$$(2\pi y)^{2s-1} \frac{2|y|^{2-2s}\pi}{\Gamma(2s)} \int_0^\infty e^{-2\pi(t+\frac{1}{t})|y|}\, t^{2s-1} \,\frac{dt}{t} = \frac{|y|}{(2\pi)^{-2s}\Gamma(2s)} \int_0^\infty e^{-2\pi(t+\frac{1}{t})|y|}\, t^{2s-1} \,\frac{dt}{t} \qquad (v \text{ complex})$$

In both cases, we have

$$\zeta_v(2s) \cdot W^E_{s,v} = (\text{invariant under } s \to 1-s)$$

[8.4] The Mellin transform The global Mellin transform of W^E factors

$$\int_{\mathbb{J}} |a|^{s'} W^E_{s,\chi} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} da = \prod_v \int_{k_v^\times} |a|^{s'} W^E_{s,\chi,v} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} da$$

To compute this, we cannot simply change the order of integration, since this would produce a divergent integral along the way. Instead, we present[38] the vectors φ_v in a different form. Let Φ_v be any Schwartz function on k_v^2 invariant under K_v (under the obvious right action of GL_2), and put[39]

$$\varphi'_v(g) = \chi(\det g)|\det g|^s \cdot \int_{k_v^\times} \chi(t)^2 |t|^{2s} \cdot \Phi(t \cdot e_2 \cdot g)\, dt$$

where e_2 is the second basis element in k^2. This φ'_v has the same left P_v-equivariance as φ_v, namely

$$\varphi'_v(\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \cdot g) = \varphi'_v(g) \cdot |a/d|^s\, \chi(a/d)$$

For Φ invariant under the standard maximal compact K_v of $GL_2(k_v)$, this function φ'_v is right K_v-invariant. By the Iwasawa decomposition, up to constant multiples there is only one such function, so

$$\varphi'_v(g) = \varphi'_v(1) \cdot \varphi_v(g) \qquad (\text{since } \varphi_v(1) = 1)$$

and

$$\varphi'_v(1) = \int_{k_v^\times} \chi(t)^2 |t|^{2s} \cdot \Phi(t \cdot e_2)\, dt = \zeta_v(2s, \chi^2, \Phi(0,*)) \qquad (\text{a Tate-Iwasawa zeta integral})$$

Thus, it suffices to compute the local Mellin transform of

$$\varphi'_v(1) \cdot W^E_{s,\chi,v}(m) = \int_{N_v} \overline{\psi}(n)\, \varphi'_v(w_\circ nm)\, dn = \chi(a)|a|^s \int_{N_v} \overline{\psi}(n) \int_{k_v^\times} \chi^2(t)|t|^{2s}\, \Phi(t \cdot e_2 \cdot w_\circ nm)\, dt\, dn$$

$$= \chi(a)|a|^s \int_{k_v} \overline{\psi}(x) \int_{k_v^\times} \chi^2(t)|t|^{2s}\, \Phi(ta\ tx)\, dt\, dx \qquad (\text{with } m = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})$$

At *finite* primes v, we may as well take Φ to be

$$\Phi(t,x) = \mathrm{ch}_{\mathfrak{o}_v}(t) \cdot \mathrm{ch}_{\mathfrak{o}_v}(x) \qquad (\mathrm{ch}_X = \text{characteristic function of set } X)$$

Then $\varphi'_v(1)$ is exactly an L-factor

$$\varphi'_v(1) = \zeta_v(2s, \chi^2, \mathrm{ch}_{\mathfrak{o}_v}) = L_v(2s, \chi^2)$$

[38] This variant presentation of the vector used to form the Eisenstein series is essentially a globally split theta correspondence. At a more elementary level, this trick is a non-archimedean analogue of classical computations involving Bessel functions.

[39] The leading $\chi(\det g)|\det g|^s$ in this integral maintains the triviality of the central character.

and [40]

$$\varphi'_v(1)\cdot W^E_{s,\chi,v}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} = \chi(a)|a|^s \int_{k_v} \overline{\psi}(x)\,\mathrm{ch}_{\mathfrak{o}_v}(tx) \int_{k_v^\times} \chi^2(t)|t|^{2s}\,\mathrm{ch}_{\mathfrak{o}_v}(ta)\,dt\,dx$$

$$= \chi(a)|a|^s \,\mathrm{meas}\,(\mathfrak{o}_v) \int_{k_v^\times} \mathrm{ch}_{\mathfrak{o}_v^*}(1/t)\chi^2(t)|t|^{2s-1}\,\mathrm{ch}_{\mathfrak{o}_v}(ta)\,dt$$

$$= |\mathfrak{d}_v|_v^{1/2}\cdot\chi(a)|a|^s \int_{k_v^\times} \mathrm{ch}_{\mathfrak{o}_v^*}(1/t)\chi^2(t)|t|^{2s-1}\,\mathrm{ch}_{\mathfrak{o}_v}(ta)\,dt$$

where $\mathfrak{d}_v \in k_v^\times$ is such that $(\mathfrak{o}_v^*)^{-1} = \mathfrak{d}_v\cdot\mathfrak{o}_v$. We can compute the Mellin transform

$$\int_{k_v^\times} |a|^{s'}\cdot\Big(\chi(a)|a|^s \int_{k_v^\times} \mathrm{ch}_{\mathfrak{o}_v^*}(1/t)\chi^2(t)|t|^{2s-1}\,\mathrm{ch}_{\mathfrak{o}_v}(ta)\,dt\Big)\,da$$

Replace a by a/t, and then t by $1/t$ to obtain a product of two zeta integrals

$$\Big(\int_{k_v^\times} |a|^{s'}\cdot\chi(a)|a|^s\,\mathrm{ch}_{\mathfrak{o}_v}(a)\,da\Big)\cdot\Big(\int_{k_v^\times} \mathrm{ch}_{\mathfrak{o}_v^*}(1/t)\chi(t)|t|^{s-1-s'}\,dt\Big)$$

$$= \zeta_v(s'+s,\chi,\mathrm{ch}_{\mathfrak{o}_v})\cdot\zeta_v(s'+1-s,\overline{\chi},\mathrm{ch}_{\mathfrak{o}_v^{*-1}})$$

$$= L_v(s'+s,\chi)\cdot L_v(s'+1-s,\overline{\chi})\cdot|\mathfrak{d}_v|^{-(s'+1-s)}\,\chi(\mathfrak{d}_v)$$

Thus, dividing through by $\varphi'_v(1)$ and putting back the measure constant, the Mellin transform of $W^E_{s,\chi,v}$ is

$$\int_{k_v^\times} |a|^{s'}\,W^E_{s,\chi,v}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} da = |\mathfrak{d}_v|_v^{1/2}\cdot\frac{L_v(s'+s,\chi)\cdot L_v(s'+1-s,\overline{\chi})}{L_v(2s,\chi^2)}\cdot|\mathfrak{d}_v|^{-(s'+1-s)}\,\chi(\mathfrak{d}_v)$$

Let $\mathfrak{d}$ be the idele whose v^{th} component is $\mathfrak{d}_v$ for finite v and whose archimedean components are all 1. The product over all finite primes v of these local factors is

$$\int_{\mathbb{J}^{\mathrm{fin}}} |a|^{s'}\,W^E_{s,\chi}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} da = |\mathfrak{d}|^{1/2}\cdot\frac{L(s'+s,\chi)\cdot L(s'+1-s,\overline{\chi})}{L(2s,\chi^2)}\cdot|\mathfrak{d}|^{-(s'+1-s)}\,\chi(\mathfrak{d})$$

In our application, we will replace s by $1-s$ and χ by $\overline{\chi}$, giving

$$\int_{\mathbb{J}^{\mathrm{fin}}} |a|^{s'}\,W^E_{1-s,\overline{\chi}}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} da = |\mathfrak{d}|^{1/2}\cdot\frac{L(s'+1-s,\overline{\chi})\cdot L(s'+s,\chi)}{L(2-2s,\overline{\chi}^2)}\cdot|\mathfrak{d}|^{-(s'+s)}\,\overline{\chi}(\mathfrak{d})$$

In particular, with χ trivial,

$$\int_{\mathbb{J}^{\mathrm{fin}}} |a|^{s'}\,W^E_{1-s}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} da = |\mathfrak{d}|^{1/2}\cdot\frac{\zeta_k(s'+1-s)\cdot\zeta_k(s'+s)}{\zeta_k(2-2s)}\cdot|\mathfrak{d}|^{-(s'+s)}$$

[8.5] Functional equation With χ trivial, Poisson summation gives the functional equation for the spherical Eisenstein series:

$$\xi(2s,\Phi(0,*))\cdot E_s(g) = \xi(2-2s,\widehat{\Phi}(0,*))\cdot E_{1-s}(g^\iota)$$

[40] The Fourier transform of the characteristic function $\mathrm{ch}_{\mathfrak{o}_v}$ of $\mathfrak{o}_v$ is $|\mathfrak{d}_v|^{1/2}$ times the characteristic function of $\mathfrak{o}_v^*$.

where g^ι is g-transpose-inverse. Peculiar to GL_2 is that

$$wg^\iota w^{-1} = (\det g)^{-1} \cdot g \qquad \text{(where } w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\text{)}$$

Since w lies in the standard maximal compact locally everywhere, and is in $GL_2(k)$, and since the Eisenstein series is spherical everywhere and has trivial central character,

$$\xi(2s, \Phi(0,*)) \cdot E_s = \xi(2-2s, \widehat{\Phi}(0,*)) \cdot E_{1-s}$$

Take Φ to be K_v-invariant locally everywhere, Taking Φ to be the standard Gaussian at archimedean places, and the characteristic function of $\mathfrak{o}_v^2$ at finite places, this gives

$$\xi(2s) \cdot E_s = |\mathfrak{d}| \cdot |\mathfrak{d}|^{2s-2} \cdot \xi(2-2s) \cdot E_{1-s} = |\mathfrak{d}|^{2s-1} \cdot \xi(2-2s) \cdot E_{1-s}$$

where the first factor $|\mathfrak{d}|$ is the product of the measures of the $\mathfrak{o}_v^2$. Recall the functional equation of ξ

$$\xi(s) = |\mathfrak{d}|^{s-\frac{1}{2}} \cdot \xi(1-s)$$

Then

$$\xi(2s) \cdot E_s = \xi(2s-1) \cdot E_{1-s}$$

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9. *Appendix: residues of degenerate Eisenstein series for* $P^{n-1,1}$

We prove meromorphic continuation and determine some residues of some very degenerate Eisenstein series (sometimes called *Epstein zeta functions*). We need to recall some specifics about these well-known examples. Let

$$P = P^{n-1,1} = \{\begin{pmatrix} (n-1)\text{-by-}(n-1) & * \\ 0 & 1\text{-by-}1 \end{pmatrix}\}$$

View $\mathbb{A}^n$ and k^n as *row* vectors. Let $e_1, \ldots, e_n$ the standard basis for k^n. The parabolic P is the stabilizer in GL_n of the line ke_n. Given a Hecke character of the form $\chi(\alpha) = |\alpha|^s$ and a Schwartz function Φ on $\mathbb{A}^n$, let

$$\varphi(g) = |\det g|^s \int_{\mathbb{J}} |t|^{ns}\, \Phi(t \cdot e_n \cdot g)\, dt$$

The factor $|t|^{ns}$ in the integrand and the leading factor $|\det g|^s$ combine to give the invariance $\varphi(zg) = \varphi(g)$ for z in the center $Z_{\mathbb{A}}$ of $G = GL_n$. By changing variables in the integral observe the left equivariance

$$\varphi(pg) = |\det pg|^s \int_{\mathbb{J}} |t|^{ns}\, \Phi(t \cdot e_n \cdot pg)\, dt = |\det A|^s |d|^{-(n-1)s} \cdot \varphi(g) \qquad \text{(for } p = \begin{pmatrix} A & * \\ & d \end{pmatrix} \in P_{\mathbb{A}}\text{)}$$

The normalization is *not* $\varphi(1) = 1$ but

$$\varphi(1) = \int_{\mathbb{J}} |t|^{ns}\, \Phi(t \cdot e_n)\, dt \qquad \text{(Tate-Iwasawa zeta integral at } ns\text{)}$$

Denote this zeta integral by $\xi = \xi(ns, \Phi(0,*))$, indicating that it only depends upon the values of Φ along the last coordinate axis. Thus, by comparison to the standard spherical Eisenstein series $E_s(g)$ corresponding to this s^{th} degenerate principal series, the Eisenstein series associated to φ has a factor of $\xi(ns, \Phi(0,*))$ included, namely

$$\xi(ns, \Phi(0,*)) \cdot E_s(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi(\gamma g)$$

Poisson summation proves the meromorphic continuation of this Eisenstein series, as follows. Let

$$\mathbb{J}^+ = \{t \in \mathbb{J} \ : \ |t| \geq 1\} \qquad \mathbb{J}^- = \{t \in \mathbb{J} \ : \ |t| \leq 1\}$$

and $g^\iota = (g^\top)^{-1}$ (transpose inverse)

$$\xi(ns, \Phi(0,*)) \cdot E_s(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi(\gamma g) = |\det g|^s \sum_{\gamma \in P_k \backslash G_k} \int_{\mathbb{J}} |t|^{ns}\, \Phi(t \cdot x \cdot \gamma g)\, dt$$

$$= |\det g|^s \sum_{\gamma \in P_k \backslash G_k} \int_{k^\times \backslash \mathbb{J}} |t|^{ns} \sum_{\lambda \in k^\times} \Phi(t \cdot \lambda e_n \cdot \gamma g)\, dt = |\det g|^s \int_{k^\times \backslash \mathbb{J}} |t|^{ns} \sum_{x \in k^n - 0} \Phi(t \cdot x \cdot g)\, dt$$

Let

$$\Theta(g) = \sum_{x \in k^n} \Phi(t \cdot x \cdot g)$$

Then

$$\xi(ns, \Phi(0,*)) \cdot E_s(g) = \ |\det g|^s \int_{k^\times \backslash \mathbb{J}^+} |t|^{ns}\, [\Theta(g) - \Phi(0)]\, dt \ + \ |\det g|^s \int_{k^\times \backslash \mathbb{J}^-} |t|^{ns}\, [\Theta(g) - \Phi(0)]\, dt$$

The usual estimate shows that the integral over $k^\times \backslash \mathbb{J}^+$ converges absolutely for all $s \in \mathbb{C}$. Rewrite the second part of the integral as an analogous integral over $k^\times \backslash \mathbb{J}^+$. Poisson summation gives

$$\sum_{x \in k^n - 0} \Phi(t \cdot x \cdot g) + \Phi(0) = |t|^{-n} |\det g|^{-1} \sum_{x \in k^n - 0} \widehat{\Phi}(t^{-1} \cdot x \cdot g^\iota) + |t|^{-n} |\det g|^{-1} \widehat{\Phi}(0)$$

Let

$$\Theta'(g^\iota) = \sum_{x \in k^n} \widehat{\Phi}(t \cdot x \cdot g^\iota) \cdot$$

Removing the $\Phi(0)$ and $\widehat{\Phi}(0)$ terms and replacing t by t^{-1} in the integral over $k^\times \backslash \mathbb{J}^-$ turns this integral into

$$|\det g|^{s-1} \int_{k^\times \backslash \mathbb{J}^+} |t|^{n(1-s)}\, [\Theta'(g^\iota) - \widehat{\Phi}(0)]\, dt$$

$$- \ |\det g|^s\, \Phi(0) \int_{k^\times \backslash \mathbb{J}^-} |t|^{ns}\, dt \ + \ |\det g|^{s-1}\, \widehat{\Phi}(0) \int_{k^\times \backslash \mathbb{J}^-} |t|^{n(s-1)}\, dt$$

The integral over $k^\times \backslash \mathbb{J}^+$ is entire. Thus, the non-elementary part of the integral is converted into two *entire* integrals over $k^\times \backslash \mathbb{J}^+$ together with two elementary integrals that give the only possible poles:

$$\xi \cdot E_s(g) = (\text{entire}) - |\det g|^s \Phi(0) \int_{k^\times \backslash \mathbb{J}^-} |t|^n\, dt + |\det g|^{s-1} \widehat{\Phi}(0) \int_{k^\times \backslash \mathbb{J}^-} |t|^{n(s-1)}\, dt$$

With

$$\kappa = \int_{k^\times \backslash \mathbb{J}^1} 1\, dt$$

the relatively elementary integrals can be evaluated

$$\int_{k^\times \backslash \mathbb{J}^-} |t|^{ns}\, dt = \left(\int_{k^\times \backslash \mathbb{J}^1} 1\, dt \right) \cdot \left(\int_0^1 t^{ns}\, dt \right) = \frac{\kappa}{ns}$$

Similarly,

$$\int_{k^\times \backslash \mathbb{J}^-} |t|^{n(s-1)}\, dt = \frac{\kappa}{n(s-1)}$$

That is,

$$\xi(ns, \Phi(0,*)) \cdot E_s = (\text{entire}) - \frac{\kappa\,\Phi(0)}{ns} + \frac{\kappa\,\widehat{\Phi}(0)}{n(s-1)}$$

Thus, the residue at $s = 1$ of E_s is

$$\text{Res}_{s=1} E_s = \frac{\kappa\,\widehat{\Phi}(0)}{n \cdot \xi(n, \Phi(0,*))}$$

Let $\mathfrak{d}$ be an idele such that $\mathfrak{d}_v$ generates the local different at a finite place v, and is trivial at archimedean places. Let Φ be the standard Gaussian at archimedean places (so its integral is 1), and the characteristic function of $\mathfrak{o}_v^n$ at finite places v. With the standard measure on $\mathbb{A}$ we have

$$\widehat{\Phi}(0) = |\mathfrak{d}|^{n/2} \qquad \xi(n, \Phi(0,*)) = \xi(n)$$

where ξ is the usual zeta function with standard gamma factors, but without any epsilon factor or accounting for conductors. The residue at $s = 1$ is

$$\text{Res}_{s=1} E = \frac{\kappa \cdot |\mathfrak{d}|^{n/2}}{n \cdot \xi(n)}$$

At $s = 0$, the relevant residue is

$$\text{Res}_{s=0}\, \xi(ns, \Phi(0,*)) \cdot E_s = -\frac{\kappa\,\Phi(0)}{n}$$

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10. *Appendix: degenerate Eisenstein series for $P^{r-2,1,1}$*

Let $P = P^{r-2,1,1}$, and

$$\varphi_{s_1,s_2,s_3}\begin{pmatrix} A & * & * \\ & a & * \\ & & d \end{pmatrix} = |\det A|^{s_1} \cdot |a|^{s_2} \cdot |d|^{s_3} \qquad (\text{with } A \in GL_{r-2},\ a, d \in GL_1)$$

and extend φ to a function on $G(\mathbb{A}) = GL_r(\mathbb{A})$ by requiring right $K_{\mathbb{A}}$-equivariance. Define an Eisenstein series on $G = GL_r$ by

$$E_{s_1,s_2,s_3}(g) = \sum_{\gamma \in P_k^{r-2,1,1} \backslash G_k} \varphi_{s_1,s_2,s_3}(\gamma g)$$

[10.1] Symmetry in s_2 and s_3 This can be rewritten as an interated sum

$$E_{s_1,s_2,s_3}(g) = \sum_{\gamma \in P_k^{r-2,2} \backslash G_k} \varphi_{s_1 \otimes E^{1,1}_{s_2,s_3}}(\gamma g)$$

where

$$\varphi_{s_1 \otimes E^{1,1}_{s_2,s_3}}\begin{pmatrix} A & * \\ & D \end{pmatrix} = |\det A|^{s_1} \cdot E^{1,1}_{s_2,s_3}(D) \qquad (\text{with } A \in GL_{r-2} \text{ and } D \in GL_2)$$

and $E^{1,1}_{s_2,s_3}$ is the GL_2 Eisenstein series

$$E^{1,1}_{s_2,s_3}(g) = \sum_{\gamma \in P_k^{1,1} \backslash GL_2(k)} \varphi_{s_2,s_3}(\gamma g) \qquad \left(\text{with } \varphi_{s_2,s_3}\begin{pmatrix} a & * \\ & d \end{pmatrix} = |a|^{s_2}|d|^{s_3}\right)$$

Since

$$|a|^{s_2}|d|^{s_3} = |ad|^{\frac{s_2+s_3}{2}} \cdot |a/d|^{\frac{s_2-s_3}{2}}$$

this GL_2 Eisenstein series can be expressed in terms of an Eisenstein series with trivial central character, namely

$$E^{1,1}_{s_2,s_3}(g) = |\det g|^{\frac{s_2+s_3}{2}} \cdot E_{\frac{s_2-s_3}{2}}(g)$$

where

$$E_s(g) = \sum_{\gamma \in P_k^{1,1}\backslash GL_2(k)} \varphi_s(\gamma g) \qquad \text{with} \qquad \varphi_s\begin{pmatrix} a & * \\ & d \end{pmatrix} = |a/d|^s$$

From the functional equation

$$\xi(2s)\cdot E_s = \xi(2s-1)\cdot E_{1-s}$$

the $P^{1,1}$ Eisenstein series $E^{1,1}_{s_2,s_3}$ has a functional equation under

$$(s_2,s_3) = (\tfrac{1}{2},-\tfrac{1}{2}) + (s_2-\tfrac{1}{2}, s_3+\tfrac{1}{2}) \to (\tfrac{1}{2},-\tfrac{1}{2}) + (s_3+\tfrac{1}{2}, s_2-\tfrac{1}{2}) = (s_3+1, s_2-1)$$

given by

$$E^{1,1}_{s_2,s_3} = \frac{\xi(s_2-s_3-1)}{\xi(s_2-s_3)} \cdot E^{1,1}_{s_3+1,s_2-1}$$

Thus,

$$E^{r-2,1,1}_{s_1,s_2,s_3} = \frac{\xi(s_2-s_3-1)}{\xi(s_2-s_3)} \cdot E^{r-2,1,1}_{s_1,s_3+1,s_2-1}$$

[10.2] Pole at $s_1 - s_2 = r-1$

There is another iterated sum expression

$$E^{r-2,1,1}_{s_1,s_2,s_3}(g) = \sum_{\gamma \in P_k^{r-1,1}\backslash G_k} \varphi_{E^{r-2,1}_{s_1,s_2}\otimes s_3}(\gamma g)$$

where

$$\varphi_{E^{r-2,1}_{s_1,s_2}\otimes s_3}\begin{pmatrix} A & * \\ & d \end{pmatrix} = E^{r-2,1}_{s_1,s_2}(A) \qquad \text{(with } A \in GL_{r-1} \text{ and } d \in GL_1\text{)}$$

and $E^{r-2,1}_{s_1,s_2}$ is the GL_{r-1} Eisenstein series

$$E^{r-2,1}_{s_1,s_2}\begin{pmatrix} A & * \\ & d \end{pmatrix} = \sum_{\gamma \in P_k^{r-2,1}\backslash GL_{r-1}(k)} \varphi_{s_1,s_2}(\gamma g) \qquad \left(\text{with } \varphi_{s_1,s_2}\begin{pmatrix} A & * \\ & d \end{pmatrix} = |A|^{s_1}|d|^{s_2}\right)$$

Since

$$|A|^{s_1}|d|^{s_2} = \left|\frac{\det A}{d^{r-2}}\right|^{\frac{s_1-s_2}{r-1}} \cdot |\det A\cdot d|^{\frac{(r-2)s_1+s_2}{r-1}}$$

we can express $E^{r-2,1}_{s_1,s_2}$ in terms of an Eisenstein series with trivial central character, as

$$E^{r-2,1}_{s_1,s_2}(h) = |\det h|^{\frac{(r-2)s_1+s_2}{r-1}} \cdot E_{\frac{s_1-s_2}{r-1}}(g)$$

where

$$E_s(h) = \sum_{P_k^{r-2,1}\backslash GL_{r-1}(k)} \varphi_s(\gamma h)$$

with

$$\varphi_s\begin{pmatrix} A & * \\ & d \end{pmatrix}(g) = \left|\frac{\det A}{d^{r-2}}\right|^s \qquad \text{(for } A \in GL_{r-2} \text{ and } d \in GL_1\text{)}$$

From the previous appendix, the Eisenstein series E_s for $P^{r-2,1}$ has a pole at $s = 1$ with constant residue

$$\mathrm{Res}_{s=1} E_s = \frac{\kappa \cdot |\mathfrak{d}|^{\frac{r-1}{2}}}{(r-1) \cdot \xi(r-1)}$$

Thus, at $\frac{s_1 - s_2}{r-1} = 1$, that is, at $s_1 - s_2 = r - 1$, $E^{r-2,1}_{s_1,s_2}(h)$ has residue

$$\mathrm{Res}_{s_1-s_2=r-1} E^{r-2,1}_{s_1,s_2}(h) = |\det h|^{\frac{(r-2)s_1+(s_1-(r-1))}{r-1}} \cdot \frac{\kappa \cdot |\mathfrak{d}|^{\frac{r-1}{2}}}{(r-1) \cdot \xi(r-1)} = |\det h|^{s_1-1} \cdot \frac{\kappa \cdot |\mathfrak{d}|^{\frac{r-1}{2}}}{(r-1) \cdot \xi(r-1)}$$

Thus, since

$$|\det A|^{s_1} \cdot |a|^{s_1-(r-1)} = |\det \begin{pmatrix} A & \\ & a \end{pmatrix}|^{s_1-1} \cdot \left| \frac{\det A}{a^{r-2}} \right|^1 \qquad (\text{for } A \in GL_{r-2} \text{ and } a \in GL_1)$$

the residue is

$$\mathrm{Res}_{s_1-s_2=r-1} E^{r-2,1,1}_{s_1,s_2,s_3} = \frac{\kappa \cdot |\mathfrak{d}|^{\frac{r-1}{2}}}{(r-1) \cdot \xi(r-1)} \cdot E^{r-1,1}_{s_1-1,s_3}$$

[10.3] Residue at $s_1 - s_3 = r$

The functional equation in s_2 and s_3 (from GL_2) is

$$E^{r-2,1,1}_{s_1,s_2,s_3} = \frac{\xi(s_2 - s_3 - 1)}{\xi(s_2 - s_3)} \cdot E^{r-2,1,1}_{s_1,s_3+1,s_2-1}$$

This functional equation and the pole of $E^{r-2,1,1}_{s_1,s_2,s_3}$ at $s - 1 - s_2 = r - 1$ give a pole of $E^{r-2,1,1}_{s_1,s_2,s_3}$ at $s_1 - s_3 = r$ with residue

$$\mathrm{Res}_{s_1-s_3=r} E^{r-2,1,1}_{s_1,s_2,s_3} = \mathrm{Res}_{s_1-s_3=r} \left(\frac{\xi(s_2 - s_3 - 1)}{\xi(s_2 - s_3)} \cdot E^{r-2,1,1}_{s_1,s_3+1,s_2-1} \right)$$

$$= \frac{\kappa \cdot |\mathfrak{d}|^{\frac{r-1}{2}}}{(r-1) \cdot \xi(r-1)} \cdot \frac{\xi(s_2 - s_3 - 1)}{\xi(s_2 - s_3)} \cdot E^{r-1,1}_{s_1-1,s_2-1}$$

$$= \frac{\kappa \cdot |\mathfrak{d}|^{\frac{r-1}{2}}}{(r-1) \cdot \xi(r-1)} \cdot \frac{\xi(s_2 - (s_1 - r) - 1)}{\xi(s_2 - (s_1 - r))} \cdot E^{r-1,1}_{s_1-1,s_2-1}$$

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