Base Change (after Kuznetsov and Abramovich-Polishchuk)

Motivation:

$$
\begin{array}{lll}
X \text{ variety} & \rightsquigarrow & X \times S \\
\wr & & \\
coh(X) & \rightsquigarrow & coh(X \times S) \\
D^b X & \rightsquigarrow & D^b(X \times S)
\end{array}
$$

$$
\begin{array}{lll}
\mathcal{D} \text{ triang. cat.} & \rightsquigarrow & \mathcal{D}_S = \text{"}\mathcal{D} \boxtimes D^b S\text{"} \quad ? \\
\cup\!\! \mid & & \cup\!\! \mid \\
\mathcal{A} \text{ heart of bd t-str. (noeth.)} & \rightsquigarrow & \mathcal{A}_S \text{ heart of bd t-str. (noeth.)} \quad ?
\end{array}
$$

Why? (1) HPD (last time)

w/ Lefschetz deco. (may not be $D^b X$!) $\rightsquigarrow \mathcal{D}_{\ } \underset{HPD}{\rightsquigarrow} \mathcal{D}^\vee$

define this!

Want: Compare $\mathcal{D}_L$ and $\mathcal{D}^\vee_{L^\vee}$ and their Kuznetsov components

(2) Moduli spaces: S curve

$\exists\ \mathcal{A}_S \Rightarrow$ semistable reduction for stable objs. in $\mathcal{A}$ $\Rightarrow$ moduli spaces are proper

1. Triangulated categories

(Simplified) setting:

- X, S smooth proj / $\mathbb{C}$
- $D^bX = \langle \mathcal{D}_1, \dots, \mathcal{D}_m \rangle$ sod

Want: $D^b(X \times S) = \langle \mathcal{D}_{1,S}, \dots, \mathcal{D}_{m,S} \rangle$

Ex $D^bX = \langle E_1, \dots, E_m \rangle$ full exc. coll.

$\rightsquigarrow D^b(X \times S) = \langle \underbrace{E_1 \boxtimes D^bS}_{\cong D^bS}, \dots, \underbrace{E_m \boxtimes D^bS}_{\cong D^bS} \rangle$

Indeed: • orthogonality: $\forall A, B \in D^bS$, $\forall i > j$, $\forall m$

$$\mathrm{Ext}^m_{X \times S}(E_i \boxtimes A, E_j \boxtimes B) \underset{\text{Künneth}}{\cong} \bigoplus_{p+q=m} \underbrace{\mathrm{Ext}^p_X(E_i, E_j)}_{0} \otimes \mathrm{Ext}^q_S(A, B)$$

$$= 0$$

• fullness: Each category $E_i \boxtimes D^bS$ is admissible

$\rightsquigarrow$ take the left orthogonal

$$\mathcal{C}_S := \langle E_1 \boxtimes D^bS, \dots, E_m \boxtimes D^bS \rangle^{\perp}$$

$\Phi : D^bS \to D^b(X \times S)$
$A \mapsto E_i \boxtimes A$

$\Phi^R : D^b(X \times S) \to D^bS$
$B \mapsto \underbrace{p_{S*}\mathcal{H}om(p_X^* E_i, B)}_{= \mathcal{H}om_S}$

But, $\forall F \in D^b(X \times S)$,

$\exists$ morph. $P_X \boxtimes P_S \to F$ $\neq 0$

and $P_X \boxtimes P_S \in \langle E_1 \boxtimes D^bS, \dots, E_m \boxtimes D^bS \rangle$

since $P_X \in \langle E_1, \dots, E_m \rangle$

$\rightsquigarrow \mathcal{C}_S = \emptyset$

argument is general

Def $\mathcal{X}, S$ smooth schemes, $q: \mathcal{X} \to S$

- Say $\mathcal{D} \subseteq D^b\mathcal{X}$ triang. subcat. is S-linear if
  $\forall A \in \mathcal{D}$, $A \otimes q^*F \in \mathcal{D}$, $\forall F \in D_{perf}(S)$

- A sod $D^b\mathcal{X} = \langle \mathcal{D}_1, \dots, \mathcal{D}_m \rangle$ is S-linear if $\mathcal{D}_i$ is, $\forall i = 1, \dots, m$.

Prop Our setting

Set $\mathcal{D}_{i,S} :=$ minimal triang. subcateg. $\subseteq D^b(X \times S)$ idempotent complete and containing $A \boxtimes B$, $\forall A \in \mathcal{D}_i$, $\forall B \in D^b S$ $\quad i = 1, \dots, m$

closed by direct summands

Then
- $\mathcal{D}_{i,S}$ S-linear
- $D^b(X \times S) = \langle \mathcal{D}_{1,S}, \dots, \mathcal{D}_{m,S} \rangle$.

Pf:
- orthogonality: same as in Ex. ✓
- fullness: more complicate, since not clear a priori that $\mathcal{D}_{i,S}$ admissible (!)

Kontsevich's trick: $F \in D^b(X \times S)$
- $X \times S$ proj $\Rightarrow \exists$ bd. above locally-free resol'n $P^\bullet \cong F$
- $X \times S$ sm. $\Rightarrow$ for $m \gg 0$, $\exists$ split triangle
  $$G \to \sigma^{\geq -m} P^\bullet \to F$$
  w/ $G \in D^b(X \times S)$, i.e.,
  $$\sigma^{\geq -m} P^\bullet \cong F \oplus G.$$

stupid truncation

By def'n, $\sigma^{\geq -m} P^\bullet \in \langle \mathcal{D}_{1,S}, \dots, \mathcal{D}_{m,S} \rangle$

Claim $F \in \langle \mathcal{D}_{1,S}, \dots, \mathcal{D}_{m,S} \rangle$

Idea: by induction on m $\rightsquigarrow$ only need $m = 2$.

Take:

$$D_2 \xrightarrow{f} P := \sigma^{\geq -m} P^\bullet \xrightarrow{g} D_1 \quad , \quad D_i \in \mathcal{D}_{i,S}$$

Consider $q : P \to P$ idemp. giving $P \cong F \oplus G$.
$q^2 = q$

By orth.: $g \circ q \circ f = 0$.

$\rightsquigarrow$ $q \circ f$ induces $p : D_2 \to D_2$ idemp.

By def. of D_2 this splits $\rightsquigarrow$ can conclude.

$\blacksquare$

Cor. X, Y sm. proj.

$\mathcal{D}_X \hookrightarrow D^b X$, $\mathcal{D}_Y \hookrightarrow D^b Y$ admissible $\Rightarrow$ $\mathcal{D}_X \boxtimes \mathcal{D}_Y :=$ min. tr. subcat. $\subseteq D^b(X \times Y)$ id. closed and containing $A \boxtimes B$, $A \in \mathcal{D}_X$, $B \in \mathcal{D}_Y$ $\hookrightarrow D^b(X \times Y)$ admissible

"non-comm. product"

$\square$

In partic., $\mathcal{D}_S$ well-def.

base change to S smooth proj. "$\mathcal{D} \boxtimes D^b S$"

$\blacksquare$

ical result :

Thm (Kuznetsov, Perry, BLMNPS)

$\mathcal{X}, S, T$ <u>noetherian</u> schemes (w/ affine diagonal)

($\mathcal{X}$ finite Krull dimension)

$g : \mathcal{X} \to S$ flat

$\phi : T \to S$ morph. (finite Tor-dimension)

$$\begin{tikzcd} \mathcal{X}_T \arrow[r, "\phi'"] \arrow[d, "g'"'] & \mathcal{X} \arrow[d, "g"] \\ T \arrow[r, "\phi"'] & S \end{tikzcd}$$

$D^b\mathcal{X} = \langle \mathcal{D}_1, \dots, \mathcal{D}_m \rangle$ (strong,) <u>S-linear</u> <u>sod</u> (w/ <u>finite cohom. amplitude</u>)

$\mathcal{D}_i \subset D^b\mathcal{X}$ right admissible

proj. functors $pr_i : D^b\mathcal{X} \to \mathcal{D}_i \to D^b\mathcal{X}$

$\exists a, b$ st. $pr_i(D^{[p,q]}\mathcal{X}) \subseteq D^{[a+p, b+q]}\mathcal{X}$ $\forall p, q$

equiv. $\underline{Hom}_S(F, G) = 0$ $\forall F \in \mathcal{D}_i, \forall G \in \mathcal{D}_j$ $\forall i > j$

Then

$D^b\mathcal{X}_T = \langle \mathcal{D}_{1,T}, \dots, \mathcal{D}_{m,T} \rangle$ (strong,) T-linear sod (w/ f. coh. ampl.)

and the functors ϕ'_* and $\phi^{*\prime}$ are <u>compatible w/ the decomp.</u>

<u>Idea Pf</u> :

$$\begin{tikzcd} D^b\mathcal{X} \arrow[r, squiggly] & D_{qc}\mathcal{X} \arrow[r, squiggly] & D_{perf}\mathcal{X} \arrow[d, squiggly] \\ D^b\mathcal{X}_T & D_{qc}\mathcal{X}_T \arrow[l, squiggly] & D_{perf}\mathcal{X}_T \arrow[l, squiggly] \end{tikzcd}$$

similar def'n as in the smooth case ...

$\boxtimes$

. .

2. t-structures

(Simplified) setting:

- X sm. proj $/\mathbb{C}$
- $\mathcal{D} \hookrightarrow D^bX$ admissible subcategory
- $\mathcal{A} \subseteq \mathcal{D}$ heart of bd t-structure, noetherian

Want: $\mathcal{A}_{\mathbb{P}^n} \subseteq \mathcal{D}_{\mathbb{P}^n}$ heart of bd t-structure, noetherian, $\forall n \geq 1$.
$\mathbb{P}^n$-local

Ex $X = \mathbb{P}^1$ $\mathcal{D} = D^b\mathbb{P}^1$

hearts of bd t-str.'s: (∞) $\mathrm{coh}\,\mathbb{P}^1 \subseteq D^b\mathbb{P}^1$

(m) $\mathcal{A}_m := \langle \mathcal{O}(-m-1)[1], \mathcal{O}(-m) \rangle \subseteq D^b\mathbb{P}^1$ $\forall m \in \mathbb{Z}$

ext'n closed

$\mathcal{A}_m \simeq \mathrm{mod}\text{-}A$, $A = \left\{ \begin{pmatrix} a & bx+cy \\ 0 & d \end{pmatrix} : a,b,c,d \in \mathbb{C} \right\}$

"t-structure" $= \mathrm{End}(\mathcal{O}(-m) \oplus \mathcal{O}(-m+1))$

$D^b\mathcal{A}_m \cong D^b\mathbb{P}^1$ [Beilinson]

Rmk $\lim_{m \to \infty} \mathcal{A}_m = \mathcal{A}_\infty = \mathrm{coh}\,\mathbb{P}^1$

Fix $m_0 \in \mathbb{Z}$ $\rightsquigarrow$ $(\mathcal{A}_{m_0})_{\mathbb{P}^1} = ?$ $\sim$ "$\mathcal{A}_{m_0} \boxtimes \mathrm{coh}\,\mathbb{P}^1$"

Idea: • $\mathcal{D}_{\mathbb{P}^1} = D^b(\mathbb{P}^1 \times \mathbb{P}^1)$

• $\mathcal{A}_{m_0,m} = \langle \mathcal{O}(-m_0-1,-m-1)[2], \mathcal{O}(-m_0-1,-m)[1], \mathcal{O}(-m_0,-m-1)[1], \mathcal{O}(-m_0,-m) \rangle \subseteq D^b(\mathbb{P}^1 \times \mathbb{P}^1)$

strong exc. collection (!)

$\cong \mathrm{mod}\text{-}\tilde{A}$, $\tilde{A} = \mathrm{End}(\dots)$

- $(\mathfrak{K}_{m_0})_{\mathbb{P}^1} := \lim\limits_{m\to+\infty} \mathfrak{K}_{m_0,m} = \{F\in D^b(\mathbb{P}^1\times\mathbb{P}^1) : (p_1)_*(F\otimes p_2^*\mathcal{O}(m))\in \mathfrak{K}_{m_0} \ \forall m \gg 0\}$

- $E\in(\mathfrak{K}_{m_0})_{\mathbb{P}^1} \rightsquigarrow E\otimes p_2^*\mathcal{O}(m)\in(\mathfrak{K}_{m_0})_{\mathbb{P}^1}, \ \forall m\in\mathbb{Z}$

  "$\mathbb{P}^1$-local"

Def $\mathcal{D}$ triang. categ.

$\mathfrak{A}\subseteq\mathcal{D}$ full additive subcategory

Say $\mathfrak{A}$ heart of bounded t-structure if

(1) $\forall F,G\in\mathfrak{A}$, $\mathrm{Hom}(F,G[-k])=0$, $\forall k>0$

(2) $\forall E\in\mathcal{D}$, $\exists\ m_E\leqslant M_E$ and $\exists$ coll. of triangles

$$\begin{tikzcd}[column sep=small]
0=E_{m_E-1} \arrow[rr] & & E_{m_E} \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{M_E-1} \arrow[rr] & & E_{M_E} = E \arrow[dl] \\
& A_{m_E} \arrow[ul, "{[1]}"] & & & & A_{M_E} \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

s.t. $A_i[i]\in\mathfrak{A}$ "cohomology objects"

Prop $\mathfrak{A}$ heart. Then

(1) $\mathfrak{A}$ abelian

(2) $\mathcal{D}^{\leq 0} := \{E\in\mathcal{D} : M_E\leq 0\}$

$\mathcal{D}^{\geq 0} := \{E\in\mathcal{D} : m_E\geq 0\}$

$\rightsquigarrow$ • $\forall F\in\mathcal{D}^{\leq 0}$, $\forall G\in\mathcal{D}^{\geq 1}$, $\mathrm{Hom}(F,G)=0$

• $\forall E\in\mathcal{D}$, $\exists$ triangle

$$\tau^{\leq 0}E\to E\to\tau^{\geq 1}E$$

st. $\tau^{\leq 0}E \in \mathcal{D}^{\leq 0}$, $\tau^{\geq 1} \in \mathcal{D}^{\geq 0}[-1]$ ← t-structure

- $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$

- $\mathcal{D} = \bigcup_{n,m \in \mathbb{Z}} \mathcal{D}^{\leq 0}[-n] \cap \mathcal{D}^{\geq 0}[-m]$ ← bounded

■

Ex $\mathcal{D} \cong D^b\mathcal{A}$, $\mathcal{A}$ abelian $\Rightarrow$ $\mathcal{A}$ heart of bd t-structure

$\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}$ = complexes w/ only positive/negative cohomologies

e.g., $\mathcal{A}_n \subseteq D^b\mathbb{P}^1$, $\forall n \in \mathbb{Z} \cup \{\infty\}$

▲

Def $\mathcal{A} \subseteq \mathcal{D}$ heart
Say $\mathcal{A}$ noetherian if $\forall A \in \mathcal{A}$ and

$\forall \; A_0 \subseteq A_1 \subseteq \_\_ \subseteq A$ in $\mathcal{A}$

$\exists \; j_A \geq 0$ st. $A_i = A_{j_A}$, $\forall i \geq j_A$.

Def $\mathcal{X}, S$ noeth. (w/ affine diagonal)
($\mathcal{X}$ finite Krull dim.)
$g : \mathcal{X} \to S$ flat
$\mathcal{D} \hookrightarrow D^b\mathcal{X}$ S-linear(right admissible w/ finite cohom. amplitude)
$\mathcal{A} \subseteq \mathcal{D}$ heart of bd t-structure

Say $\mathfrak{A}$ <u>S-local</u> if, $\forall U \overset{j}{\hookrightarrow} S$ <u>open</u>,

$\exists\, \mathfrak{A}_U \subseteq \mathcal{D}_U$ <u>heart</u> of bd t-structure st.

$(j')^*: \mathfrak{A} \to \mathfrak{A}_U$ <u>t-exact</u>

e.g. S q.proj.

Prop Assume $\exists$ enough line bdles on S.

Then $\mathfrak{A}$ S-local iff $\otimes g^* L$ <u>t-exact</u>, $\forall L$ line bdle / S.

e.g., always true if S affine

main ingredient in the pf of general thm we will see!

Prop Our setting, $r \geq 1$.

Set

$$\mathfrak{A}_{\mathbb{P}^r} := \{ F \in \mathcal{D}_{\mathbb{P}^r} : p_{\mathcal{D}*}(F(n)) \in \mathfrak{A}, \forall n \gg 0 \}$$

Then $\mathfrak{A}_{\mathbb{P}^r}$ heart of bd t-str., noetherian, $\mathbb{P}^r$-local

Pf: • $D^b\mathbb{P}^r = \langle \mathcal{O}(-r-n), \ldots, \mathcal{O}(-n) \rangle$ full strong exc. collection

$$\rightsquigarrow \mathcal{D}_{\mathbb{P}^r} = \langle \mathcal{D} \boxtimes \mathcal{O}(-r-n), \ldots, \mathcal{D} \boxtimes \mathcal{O}(-n) \rangle$$

• $\mathfrak{A}_n := \langle \mathfrak{A} \boxtimes \mathcal{O}(-r-n)[r], \ldots, \mathfrak{A} \boxtimes \mathcal{O}(-n) \rangle \subseteq \mathcal{D}_{\mathbb{P}^r}$

<u>ext. closure</u>

$$= \left\{ F \in \mathcal{D}_{\mathbb{P}^r} : \begin{array}{c} p_{\mathcal{D}*}(F(n)) \in \mathfrak{A} \\ \vdots \\ p_{\mathcal{D}*}(F(n+r)) \in \mathfrak{A} \end{array} \right\}$$

heart of bd t-str., noetherian, $\forall n \in \mathbb{Z}$.

use "gluing" (!)

- $\mathcal{H}_{pr} := \lim_{n\to\infty} \mathcal{H}_n$

heart of bd t-str., noetherian, $\mathbb{P}^r$-local

use noetherianity (!) (to stabilize cohomology objs...)

analogue of Hilbert basis thm.

ok, by Prop before!

■

Thm (Abramovich-Polishchuk, Perry, BLMNPS)

$\mathcal{X}, S, T$ noeth. (w/ affine diag.)

($\mathcal{X}$ finite Krull dimension)

$g : \mathcal{X} \to S$ flat (essentially of) finite-type ($\leftarrow$ actually ess. perfect)

$\phi : T \to S$ morph. (finite Tor dim.)

$\mathcal{D} \hookrightarrow D^b\mathcal{X}$ S-linear (, strong w/ finite cohom. amplitude)

$\mathcal{H} \subseteq \mathcal{D}$ heart of | bd t-structure S-local (tilted-) noetherian (locally on S).

Then: (1) $\exists\, \mathcal{H}_T \subseteq \mathcal{D}_T$ | heart of T-local bd t-str. (tilted-) noeth. (loc on T)

(2) If ϕ proj, L ϕ-ample line bdle on T, then

$$\mathcal{H}_T = \{ F \in \mathcal{D}_T : \phi'_*(F \otimes g'^* L^n) \in \mathcal{H},\ \forall n \gg 0 \}$$

indep. on L a posteriori ...

(3) $\psi : T' \to S$ morph. (finite Tor dim.)

$f : T' \to T$ S-morph. ...

Then

- $f'^* : \mathcal{D}_T \to \mathcal{D}_{T'}$ right t-exact
- f flat $\Rightarrow$ f'^* t-exact
- f finite $\Rightarrow$ f'_* t-exact

■

properness

X sm. proj. variety $/\mathbb{C}$

$\mathcal{D} \hookrightarrow D^b X$ admissible

$\mathcal{A} \subseteq \mathcal{D}$ heart of bd t-str., noeth.

(DVR ess. of finite type...)

C sm. q.proj. curve $\rightsquigarrow$ $\mathcal{A}_C \subseteq \mathcal{D}_C$ noeth

Def $F \in \mathcal{A}_C$ any S integral

- F C-torsion if F push-forward of an obj. in $\mathcal{A}_Z$, for $Z \subsetneq C$ proper closed subscheme
- F C-torsion free if $\nexists\, F' \subseteq F$, F' C-torsion. ($F' \neq 0$)
- F C-flat if $F_c \in \mathcal{A}_c$, $\forall c \in C$

derived restriction

Lem 1 F C-flat iff F C-torsion free

usual pf...

▲

Lem 2 $F \in \mathcal{D}_C$, $c_0 \in C$.

$$F_{c_0} \in \mathcal{A}_{c_0} \Rightarrow \exists\, U \underset{\text{op.}}{\subseteq} C,\ c_0 \in U \text{ st. } \begin{cases} \cdot\ F_U \in \mathcal{A}_U \\ \cdot\ F_c \in \mathcal{A}_c,\ \forall c \in U \end{cases}$$

openness of heart

openness of flatness

Pf: Take cohom. & use Lem 1

▲

Cor (1) $F \in \mathcal{D}_C$ st. $F_c \in \mathcal{A}_c$ $\forall c \in C$ $\Rightarrow$ $F \in \mathcal{A}_C$ C-flat.

(2) $C' = C \setminus \{c_0\}$, $E_{C'} \in \mathcal{A}_{C'}$ C'-flat. Then $\exists\, E \in \mathcal{A}_C$ C-flat st. $E|_{C'} = E_{C'}$.

valuative crit. for properness, first part...

Pf: (1) Sheaf property

(2) $\exists \widetilde{E} \in \mathcal{D}_C$ st. $\widetilde{E}|_{C'} \cong E'_C$ $\rightsquigarrow$ take cohom. & C-torsion free part.

∎

Extra: application to moduli spaces on curves.

C sm. proj. curve / $\mathbb{C}$

(r,d) coprime $\rightsquigarrow$ $M := M_C(r, L)$ moduli space of stable v.bdles

w/ $rk = r$, $\det = L$, $\deg L = d$

$\mu := \frac{d}{r}$

$D^bC \rightsquigarrow \operatorname{coh} C \subseteq D^bC$ heart

$\operatorname{coh}^{\mu} C$ = tilt at μ

Using $\operatorname{coh}^{\mu} C$ + base-change $\rightsquigarrow$
- moduli space proper alg. space
- theta-divisor is strictly nef

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