Stability conditions (C. Voisin)

$\mathcal{D}$:= triang. category

I. Slicing
II. Stability cond.
III. Structure of Stab $\mathcal{D}$

Fix $K_0(\mathcal{D}) \to \Lambda$ ← free f.gen. ab. gp.

I. Slicing

Def Slicing of $\mathcal{D}$ is:

$\forall \phi \in \mathbb{R}$, $P(\phi) \subseteq \mathcal{D}$ full subcategory st.

(1) $A \in P(\phi)$, $B \in P(\psi)$, $\phi > \psi \Rightarrow Hom(A, B) = 0$

(2) $P(\phi + 1) = P(\phi)[1]$

(3) $\forall E \in \mathcal{D}$, $\exists$ triangles

$$\begin{tikzcd}
E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{m-1} \arrow[r] & E_m \arrow[d] \\
& A_1 \arrow[ul, "{[1]}"] & & & A_m \arrow[ul, "{[1]}"]
\end{tikzcd}$$

w/ $A_i \in P(\phi_i)$ & $\phi_1 > \cdots > \phi_m$.

Slicing $\rightsquigarrow$ t-structure (bounded)

Def t-structure on $\mathcal{D}$ is data of $\mathcal{F} \subseteq \mathcal{D}$ full additive w/

(1) $\mathcal{F}[1] \subseteq \mathcal{F}$

(2) $\mathcal{F}^{\perp} := \{ B \in \mathcal{D} : \mathrm{Hom}(A, B) = 0 \ \forall A \in \mathcal{F} \}$

$\forall E \in \mathcal{D}$, $\exists$ triangle

$$A \to E \to B$$

where $A \in \mathcal{F}$, $B \in \mathcal{F}^{\perp}$.

Heart: $\mathcal{A} = \mathcal{F} \cap \mathcal{F}^{\perp}[1]$

Lem (BBD) $\mathcal{A}$ abelian category & ex. seq. in $\mathcal{A}$ are triangles in $\mathcal{D}$ w/ vertices in $\mathcal{A}$. $\blacktriangle$

Properties: $\forall A \in \mathcal{A}[i]$, $B \in \mathcal{A}[j]$

$$\mathrm{Hom}(A, B) = 0 \quad \forall i > j. \qquad [*]$$

- **Prop** $\mathcal{A} \subseteq \mathcal{D}$ ab. sub. w/ $[*]$ is heart of t-str. iff $\forall E \in \mathcal{D}$, $\exists$ triangles

$$\begin{tikzcd} 0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_m = E \arrow[dl] \\ A_1 \arrow[u, dashed, "{[1]}"] & & A_m \arrow[u, dashed, "{[1]}"] & \end{tikzcd}$$

w/ $A_i \in \mathcal{A}[a_i]$.

$a_1 > \cdots > a_m$

If P slicing $\rightsquigarrow$ $\forall\, I \subseteq \mathbb{R}$ interval,

$$P(I) := \{ \text{full sub. stable under ext'n \& gen. by } P(\phi),\ \phi \in I \}$$

$\rightsquigarrow$ $\mathcal{F} := P((0;+\infty))$

$\mathcal{A} := P((0;1])$.

II Stability condition

Λ = fixed free f.gen. gp.

$\mathcal{A}$ = heart of bdd t-str.   Rmk $K_0(\mathcal{A}) = K_0(\mathcal{D})$

$v : \mathcal{A} \to \Lambda$
$\quad\ \ \mathcal{D} \to$

Def $Z : \Lambda \to \mathbb{C}$ morph. of ab. gps.
s.t.

$\forall A \in \mathcal{A}$, $\underbrace{Z(v(A))}_{\text{central charge}} \in \mathbb{R}_{>0} \cdot e^{i\pi\varphi(A)}$, $\varphi(A) \in (0;1]$.

$\varphi(A)$ = phase

Given Z on $\mathcal{A}$ $\rightsquigarrow$ notion of <u>semistable</u> obj :

$A \in \mathcal{A}$ <u>semistable</u> if

$\forall\ 0 \neq B \subsetneq A$,

$\phi(B) \leq \phi(A)$

<u>Def</u> $(Z, \mathcal{A})$ has HN property if

$\forall A \in \mathcal{A}$, $\exists$ HN <u>filtration</u>

$0 = A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_n = A$

st. $B_i := A_i / A_{i-1}$ <u>semistable</u> of phase ϕ_i

and $\phi_1 > \cdots > \phi_n$.

<u>Lem</u> Z has HN iff $\forall A \in \mathcal{A}$

(i) $\nexists$ ∞ chain $\cdots \hookrightarrow A_i \hookrightarrow A_{i-1} \hookrightarrow \cdots \hookrightarrow A_0 = A$
w/ $\phi(A_i) > \phi(A_{i-1})$ ~~max~~ $\forall i$.

(ii) $\nexists$ ∞ chain $A = Q_0 \twoheadrightarrow Q_1 \twoheadrightarrow Q_2 \twoheadrightarrow \cdots$
w/ $\phi(Q_i) < \phi(Q_{i+1})$ $\forall i$.

Pf: (i) $\Rightarrow \exists A_0 \subseteq A$ max

(ii) $\Rightarrow$ if replace A w/ $A/A_0 \rightsquigarrow$ terminate.

$\blacktriangle$

Def Stability condition on $\mathcal{D}$ is $\sigma = (Z, P)$ where

- P slicing of $\mathcal{D}$
- $Z : \Lambda \to \mathbb{C}$ gp. homom st.

$$\forall E \in P(\phi), \quad Z(E) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi}$$

Prop ~~[illegible]~~ A stab. cond on $\mathcal{D}$ is the same as

- heart of bd t-str. $\mathcal{A}$ on $\mathcal{D}$
- $Z : \Lambda \to \mathbb{C}$ st. funct. on $\mathcal{A}$ w/ HN property

t-structure + stab. function

$\Downarrow$

$\forall \phi \in (0;1]$

$P(\phi) := \{A \in \mathcal{A} : A$ st. of phase $\phi\}$ abelian categ.

$\rightsquigarrow P(\phi+1) = P(\phi)[1] \ldots$

$\blacksquare$

III. Topological aspects

Distance on space of slicings:

$E \in \mathcal{D} \rightsquigarrow \phi_1 > \dots > \phi_m$ & triangles

$$\phi^+(E) := \phi_1 \quad , \quad \phi^-(E) := \phi_m .$$

$$d(P,Q) := \sup_{0 \neq E} \left\{ |\phi_P^+(E) - \phi_Q^+(E)| \, , \, |\phi_P^-(E) - \phi_Q^-(E)| \right\} \in [0;+\infty]$$

<u>Lem</u> $d(P,Q) = \sup_{E \in P(\phi)} \left\{ |\phi_Q^+(E) - \phi| \, , \, |\phi_Q^-(E) - \phi| \right\}$ $\blacktriangle$

<u>Cor</u> d <u>quasi-distance</u>, i.e., $d(P,Q) = 0 \Leftrightarrow P = Q$. $\blacksquare$

$$\underset{\text{dist. top.}}{\text{Stab}} \subseteq \text{Slicings} \times \text{Hom}(\Lambda, \mathbb{C})$$

dist. top. (on Slicings), std. topology (on $\text{Hom}(\Lambda, \mathbb{C})$)

$\rightsquigarrow$ <u>induced topology</u> on Stab

<u>Rmk</u> f: $\text{Stab} \longrightarrow \text{Hom}(\Lambda, \mathbb{C})$ <u>continuous</u>.

$(Z,P) \longmapsto Z$

$\blacktriangle$

Fact f. local homeo onto its image.

Lem f locally inj; more precisely, if

$\sigma=(Z,P)$ & $d(P,Q)<1 \Rightarrow \sigma=\tau$.
$\tau=(Z,Q)$

▲

More gen., Lem. $d(P,Q)<\frac{1}{2}$ and $\forall E \in \mathcal{A}$

$$\frac{|W(v(E))-Z(v(E))|}{|Z(v(E))|}<\sin(\pi\varepsilon)$$

Then $d(P,Q)<\varepsilon$.

▲

Action of $\widetilde{Gl(2,\mathbb{R})^+}$ - univ. cover of $Gl(2,\mathbb{R})^+$

on $Stab(\mathcal{D})$.

Support property :

$Q: \Delta_{\mathbb{R}} \to \mathbb{R}$ quadratic form

Say Z satisfies supp. prop. if

(i) Q neg. def. on $\ker Z$
(ii) $Q(v(E)) \geq 0$, $\forall E \in P(\phi)$,

Thm (BMS)
If (P, Z) stab. cond. satisf. supp. property wrt Q.
then

$$f: \mathrm{Stab} \to \mathrm{Hom}(\Lambda, \mathbb{C})$$

is local homeom. ~~onto~~ of image containing all Z' st. Q neg. def. on $\ker Z'$. ∎

Cor Stab complex manifold, of $\dim = \mathrm{rk}\, \Lambda$. ∎

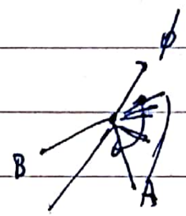
Sketch Pf of Lem : Stat: $P = Q$

$E \in P(\phi) \rightsquigarrow$ want $E \in Q(\phi)$.

If not, $\exists\ A \in Q(>\phi),\ B \in Q(<\phi)$ &

$$A \to E \to B$$

since $E \in P(\phi) \Rightarrow E \in Q((\phi-1, \phi+1))$



$\rightsquigarrow$ can have $A \in Q((\phi, \phi+1))$

Wrt P : Claim $A \notin P(\leq \phi)$

$$\rightsquigarrow \exists\ C \to A \to E$$

$$\rightsquigarrow C \to B[-1]$$ factorize

impossible, since $B \in Q(<\varphi) \Rightarrow B[-1] \in Q(\leq \varphi - 1) \subseteq P(\leq \varphi)$

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