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Kuznetsov components and some examples of fractional Calabi-Yau categories.

Everything in this talk, unless otherwise mentioned, is taken from Alexander Kuznetsov's article: "Calabi-Yau and fractional Calabi-Yau categories".

Recall that if X is a smooth projective variety of dimension n, then X has a Serre functor, namely: $S_X = \omega_X[n] \otimes$ — where ω_X is the canonical bundle.

In particular, if X is Calabi-Yau then $S_X = [\dim X]$. This motivates the following definitions:

Def: let $\mathcal{C}$ be a triangulated category.

1) $\mathcal{C}$ is a n-Calabi-Yau category if it has a Serre functor $S_{\mathcal{C}}$ and if there exists $n \in \mathbb{Z}$ s.t. $S_{\mathcal{C}} = [n]$.

Such a n is called the Calabi-Yau dimension of $\mathcal{C}$.

2) $\mathcal{C}$ is a fractional Calabi-Yau category if $\mathcal{C}$ has a Serre functor $S_{\mathcal{C}}$ and there exists $p \in \mathbb{Z}$, $q \in \mathbb{Z}^*$ s.t. $S_{\mathcal{C}}^q \cong [p]$.

The main theorem of Kuznetsov's article gives several examples of CY-categories and fractional CY-categories, which will be pieces in the semi-orthogonal decomposition of some varieties.

Why are we interested in such categories?

- If X is a variety with a semi-orthogonal component which is a 2-CY-category, then any moduli space of coherent sheaves on X has a closed 2-form. In some cases this gives interesting hyper-kähler varieties.
- More generally, some geometric properties of X can be deduced from the existence of a fractional CY-component in its semi-orthogonal decomposition.

② Notations and formulas:

- Let ϕ be a functor. We will denote by ϕ^* and $\phi^!$ its left, respectively right adjoints, if they exist. Then we will denote:

$$\eta_{\phi,\phi^*} : id \longrightarrow \phi\circ\phi^* \qquad \varepsilon_{\phi^*,\phi} : \phi^*\circ\phi \longrightarrow id.$$

Moreover
$$(\phi\circ\varepsilon_{\phi^*,\phi})\circ(\eta_{\phi,\phi^*}\circ\phi) = id \quad (1)$$
$$(\varepsilon_{\phi^*,\phi}\circ\phi^*)\circ(\phi^*\circ\eta_{\phi,\phi^*}) = id \quad (2)$$

ϕ is fully faithful iff $\varepsilon_{\phi^*,\phi}$ is an isomorphism, iff $\eta_{\phi^!,\phi}$ is an isomorphism ("Fourier-Mukai transforms in algebraic geometry", D. Huybrechts p8).

- $\Psi = (\eta_{\phi^!,\phi}\circ\phi^*) + (\phi^!\circ\eta_{\phi,\phi^*}) : \phi^*\oplus\phi^! \longrightarrow \phi^!\circ\phi\circ\phi^*$

  $\Gamma = (\phi^*\circ\varepsilon_{\phi,\phi^!}) + (\varepsilon_{\phi^*,\phi}\circ\phi^!) : \phi^*\circ\phi\circ\phi^! \longrightarrow \phi^*\oplus\phi^!$
- We have the distinguished triangles:

  (3) $T_Y \to id \xrightarrow[\eta_{\phi,\phi^*}]{} \phi\circ\phi^*$ (4) $\phi\circ\phi^! \xrightarrow{\varepsilon_{\phi,\phi^!}} id \longrightarrow T_Y'$

  (5) $\phi^*\circ\phi \xrightarrow[\varepsilon_{\phi^*,\phi}]{} id \longrightarrow T_X$ (6) $T_X' \longrightarrow id \xrightarrow{\eta_{\phi^!,\phi}} \phi^!\circ\phi$

  where $\phi : D^b(X) \to D^b(Y)$ is spherical.
- $\phi\circ T_X \simeq T_Y\circ\phi[2]$ (7) $\phi^*\circ T_Y[2] \simeq T_X\circ\phi^*$ (8)
- $\rho = T_X\circ\mathcal{L}_X^d$ $\sigma = S_X\circ T_X\circ\mathcal{L}_X^m$
- $T_\Pi(\mathcal{B}\otimes\mathcal{L}_\Pi^i) = \mathcal{B}\otimes\mathcal{L}_\Pi^{i-d}$ (9)
- $\mathcal{L}_\Pi\circ\phi \simeq \phi\circ\mathcal{L}_X$ (10)
- $T_X\circ\mathcal{L}_X = \mathcal{L}_X\circ T_X$ (11)
- $S_{\mathcal{A}_X}^{d/c} = \rho^{-m/c}\circ\sigma^{d/c}$ (12)
- $\mathcal{A}_X = \{F\in D^b(X), \phi(F)\in \langle \mathcal{B}\otimes\mathcal{L}_\Pi^{-d}, \dots, \mathcal{B}\otimes\mathcal{L}_\Pi^{-1}\rangle\}$ (13)
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1. Spherical functors

Let X and Y be smooth projective varieties.

Def Let $\phi : D^b(X) \to D^b(Y)$ be a Fourier-Mukai functor.
In particular, ϕ has then a left and a right adjoint.
Then ϕ is called spherical if:

1) $\psi := (\eta_{\phi^!,\phi} \circ \phi^*) + (\phi^! \circ \eta_{\phi,\phi^*}) : [\phi^* \oplus \phi^!](A) \xrightarrow{\sim} \phi^! \circ \phi \circ \phi^*(A)$ is an isomorphism for all $A \in D^b(Y)$.

2) $\Gamma := (\phi^* \circ \varepsilon_{\phi,\phi^!}) + (\varepsilon_{\phi^*,\phi} \circ \phi^!) : \phi^* \circ \phi \circ \phi^!(A) \xrightarrow{\sim} (\phi^* \oplus \phi^!)(A)$ is an isomorphism for all $A \in D^b(Y)$.

Def Let $\phi : D^b(X) \to D^b(Y)$ be a spherical functor. Then we can define the functors T_X, T_X', T_Y, T_Y' as follows:
we have the following distinguished triangles, where we identify a Fourier-Mukai functor to its kernel (i.e. we will write ϕ_P for P), since $id = \phi_{\mathcal{O}_\Delta}$:

$$T_Y \to id \xrightarrow{\eta_{\phi,\phi^*}} \phi \circ \phi^* \qquad \phi \circ \phi^! \xrightarrow{\varepsilon_{\phi,\phi^!}} id \to T_Y'$$

$$\phi^* \circ \phi \xrightarrow{\varepsilon_{\phi^*,\phi}} id \to T_X \qquad T_X' \to id \xrightarrow{\eta_{\phi^!,\phi}} \phi^! \circ \phi$$

Then T_X and T_Y are called spherical twists.

Prop: With the same notations:
T_X, T_X' and T_Y, T_Y' are mutually inverse autoequivalences of $D^b(X)$, respectively $D^b(Y)$.

Proof: Let $\delta : \phi^! \circ \phi \to T_X'[1]$. Composing (6) with ϕ^* on the right, one gets the distinguished triangle (ϕ^* is exact since it is a FM transform):

$$\phi^* \xrightarrow{\eta_{\phi^!,\phi} \circ \phi^*} \phi^! \circ \phi \circ \phi^* \xrightarrow{\delta \circ \phi^*} T_X'[1] \circ \phi^*$$

This implies that $(\delta \circ \phi^*) \circ (\eta_{\phi^!,\phi} \circ \phi^*) = 0 \quad (*)$

Let $\varphi = (\delta \circ \phi^*) \circ (\phi^! \circ \eta_{\phi,\phi^*})$. Then:

$$
\begin{tikzcd}
\phi^* \arrow[r] \arrow[d, "id"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & \phi^* \oplus \phi^! \arrow[r] \arrow[d, "\psi"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & \phi^! \arrow[d, "\varphi"] & (a) \\
\phi^* \arrow[r, "\eta_{\phi^!,\phi} \circ \phi^*"'] & \phi^! \circ \phi \circ \phi^* \arrow[r, "\delta \circ \phi^*"'] & T_X'[1] \circ \phi^* & (b)
\end{tikzcd}
$$

The right square commutes because of $(*)$ and by definition of ψ.
The left square commutes by definition of ψ.
Moreover, (a) and (b) are distinguished triangles and id and ψ are isomorphism, thus φ is an isomorphism.

Moreover, one can get the following diagram:

$$
\begin{tikzcd}
\phi^! \circ \phi \arrow[r, "\phi^! \circ \eta_{\phi,\phi^*} \circ \phi"] \arrow[dr, "\gamma"'] & \phi^! \circ \phi \circ \phi^* \circ \phi \arrow[r, "\delta \circ \phi^* \circ \phi"] \arrow[d, "\phi^! \circ \phi \circ \varepsilon_{\phi^*,\phi}"] \arrow[dl, phantom, "\circlearrowright"] & T_X'[1] \circ \phi^* \circ \phi \arrow[d, "T_X'[1] \circ \varepsilon_{\phi^*,\phi}"] \\
& \phi^! \circ \phi \arrow[r, "\delta"] & T_X'[1]
\end{tikzcd}
$$

The square commutes since we have $\underbrace{\phi^! \circ \phi}_{\delta \circlearrowright} \circ \underbrace{\phi^* \circ \phi}_{\varepsilon_{\phi^*,\phi} \circlearrowleft}$, and $\gamma = id$

since $\gamma = \phi^! [\ \phi \circ \varepsilon_{\phi^*,\phi} \circ \eta_{\phi,\phi^*} \circ \phi] = \phi^! \circ \phi$ by (1).
As a result, $\delta = \delta \circ id = T_X'[1] \circ \varepsilon_{\phi^*,\phi} \circ \underbrace{(\delta \circ \phi^* \circ \phi \circ \phi^! \circ \eta_{\phi,\phi^*} \circ \phi)}_{=: \nu}$.

Therefore the right square in the following diagram is commutative:

$$
\begin{tikzcd}
id \arrow[r, "\eta_{\phi^!,\phi}"] \arrow[d, dashed, "\wr"'] \arrow[dr, phantom, "\circlearrowleft"] & \phi^! \circ \phi \arrow[r, "\delta"] \arrow[d, "\nu"] \arrow[dr, phantom, "\circlearrowright"] & T_X'[1] \arrow[d, "id"] & (c) \\
T_X' \circ T_X \arrow[r] & T_X'[1] \circ \phi^* \circ \phi \arrow[r, "T_X'[1] \circ \varepsilon_{\phi^*,\phi}"'] & T_X'[1] & (d)
\end{tikzcd}
$$

Moreover $\nu = \varphi \circ \phi$ is an isomorphism since φ is an isomorphism.

(5) The triangles (c) and (d) are distinguished, by definition for c) and for d) because it is (5) composed with $T_X'[1]$ on the left. Hence, there exists a map $\varphi : id \to T_X' \circ T_X$ such that the left square commutes. Finally, since id and ν are isomorphisms, so is φ.

Similarly, $T_Y' \circ T_Y \simeq id$. Using τ instead of ψ and a similar reasoning, one also gets $T_X \circ T_X' = id$ and $T_Y \circ T_Y' = id$. ∎

Rem: In "Spherical DG-functors" Rina Anno and Timothy Logvinenko give a different definition of spherical functor:

* T_X and T_X' are quasi-inverse autoequivalences
* T_Y and T_Y' are quasi-inverse autoequivalences
* $\phi^* \circ T_Y'[-1] \longrightarrow \phi^* \circ \phi \circ \phi^! \longrightarrow \phi^!$ is an isomorphism
* $\phi^! \longrightarrow \phi^! \phi \phi^* \longrightarrow T_X' \phi^*[1]$ is an isomorphism.

This definition is equivalent to the one with φ and Γ.

Prop With the same notations, $\phi \circ T_X \simeq T_Y \circ \phi[2]$ and $T_X \circ \phi^* \simeq \phi^* \circ T_Y[2]$.

Proof: Combining $\phi^! \xrightarrow[\varphi]{\sim} T_X'[1] \circ \phi^*$ and $T_X \circ T_X' = id$ one gets:

$$T_X \circ \phi^! \simeq \phi^*[1] \quad (*)$$

Similarly the triangle $id \xrightarrow{\eta_{\phi,\phi^*}} \phi \circ \phi^* \xrightarrow{\tau} T_Y[1]$ is distinguished.
Composing with $\phi^!$ on the left, one gets the distinguished triangle:

$$\phi^! \xrightarrow{\phi^! \circ \eta_{\phi,\phi^*}} \phi^! \circ \phi \circ \phi^* \xrightarrow{\phi^! \circ \tau} \phi^! \circ T_Y[1].$$

In particular, $(\phi^! \circ \tau) \circ (\phi^! \circ \eta_{\phi,\phi^*}) = 0$. $(**)$

Let $\varphi' = \phi^! \circ \tau \circ \eta_{\phi^!,\phi} \circ \phi^*$

Then

$$\begin{tikzcd}
\phi^! \arrow[r] \arrow[d, "id"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & \phi^! \oplus \phi^* \arrow[r] \arrow[d, "\psi"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & \phi^* \arrow[d, "\varphi'"] \\
\phi^! \arrow[r, "\phi^! \circ \eta_{\phi,\phi^*}"'] & \phi^! \circ \phi \circ \phi^* \arrow[r, "\phi^! \circ \tau"'] & \phi^! \circ T_Y[1]
\end{tikzcd}$$

commutes by $(**)$ and the definition of ψ. Hence φ' is an isomorphism and $\phi^* \simeq \phi^! \circ T_Y[1]$. $(***)$

⑥ Finally, $\phi^* \circ T_Y^{-1}[-1] \underset{(***)}{\simeq} \phi^! \underset{(*)}{\simeq} T_X^{-1} \circ \phi^*[1]$ (****)

Composing with T_X on the left and $T_Y[1]$ on the right gives:

$$T_X \circ \phi^* \simeq \phi^* \circ T_Y[2].$$

By Yoneda's lemma a right adjoint is unique upto isomorphism, thus taking right adjoints on both sides in (****) one gets:

$$T_Y \circ \phi[1] \simeq \phi \circ T_X[-1] \implies T_Y \circ \phi[2] \simeq \phi \circ T_X \quad ■$$

2. Kuznetsov's theorem

Let X be a smooth projective variety. Let $\mathcal{M}$ be a smooth projective variety with a rectangular Lefschetz decomposition of length m with respect to $\mathcal{L}_{\mathcal{M}}$.

Let $\phi: D^b(X) \to D^b(\mathcal{M})$ be a spherical functor.

Finally let us define $\rho = T_X \circ \mathcal{L}_X^d$ with $\mathcal{L}_X$ and d as in the following theorem, and $\sigma = S_X \circ T_X \circ \mathcal{L}_X^m$.

Theorem (Kuznetsov) With the same notations.

Assume $D^b(\mathcal{M}) = \langle \mathcal{B}, \mathcal{B} \otimes \mathcal{L}_{\mathcal{M}}, \dots, \mathcal{B} \otimes \mathcal{L}_{\mathcal{M}}^{m-1} \rangle$ and:

1) $\exists\ 1 \leq d < m$ s.t. $\forall i \in \mathbb{Z}$ $T_{\mathcal{M}}(\mathcal{B} \otimes \mathcal{L}_{\mathcal{M}}^i) = \mathcal{B} \otimes \mathcal{L}_{\mathcal{M}}^{i-d}$

2) $\exists\ \mathcal{L}_X$ a line bundle on X s.t. $\mathcal{L}_{\mathcal{M}} \circ \phi \simeq \phi \circ \mathcal{L}_X$ where $\mathcal{L}_{\mathcal{M}}$ here means the functor "tensoring with $\mathcal{L}_{\mathcal{M}}$".

3) $T_X \circ \mathcal{L}_X = \mathcal{L}_X \circ T_X$

Then $\phi^*: D^b(\mathcal{M}) \to D^b(X)$ is fully faithful on $\mathcal{B}$ and induces a semi-orthogonal decomposition:

$$D^b(X) = \langle \mathcal{A}_X, \mathcal{B}_X, \mathcal{B}_X \otimes \mathcal{L}_X, \dots, \mathcal{B}_X \otimes \mathcal{L}_X^{m-d-1} \rangle$$

with $\mathcal{B}_X = \phi^*(\mathcal{B})$ $\mathcal{A}_X = \langle \mathcal{B}_X, \dots, \mathcal{B}_X \otimes \mathcal{L}_X^{m-d-1} \rangle^\perp$

$\mathcal{A}_X$ is called the Kuznetsov component.

If $c = \gcd(d, m)$ then $\mathcal{A}_X$ has a Serre functor s.t. $S_{\mathcal{A}_X}^{d/c} \simeq \rho^{-m/c} \circ \sigma^{d/c}$

In particular if some powers of σ and ρ are shifts then $\mathcal{A}_X$ is a

7 fractional Calabi-Yau category.

Rem if $d = m$ then ϕ^* is not fully faithful but if $A_X = D^b(X)$ then
$$S_{A_X} = S_X = e^{-1} \circ \sigma.$$

3. Examples of (fractional) CY-categories given by this theorem

3.1 let X, Π be smooth projective varieties.
let $f : X \to \Pi$ be a divisorial embedding s.t. $|f(X)| \in |\mathcal{L}_\Pi^d|$, $1 \leq d \leq m$.
Then $\phi = Rf_*$ is spherical.

Pf We will denote f_* for Rf_*.
Rf_* is a FM transform of kernel $\mathcal{O}_{\Gamma_f}$.
Then f^* is a left adjoint of f_* and $f^! = \omega_X \otimes \omega_\Pi^*|_X[-1] \otimes f^*(-)$ is a right adjoint of f_* (this comes from Grothendieck-Verdier duality, see Huybrechts' "FM transforms in algebraic geometry" p 187, 87).
Now the adjunction formula gives $\omega_X = f^*(\omega_\Pi \otimes \mathcal{O}(X))$
$$\simeq f^*(\omega_\Pi \otimes \mathcal{L}_\Pi^d).$$
let us set $\mathcal{L}_X = f^*(\mathcal{L}_\Pi)$. Then $\omega_X \otimes \omega_\Pi^*|_X = f^*(\omega_\Pi^*) \otimes f^*(\omega_\Pi) \otimes f^*(\mathcal{L}_\Pi^d)$
$$= \mathcal{L}_X^d.$$

Hence $f^! = f^*(-) \otimes \mathcal{L}_X^d[-1]$. Moreover $f^!(f_*(f^*(F))) \simeq f^!(F \otimes f_* \mathcal{O}_X)$.
But there is a resolution (the Koszul resolution) of $f_*(\mathcal{O}_X)$:
$$\mathcal{O}(-X) \xrightarrow{\varphi} \mathcal{O}_\Pi \to f_*(\mathcal{O}_X) \to 0$$
$$\mathcal{O}(-X) \simeq \mathcal{L}_\Pi^{-d}$$

Therefore $f^!(f_* \mathcal{O}_X \otimes F) = f^!(F \otimes (\mathcal{L}_\Pi^{-d} \xrightarrow{\varphi} \mathcal{O}_\Pi))$
$$= f^*(F) \otimes f^*(\mathcal{L}_\Pi^{-d} \xrightarrow{\varphi} \mathcal{O}_\Pi) \otimes \mathcal{L}_X^d[-1] = (*)$$
But $\varphi|_X = 0$, hence $f^*(\mathcal{L}_\Pi^{-d} \xrightarrow{\varphi} \mathcal{O}_\Pi) = \mathcal{L}_X^{-d} \xrightarrow{0} \mathcal{O}_X \simeq \mathcal{O}_X \oplus \mathcal{L}_X[1]$
in $D^b(X)$

$$\Rightarrow (*) = f^*(F) \otimes \mathcal{L}_X^d \otimes (\mathcal{O}_X \oplus \mathcal{L}_X^{-d}[1])[-1]$$

8 $= (f^*(F) \otimes \mathcal{L}_X^d[-1]) \oplus f^*(F) = f^!(F) \oplus f^*(F)$.

Thus φ is an isomorphism.

Similarly, $f^* \circ f_* \circ f^!(F) = f^* \circ f_*(f^*(F) \otimes \mathcal{L}_X^{+d}[-1])$

$$= f^*(F \otimes f_*(\mathcal{L}_X^d))[-1]$$

$$= f^*(F \otimes \mathcal{L}_\Pi^d \otimes f_*(\mathcal{O}_X))[-1]$$

$$= f^*(F) \otimes \mathcal{L}_X^d \otimes (\mathcal{O}_X \oplus \mathcal{L}_X^{-d}[1])[-1]$$

$$= f^*(F) \oplus f^!(F)$$

And τ is an isomorphism. ∎

We would like to apply the theorem, for this we need:

Prop: With the same notations and assumptions as above:

1) T_X commutes with $\mathcal{L}_X$
2) $\exists\ p$ s.t. ρ^p is a shift
3) if $\omega_\Pi = \mathcal{L}_\Pi^{-m}$ then $\exists\, q$ s.t. σ^q is a shift.

Proof: There is a distinguished triangle $\mathcal{L}_\Pi^{-d} \to \mathcal{O}_\Pi \to f_*\mathcal{O}_X$

$\Rightarrow \forall F \in D^b(\Pi)$ $\quad F \otimes \mathcal{L}_\Pi^{-d} \longrightarrow F \longrightarrow f_*\mathcal{O}_X \otimes F$ is distinguished

$$f_*\mathcal{O}_X \otimes F = f_*(\mathcal{O}_X \otimes f^*F) = f_* f^* F$$

The triangle $F \otimes \mathcal{L}_X^{-d}[1] \longrightarrow f^* f_*(F) \to F$ is also distinguished for all $F \in D^b(X)$.

By definition of T_Π and T_X this implies $T_\Pi = \mathcal{L}_\Pi^{-d}$ and $T_X = \mathcal{L}_X^{-d}[2]$.

Thus T_X commutes with $\mathcal{L}_X$ and (11) is satisfied.

$$\rho = T_X \circ \mathcal{L}_X^d = [2]$$

$$\sigma = S_X \circ T_X \circ \mathcal{L}_X^m = \omega_X[\dim X] \otimes \mathcal{L}_X^{-d}[2] \otimes \mathcal{L}_X^m$$

$$\omega_X = f^*(\omega_\Pi \otimes \mathcal{L}_\Pi^d) = f^*(\mathcal{L}_\Pi^{d-m}) = \mathcal{L}_X^{d-m}$$

$$\Rightarrow \sigma = [\dim X + 2] = [\dim \Pi + 1]. \ ∎$$

9 To apply the theorem we still need to check (10) and (9).
Here $\phi = f_*$ and $f_*(\mathcal{L}_X \otimes F) = f_*(f^*(\mathcal{L}_\Pi) \otimes F) = \mathcal{L}_\Pi \otimes f_*(F)$
Hence (10) holds.

$$T_\Pi = \mathcal{L}_\Pi^{-d} \Rightarrow T_\Pi(\mathcal{B} \otimes \mathcal{L}_\Pi^{i}) = \mathcal{B} \otimes \mathcal{L}_\Pi^{i-d} \text{ and (9) holds.}$$

In the end in such a case we need to check:

- that the decomposition of Π exists

- that $\omega_\Pi = \mathcal{L}_\Pi^{-m}$

Examples: 1) $X \subset \mathbb{P}^n$ a smooth hypersurface of degree $d \leq n$. Then:

* $f: X \hookrightarrow \mathbb{P}^n$ and $|f(X)| \in |\mathcal{O}_{\mathbb{P}^n}(1)^d| = |\mathcal{O}_{\mathbb{P}^n}(d)|$
* $\omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1) = \mathcal{O}_{\mathbb{P}^n}(1)^{-(n+1)}$
* $D^b(\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}, \dots, \mathcal{O}_{\mathbb{P}^n}(n) \rangle$
* $\mathcal{B}_X = \langle \mathcal{O}_X \rangle \quad \mathcal{B} = \langle \mathcal{O}_{\mathbb{P}^n} \rangle$ $\quad \mathcal{L}_\Pi = \mathcal{O}_{\mathbb{P}^n}(1)$

$$T_\Pi(\mathcal{B} \otimes \mathcal{L}_\Pi^{i}) = T_\Pi(\mathcal{L}_\Pi^{i}) = \mathcal{L}_\Pi^{i-d} \qquad \mathcal{L}_X = \mathcal{O}_X(1) = f^*(\mathcal{O}_{\mathbb{P}^n}(1))$$

Hence $D^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \dots, \mathcal{O}_X(n-d) \rangle$

$$c = \gcd(d, n+1) \qquad S_{\mathcal{A}_X}^{d/c} = \left[-2\frac{(n+1)}{c} + (n+1)\frac{d}{c}\right] = \left[\left(\frac{n+1}{c}\right)(d-2)\right]$$

If $d \mid n+1$ then $c = d$ and $S_{\mathcal{A}_X} = \left[\frac{(n+1)(d-2)}{d}\right]$.

Hypersurfaces of degree 3 in $\mathbb{P}^5$ is a particular case.

2) $X \subset \mathbb{P}(w_0, \dots, w_n) =: \Pi$ a smooth hypersurface of degree $d < w := \sum_{i=0}^{n} w_i$ in a weighted projective space.

* $D^b(\Pi) = \langle \mathcal{O}, \dots, \mathcal{O}(w-1) \rangle \qquad m = w$
* $f: X \hookrightarrow \Pi$ and $|f(X)| \in |\mathcal{O}(1)^d|$
* $\omega_\Pi = \mathcal{O}(1)^{-w}$
* $\mathcal{B} = \langle \mathcal{O} \rangle$, $\mathcal{L}_\Pi = \mathcal{O}(1)$ so it is as in the example 1)

Therefore $D^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \dots, \mathcal{O}_X(w-d-1) \rangle$

10 and if $c = \gcd(d, \omega)$ then $S_{\mathcal{A}_X}^{d/c} = [-2\frac{\omega}{c} + \frac{d}{c}(n+1)]$

If $d | \omega$ then $c = d$ and $S_{\mathcal{A}_X} = [-2\frac{\omega}{d} + n+1]$

3) Let Q be a smooth quadric of dimension $n = 4s+2$. Then

$D^b(Q) = \langle \mathcal{B}, \mathcal{B}(2s+1) \rangle \qquad m = 2$

where $\mathcal{B} = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(2s), S(2s) \rangle$ and S is a spinor bundle.

Let $X \subset Q$ be a hypersurface of degree $2s+1$.

* $\mathcal{L}_M = \mathcal{O}(2s+1) \qquad |X| \in |\mathcal{O}(2s+1)| \Rightarrow d = 1$
* $\omega_Q = \mathcal{O}(2s+1)^{-2} = \mathcal{O}(-4s-2) \qquad c = \gcd(1,2) = 1$

As a result, $D^b(X) = \langle \mathcal{A}_X, \mathcal{B}_X \rangle$ since $m-d-1 = 0$, with

$\mathcal{B}_X = \langle \mathcal{O}_X, \mathcal{O}_X(1), \dots, \mathcal{O}_X(2s), S(2s)|_X \rangle$.

$S_{\mathcal{A}_X} = [-2 \times 2 + 1 \times (4s+3)] = [4s-1]$

4) Assume $\gcd(k, n) = 1$, let $X \subset Gr(k, n)$ be a hypersurface of degree $d < n$.

* $D^b(Gr(k,n)) = \langle \mathcal{B}, \mathcal{B}(1), \dots, \mathcal{B}(n-1) \rangle \qquad m = n$

$\mathcal{B} = \langle \Sigma^\alpha \mathcal{U}^\vee \mid \alpha_1 < (n-k)(k-1)/k, \alpha_2 < (n-k)(k-2)/k, \dots, \alpha_{k-1} < (n-k)/k \rangle$

* $\mathcal{L}_M = \mathcal{O}(1) \qquad |X| \in |\mathcal{O}(1)^d|$
* $\omega_{Gr(k,n)} = \mathcal{O}(1)^{-n} \qquad c = \gcd(n, d)$

Thus $D^b(X) = \langle \mathcal{A}_X, \mathcal{B}_X, \dots, \mathcal{B}_X(n-d-1) \rangle$

$S_{\mathcal{A}_X}^{d/c} = [-2\frac{n}{c} + \frac{d}{c}(k(n-k)+1)]$

If $d | n$ then $c = d$ and $S_{\mathcal{A}_X} = [-2\frac{n}{d} + k(n-k)+1]$

3.2) Let $f : X \to M$ be a double covering branched in $D \in |\mathcal{L}_M^{2d}|$, $1 \leq d \leq m$.

<u>Proposition:</u> Rf_* is spherical.

(11) Proof: $f^!(F) = f^*(F) \otimes \omega_X \otimes f^*(\omega_\Pi^*)$

$\omega_X = f^*(\omega_\Pi \otimes \mathcal{O}(X)) = f^*(\omega_\Pi) \otimes \mathcal{L}_X^d$ with $\mathcal{L}_X = f^*(\mathcal{L}_\Pi)$

$\Rightarrow f^!(F) = f^*(F) \otimes \mathcal{L}_X^d$

$\Rightarrow f^!(f_*(f^*(F))) = f^!(f_*\mathcal{O}_X \otimes F)$

but since f is a covering $f_*\mathcal{O}_X = \mathcal{O}_\Pi \oplus \mathcal{O}(-X) = \mathcal{O}_\Pi \oplus \mathcal{L}_\Pi^{-d}$

Thus $f^!(f_*(f^*(F))) = f^!(F \otimes (\mathcal{O}_\Pi \oplus \mathcal{L}_\Pi^{-d}))$

$= \mathcal{L}_X^d \otimes f^*(F) \otimes (\mathcal{O}_X \oplus \mathcal{L}_X^{-d})$

$= f^*(F) \oplus f^*(F) \otimes \mathcal{L}_X^d = f^*(F) \oplus f^!(F)$

and Ψ is an isomorphism. Similarly, Γ is an isomorphism. ∎

Prop: 1) T_X commutes with $\mathcal{L}_X$

2) some power of ρ is a shift

3) if $\omega_\Pi = \mathcal{L}_\Pi^{-m}$ there is a power of σ which is a shift.

Proof: There is a distinguished triangle:

$\mathcal{O}_\Pi \to f_*(\mathcal{O}_X) \to \mathcal{L}_\Pi^{-d}$.

Therefore for all $F \in D^b(\Pi)$ one has a distinguished triangle:

$$F \to F \otimes f_*(\mathcal{O}_X) \to F \otimes \mathcal{L}_\Pi^{-d}$$

$$F \otimes f_*(\mathcal{O}_X) = f_*(f^*F)$$

Moreover, for all $F \in D^b(X)$ the following triangle is distinguished:

$$\tau^* F \otimes \mathcal{L}_X^{-d} \to f^* f_*(F) \to F$$

where τ is the involution of the covering f.

By definition of T_Π and T_X this yields $T_\Pi = \mathcal{L}_\Pi^{-d}[-1]$ and $T_X = \tau \circ \mathcal{L}_X^{-d}[1]$.

Since $\tau(\mathcal{L}_X) \cong \mathcal{L}_X$ then $T_X = \tau \circ \mathcal{L}_X^{-d}[1]$ commutes with $\mathcal{L}_X$ and (11) holds.

$\rho = \tau \circ \mathcal{L}_X^{-d}[1] \circ \mathcal{L}_X^{d} = \tau[1] \quad \Rightarrow \rho^2 = [2]$

(12) $\sigma = \omega_X[\dim X] \circ T \circ \mathcal{L}_X^{-d}[1] \circ \mathcal{L}_X^{m}$

$$\omega_X = f^*(\omega_M \otimes \mathcal{L}_M^{d}) = f^*(\mathcal{L}_M^{d-m}) = \mathcal{L}_X^{d-m}$$

$\Rightarrow \sigma = \tau[\dim X + 1] \qquad \sigma^2 = [2(\dim X+1)] = [2(\dim M+1)]$ ▯

As in the previous case, we would like to use the theoreme and for this we still need to check (9) and (10). Again:

* $f_*(\mathcal{L}_X \otimes F) = f_*(f^*\mathcal{L}_M \otimes F) = \mathcal{L}_M \otimes f_*(F)$ and (10) holds.
* $T_M = \mathcal{L}_M^{-d}[-1] \Rightarrow T_M(\mathcal{B} \otimes \mathcal{L}_M^{i}) = \mathcal{B} \otimes \mathcal{L}_M^{i-d}[-1]$

since $\mathcal{B}$ is stable under shifts, (9) holds.

So as before we need to check that $\omega_M = \mathcal{L}_M^{-m}$ and that the decomposition of M exists.

Examples: 1) Let $X \to \mathbb{P}^n$ be a double covering ramified in a smooth hypersurface of degree $2d$, $1 \leq d \leq n$.

We have already checked that $\mathcal{O}(1)^{-m} = \omega_M$ in this case, and we have seen the semi-orthogonal decomposition of $D^b(\mathbb{P}^n)$ in 3.1.

Thus $D^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \dots, \mathcal{O}_X(n-d) \rangle$ and if $c = \gcd(n+1, d)$ then

$$S_{\mathcal{A}_X}^{d/c} = \tau^{(d-(n+1))/c}\left[-\frac{(n+1)}{c} + \frac{d}{c}(n+1)\right] = \tau^{\frac{(d-(n+1))}{c}}\left[\frac{(n+1)}{c}(d-1)\right]$$

If $d \mid n+1$ and $(n+1)/d$ is odd then $S_{\mathcal{A}_X} = \left[(n+1)\frac{(d-1)}{d}\right]$

2) Assume $\gcd(k,n)=1$ and $X \to Gr(k,n)$ is a double covering ramified in a smooth hypersurface of degree $2d$, $d < n$.

As in 3.1 we can apply the theorem and:

$$D^b(X) = \langle \mathcal{A}_X, \mathcal{B}_X, \dots, \mathcal{B}_X(n-d-1) \rangle \qquad c = \gcd(n,d)$$

$$S_{\mathcal{A}_X}^{d/c} = \tau^{(d-n)/c}\left[-\frac{n}{c} + (k(n-k)+1) \times \frac{d}{c}\right]$$

If $d \mid n$ and n/d is odd then $S_{\mathcal{A}_X} = \left[-\frac{n}{d} + k(n-k)+1\right]$

3.3 | Gushel-Mukai manifolds

This part comes from "Lectures on non-commutative K3 surfaces, Bridgeland stability and moduli spaces", E. Macrì, P. Stellari.

<u>Def</u>: let $\mathrm{Cone}(Gr(2,5)) \subset \mathbb{P}^{10}$ be the cone over $Gr(2,5) \subset \mathbb{P}^9$ (where the inclusion is the Plücker embedding).
let $\mathbb{P}^{n+4} \subset \mathbb{P}^{10}$ be a linear subspace and $Q \subset \mathbb{P}^{n+4}$ be a quadric hypersurface. let $X = \mathrm{Cone}(Gr(2,5)) \cap \mathbb{P}^{n+4} \cap Q$ $\quad 2 \leq n \leq 6$
If X is smooth of dimension n then X is a Gushel-Mukai manifold.

Since X is smooth it doesn't contain the vertex of the cone. Therefore one can consider the projection $f : X \longrightarrow Gr(2,5)$ called the Gushel map.
We will consider only the cases $n = 4, 6$.
There are two possibilities:
1) f is an embedding, its image is a quadric section of a smooth linear section of $Gr(2,5)$. In this case X is called ordinary.
2) f is a double covering onto a smooth linear section of $Gr(2,5)$ ramified along a quadric section. In this case X is called special and we will denote by τ the involution of the covering.
In both cases we will denote by M_X the smooth linear section.

<u>Lemma</u>: let $i : M \hookrightarrow Gr(2,5)$ be a smooth linear section of $\dim N \geq 3$ then M has a rectangular Lefschetz decomposition with respect to $\mathcal{O}_M(1) = \mathcal{O}_{Gr(2,5)}|_M$:

$$D^b(M) = \langle \mathcal{B}_M, \mathcal{B}_M(1), \dots, \mathcal{B}_M(N-2) \rangle$$

with $\mathcal{B}_M = \{ \mathcal{O}_M, \mathcal{U}_M^\vee \}$ $\quad \mathcal{U}$ = tautological rank 2 subbundle.

<u>Idea of proof</u>: by inverse induction on N. $\quad \dim Gr(2,5) = 2 \times 3 = 6$
If $N = 5$ then we can apply example 3.1.4): $d = 1 = c$

(13bis) $D^b(M) = \langle \mathcal{A}_X, B_X, B_X(1), \dots, B_X(\underbrace{5-1-1}_{3=5-2}) \rangle$

$S_{\mathcal{A}_X} = [2(5-2)+1-2\times 5] = [-3]$

But $\ast$ $HH_*(M)$ is concentrated in degree 0

$\ast$ $\mathcal{A}_X$ is a -3 CY-category $\Rightarrow HH_{-3}\, \mathcal{A}_X \neq 0$

$\ast$ if $\mathcal{D} = \langle \mathcal{D}_1, \dots, \mathcal{D}_r \rangle$ then $HH_i(\mathcal{D}) \simeq \bigoplus_{j=1}^{r} HH_i(\mathcal{D}_j)$

$\Rightarrow HH_{-3}(D^b(M)) \simeq HH_{-3}\, \mathcal{A}_X \oplus \dots$

$\Rightarrow \mathcal{A}_X$ has to be zero.

$\Rightarrow D^b(M) = \langle B_X, B_X(1), \dots, B_X(3) \rangle$.

Now let $M' \subset M$ be a linear section.

Again we can apply example 3.1): $d = 1 = c$

$D^b(M') = \langle \mathcal{A}_X', B_{M'}, \dots, B_{M'}(\underbrace{4-1-1}_{4-2=2}) \rangle$

$S_{\mathcal{A}_X'} = [-2\times \frac{4}{1} + 6\times 1] = [-2]$ and as before $\mathcal{A}_X' = 0$, etc. ■

$\ast$ let us first consider the case $n=6$. Then f is a double covering, branched in a quartic section. Thus the branched locus is in $\mathcal{O}(1)^2$ and $d=1$. We can apply example 3.2.2).

$D^b(X_6) = \langle \mathcal{A}_X, B_X, \dots, B_X(\underbrace{5-1-1}_{3}) \rangle$

$c=1$ $\quad S_{\mathcal{A}_X} = \tau^{5-1}[(2\times 3+1)\times 1 - 5] = [2]$.

$\ast$ let us now consider the case $n=4$. There are two possibilities:

1) f is an embedding. There is a quadric Q' s.t.

$X_4 = M_X \cap Q' \hookrightarrow M_X \hookrightarrow Gr(2,5)$

by the lemma $D^b(M_X) = \langle B_{M_X}, \dots, B_{M_X}(3) \rangle$

We can use the following lemma:

(13 ter) <u>lemma</u>: $f_*: D^b(X) \to D^b(\Pi_X)$ is spherical

$T_X = \mathcal{O}_X(-2)[2] \quad T_{\Pi_X} = \mathcal{O}_{\Pi_X}(-2)$ if X ordinary

$T_X = \tau \circ \mathcal{O}_X(-1)[1] \quad T_{\Pi_X} = \mathcal{O}_{\Pi_X}(-1)[-1]$ if X special.

this is ex. 3.1 and 3.2.

We can now apply the example 3.1.

$d=2$ since X_4 is a hypersurface of degree 2 in Π_X

$m=4$, $c=2$

$$D^b(X_4) = \langle \mathcal{A}_X, B_X, \dots, B_X(\underbrace{4-2-1}_{1}) \rangle$$

$$S_{\mathcal{A}_X} = [-2 \times \tfrac{4}{2} + 6] = [2]$$

2) f is a double covering branched in $Q \subset \Pi_X$ a quadric.

$|Q| \in |\mathcal{O}(1)|^2 \Rightarrow d=1 \quad c=1 \quad m=3 \quad D^b(\Pi_X) = \langle B_{\Pi_X}, \dots, B_{\Pi_X}(2) \rangle$ by the lemma.

Applying example 3.2) gives:

$$D^b(X_4) = \langle \mathcal{A}_X, B_X, \dots, B_X(\underbrace{3-1-1}_{1}) \rangle$$

$$S_{\mathcal{A}_X} = \tau^{1-3}[-3+5] = \tau^2[2] \qquad S_{\mathcal{A}_X} = [2].$$

Since the Debarre-Voisin varieties are hyperplane sections $X_{20} \subset Gr(3,10)$ with the example 3.1 one gets:

$d=1 \quad c=1$

$$D^b(X_{20}) = \langle \mathcal{A}_X, B_X, \dots, B_X(\underbrace{10-1-1}_{8}) \rangle$$

$$S_{\mathcal{A}_X} = [-2 \times 10 + (3 \times 7 + 1) \times 1] = [-20 + 22] = [2].$$

4. Idea of the proof of the theorem

4.1] The semi-orthogonal decomposition

With the same notations as in the theorem.

Let $\beta_\pi : \mathcal{B} \hookrightarrow D^b(\mathcal{M})$. $\mathcal{B}$ is admissible because it is a piece of the orthogonal decomposition of $D^b(\mathcal{M})$. Hence, β_π has a right adjoint $\beta_\pi^!$. Then, $\beta_\pi^! \circ \phi$ is a right adjoint to $\phi^* \circ \beta_\pi$.

To show that ϕ^* is fully faithful on $\mathcal{B}$ it is enough to show that $\phi \circ \phi^*|_{\mathcal{B}} \simeq id$, that is to say $\beta_\pi^! \circ \phi \circ \phi^* \circ \beta_\pi \simeq id$.

Composing the triangle (3) by $\beta_\pi^!$ on the left and β_π on the right, one gets the following distinguished triangle:

$$\beta_\pi^! \circ T_\pi \circ \beta_\pi (A) \longrightarrow \beta_\pi^! \circ \beta_\pi (A) \longrightarrow \beta_\pi^! \circ \phi \circ \phi^* \circ \beta_\pi (A)$$

for all $A \in \mathcal{B}$ since β_π is exact and thus $\beta_\pi^!$ too.

But $\beta_\pi^! \circ \beta_\pi \simeq id$ since β_π is fully faithful, therefore it is enough to show that $\beta_\pi^! \circ T_\pi \circ \beta_\pi = 0$.

Let us show that $\ker \beta_\pi^! = \mathcal{B}^\perp$.

$$\begin{aligned} A \in \ker \beta_\pi^! &\iff Hom(B, \beta_\pi^! A) = 0 \quad \forall B \in \mathcal{B} \\ &\iff Hom(\beta_\pi B, A) = 0 \quad \forall B \in \mathcal{B} \\ &\iff A \in \mathcal{B}^\perp \end{aligned}$$

Thus it is enough to show that $Im(T_\pi \circ \beta_\pi) \subset \mathcal{B}^\perp$.

By hypothesis, $T_\pi(\mathcal{B}) = \mathcal{B} \otimes \mathcal{L}_\pi^{-d}$

Let $B, B_1 \in \mathcal{B}$ then: $Hom(B, B_1 \otimes \mathcal{L}_\pi^{-d}) \simeq Hom(B \otimes \mathcal{L}_\pi^{d}, B_1) = 0$

since $1 \leq d \leq m-1$ and by definition of the semi-orthogonal decomposition of $D^b(\mathcal{M})$.

As a result, $T_\pi(\mathcal{B}) \subset \mathcal{B}^\perp$ and $\phi^*|_{\mathcal{B}}$ is fully faithful.

* Now we need to show that $\mathcal{B}_X, \dots, \mathcal{B}_X \otimes \mathcal{L}_X^{m-d-1}$ is semi-orthogonal.

(15) Let $B_1, B_2 \in \mathcal{B}_X$ $\quad 0 \leq i < j \leq m-d-1$.

$\mathcal{B}_X = \phi^*(\mathcal{B})$ so $B_1 = \phi^* B_3$ $\quad B_2 = \phi^* B_4$

$$\mathrm{Hom}(B_1 \otimes \mathcal{L}_X^j, B_2 \otimes \mathcal{L}_X^i) \simeq \mathrm{Hom}(B_1, B_2 \otimes \mathcal{L}_X^{i-j})$$

$$\simeq \mathrm{Hom}(B_3, \phi\phi^*(B_4 \otimes \mathcal{L}_\pi^{i-j}))$$

$$= \mathrm{Hom}(B_3, \phi \circ \mathcal{L}_X^{i-j} \circ \phi^*(B_4))$$

Hence it is enough to show $\beta_\pi^! \circ \phi \circ \mathcal{L}_X^{i-j} \circ \phi^* \circ \beta_\pi = 0 \quad \forall\ 1+d-m \leq i-j < 0$.

By assumption $\phi \circ \mathcal{L}_X = \mathcal{L}_\pi \circ \phi \Rightarrow \mathcal{L}_X \circ \phi^* = \phi^* \circ \mathcal{L}_\pi$ taking left adjoints

$\Rightarrow \beta_\pi^! \circ \phi \circ \mathcal{L}_X^{i-j} \circ \phi^* \circ \beta_\pi = \beta_\pi^! \circ \mathcal{L}_\pi^{i-j} \circ \phi \circ \phi^* \circ \beta_\pi$

Composing (3) with $\beta_\pi^! \circ \mathcal{L}_\pi^{i-j}$ on the left, β_π on the right, one gets the distinguished triangle:

$$\beta_\pi^! \circ \mathcal{L}_\pi^{i-j} \circ T_\pi \circ \beta_\pi \longrightarrow \beta_\pi^! \circ \mathcal{L}_\pi^{i-j} \circ \beta_\pi \longrightarrow \beta_\pi^! \circ \mathcal{L}_\pi^{i-j} \circ \phi \circ \phi^* \circ \beta_\pi \quad (*)$$

$\mathrm{Im}(\mathcal{L}_\pi^{i-j} \circ \beta_\pi) = \mathcal{B} \otimes \mathcal{L}_\pi^{i-j}$

$\mathrm{Im}(\mathcal{L}_\pi^{i-j} \circ T_\pi \circ \beta_\pi) = \mathcal{L}_\pi^{i-j} \otimes \mathcal{B} \otimes \mathcal{L}_\pi^{-d} = \mathcal{B} \otimes \mathcal{L}^{i-j-d}$

$\Rightarrow$ these two images are in $\mathcal{B}^\perp = \ker \beta^!$

$\Rightarrow$ the triangle $(*)$ is in fact $0 \to 0 \to \beta_\pi^! \circ \mathcal{L}_\pi^{i-j} \circ \phi \circ \phi^* \circ \beta_\pi$

$$\Rightarrow \beta_\pi^! \circ \mathcal{L}_\pi^{i-j} \circ \phi \circ \phi^* \circ \beta_\pi = 0.$$

Finally, $\mathcal{B}_X \hookrightarrow D^b(X)$ has a right adjoint, namely $\beta_\pi^! \circ \phi$.

$(\mathcal{B}_X = \phi^* \circ \beta_\pi(\mathcal{B}),\ \beta_\pi^! \circ \phi(\mathcal{B}_X) \cong \mathcal{B} \xrightarrow[\phi^*]{} \mathcal{B}_X$).

ϕ^* fully faithful on $\mathcal{B}$

Thus $\mathcal{B}_X$ is admissible, and so is $\langle \mathcal{B}_X, \dots, \mathcal{B}_X \otimes \mathcal{L}_X^{m-d-1} \rangle$

$$\Rightarrow D^b(X) = \langle \mathcal{A}_X, \mathcal{B}_X, \dots, \mathcal{B}_X \otimes \mathcal{L}_X^{m-d-1} \rangle$$

$$\mathcal{A}_X = \langle \mathcal{B}_X, \dots, \mathcal{B}_X \otimes \mathcal{L}_X^{m-d-1} \rangle^\perp$$

∎

(16) <u>4.2</u> Let us denote $\beta_X = \phi^* \circ \beta_M$ $\beta_X^! = \beta_M^! \circ \phi$.

We have now to compute $S_{\mathcal{A}_X}$ (and show it exists).

<u>A. Mutation functors</u>

<u>Def</u> Let $\mathcal{B} \subset D^b(M)$ be admissible, let $\beta : \mathcal{B} \hookrightarrow D^b(M)$.
The left mutation functor associated to $\mathcal{B}$, $\mathbb{L}_{\mathcal{B}}$ is the functor defined

by the distinguished triangle:

$$\beta\beta^! \xrightarrow{\varepsilon} id \longrightarrow \mathbb{L}_{\mathcal{B}}$$

(where as before each functor is a FM transform and the triangle given by their kernels is distinguished).
The right mutation functor is given by the distinguished triangle:

$$\mathbb{R}_{\mathcal{B}} \to id \xrightarrow{\eta} \beta\beta^*$$

Here are some properties of $\mathbb{L}_{\mathcal{B}}$ which can be found in "Homological projective duality", A. Kuznetsov, p4,5. (for the proof he refers to "Representation of associative algebras and coherent sheaves" A. Bondal).

<u>Prop</u>: * $\mathbb{L}_{\mathcal{B}} = i_{\mathcal{B}^\perp} \circ i^*_{\mathcal{B}^\perp}$

* $\mathbb{L}_{\mathcal{B}}(\mathcal{B}) = 0$ $\quad \mathbb{L}_{\mathcal{B}} : {}^\perp\mathcal{B} \longrightarrow \mathcal{B}^\perp$ is an equivalence

* if $\langle \mathcal{A}_0, \dots, \mathcal{A}_m \rangle$ is semi-orthogonal then

$$\mathbb{L}_{\langle \mathcal{A}_0, \dots, \mathcal{A}_m \rangle} = \mathbb{L}_{\mathcal{A}_0} \circ \dots \circ \mathbb{L}_{\mathcal{A}_m}.$$

<u>Lemma</u>: Let $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semi-orthogonal decomposition of a triangulated category $\mathcal{C}$. In particular, $\mathcal{A}$ and $\mathcal{B}$ are admissible. If $\mathcal{C}$ has a Serre functor then so do $\mathcal{A}$ and $\mathcal{B}$. Moreover

$$S_{\mathcal{B}} = \mathbb{R}_{\mathcal{A}} \circ S_{\mathcal{C}} \qquad S_{\mathcal{A}}^{-1} = \mathbb{L}_{\mathcal{B}} \circ S_{\mathcal{C}}^{-1}$$

<u>Idea of proof</u>: $\forall A, B \in \mathcal{C}$ $\quad \mathrm{Hom}(A, B) \simeq \mathrm{Hom}(B, S_{\mathcal{C}}(A))^*$

$\Leftrightarrow \mathrm{Hom}(S_{\mathcal{C}}^{-1}(A), B) \simeq \mathrm{Hom}(B, A)$ since $S_{\mathcal{C}}$ is an equivalence.

(17) Let $A, B \in \mathcal{A}$. Since $\rho\rho^! A \to A \to \mathbb{L}_{\mathcal{B}} A$ is distinguished, there is a long exact sequence:

$$\mathrm{Hom}(\rho\rho^! A[-1], B) \to \mathrm{Hom}(\mathbb{L}_{\mathcal{B}} A, B) \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(\rho\rho^! A, B)$$

but $B \in \mathcal{A}$ and $\rho\rho^! A \in \mathcal{B}$ $\Rightarrow$ $\mathrm{Hom}(\rho\rho^! A[i], B) = 0$ $\quad i = -1, 0$

$\Rightarrow \mathrm{Hom}(\mathbb{L}_{\mathcal{B}} A, B) \simeq \mathrm{Hom}(A, B)$.

Since S_{φ} is an equivalence we can write $A \simeq S_{\varphi}^{-1} C$. Hence

$$\mathrm{Hom}(\mathbb{L}_{\mathcal{B}} A, B) \simeq \mathrm{Hom}(\mathbb{L}_{\mathcal{B}} \circ S_{\varphi}^{-1} C, B)$$
$$\simeq \mathrm{Hom}(S_{\varphi}^{-1} C, B) \simeq \mathrm{Hom}(B, C). \quad \blacksquare$$

In the situation of the theorem this gives that $\mathcal{A}_X$ has a Serre functor and $S_{\mathcal{A}_X}^{-1} = \mathbb{L}_{\langle \mathcal{B}_X, \dots, \mathcal{B}_X \otimes \mathcal{L}_X^{m-d-1} \rangle} \circ S_X^{-1}$.

<u>Lemma</u>: $\mathcal{A}_X = \{ F \in D^b(X),\ \phi(F) \in \langle \mathcal{B} \otimes \mathcal{L}_{\Pi}^{-d}, \dots, \mathcal{B} \otimes \mathcal{L}_{\Pi}^{-1} \rangle \}$

<u>Proof</u>: $\mathcal{A}_X = \{ F \in D^b(X) \mid \mathrm{Hom}(\phi^*(\mathcal{B}), F) = \dots = \mathrm{Hom}(\phi^*(\mathcal{B} \otimes \mathcal{L}_{\Pi}^{m-d-1}), F) = 0 \}$

Since by (10) and since ϕ^* is fully faithful on $\mathcal{B}$ one gets:

$$\phi^* \circ \phi \simeq \mathrm{id} \quad \text{and} \quad \begin{array}{ccc} \phi^* \circ \mathcal{L}_{\Pi} \circ \phi(\mathcal{B}_X) & \simeq & \phi^* \circ \phi \circ \mathcal{L}_X(\mathcal{B}_X) \\ \wr\wr & & \wr\wr \\ \phi^*(\mathcal{L}_{\Pi} \otimes \phi(\mathcal{B}_X)) & & \mathcal{B}_X \otimes \mathcal{L}_X \\ \wr\wr & & \\ \phi^*(\mathcal{L}_{\Pi}) \otimes \mathcal{B}_X & \simeq & \phi^*(\mathcal{L}_{\Pi}) \otimes \phi^*(\mathcal{B}) \end{array}$$

$$\Rightarrow \mathcal{A}_X = \{ F \in D^b(X),\ \mathrm{Hom}(\mathcal{B}, \phi(F)) = \dots = \mathrm{Hom}(\mathcal{B} \otimes \mathcal{L}_{\Pi}^{m-d-1}, \phi(F)) = 0 \}$$
$$= \{ F \in D^b(X),\ \phi(F) \in \langle \mathcal{B}, \dots, \mathcal{B} \otimes \mathcal{L}_{\Pi}^{m-d-1} \rangle^{\perp} \}$$
$$= \{ F \in D^b(X),\ \phi(F) \in \langle \mathcal{B} \otimes \mathcal{L}_{\Pi}^{-d}, \dots, \mathcal{B} \otimes \mathcal{L}_{\Pi}^{-1} \rangle \}$$

Since by tensoring the semi-orthogonal decomposition of $D^b(\Pi)$ by $\mathcal{L}_{\Pi}^{-d}$ one gets a new semi-orthogonal decomposition:

$$D^b(\Pi) = \langle \mathcal{B} \otimes \mathcal{L}_{\Pi}^{-d}, \dots, \mathcal{B} \otimes \mathcal{L}_{\Pi}^{m-d-1} \rangle \quad \blacksquare$$

(18) <u>Lemma</u>: $\rho \circ \phi^* \simeq \phi^* \circ T_\Pi \circ \mathcal{L}_\Pi^d [2]$

$\sigma \circ \phi^* = \phi^* \circ \mathcal{L}_\Pi^m \circ S_\Pi [1]$

In particular ρ and σ preserve each piece of $\langle \mathcal{B}_X, \dots, \mathcal{B}_X \otimes \mathcal{L}_X^{m-d-1} \rangle = \mathcal{C}$.

<u>Proof</u>: $\rho \circ \phi^* = T_X \circ \mathcal{L}_X^d \circ \phi^*$

$= T_X \circ \phi^* \circ \mathcal{L}_\Pi^d$ by taking left adjoint in (10).

$= \phi^* \circ T_\Pi \circ \mathcal{L}_\Pi^d [2]$ by (8)

$\sigma \circ \phi^* = S_X \circ T_X \circ \mathcal{L}_X^m \circ \phi^* = S_X \circ T_X \circ \phi^* \circ \mathcal{L}_\Pi^m$

$= S_X \circ \phi^* \circ T_\Pi \circ \mathcal{L}_\Pi^m [2]$ by (8)

Moreover if $\alpha : \mathcal{C}_1 \to \mathcal{C}_2$ is a functor between triangulated categories with Serre functors, then if α^* and $\alpha^!$ exists one has:

$\alpha^! \circ S_{\mathcal{C}_2} = S_{\mathcal{C}_1} \circ \alpha^*$. $(*)$

Applying $(*)$ to $T_\Pi \circ \mathcal{L}_\Pi^m$ one gets:

$\mathcal{L}_\Pi^{-m} \circ T_\Pi^{-1} \circ S_\Pi = S_\Pi \circ \mathcal{L}_\Pi^{-m} \circ T_\Pi^{-1}$

$\Rightarrow S_\Pi^{-1} \circ T_\Pi \circ \mathcal{L}_\Pi^m = T_\Pi \circ \mathcal{L}_\Pi^m \circ S_\Pi^{-1} \Rightarrow T_\Pi \circ \mathcal{L}_\Pi^m = S_\Pi^{-1} \circ T_\Pi \circ \mathcal{L}_\Pi^m \circ S_\Pi$

$\Rightarrow \sigma \circ \phi^* = S_X \circ \phi^* \circ T_\Pi \circ \mathcal{L}_\Pi^m [2] = S_X \circ \phi^* \circ S_\Pi^{-1} \circ T_\Pi \circ \mathcal{L}_\Pi^m \circ S_\Pi [2]$

$= \phi^! \circ T_\Pi \circ \mathcal{L}_\Pi^m \circ S_\Pi [2]$ by $(*)$ applied to ϕ

$= \phi^* \circ T_\Pi^{-1} \circ T_\Pi \circ \mathcal{L}_\Pi^m \circ S_\Pi [1]$ by $(****)$ p. (6).

$= \phi^* \circ \mathcal{L}_\Pi^m \circ S_\Pi [1]$.

Then $\rho(\mathcal{B}_X) = \rho \circ \phi^* (\mathcal{B}) = (\phi^* \circ T_\Pi \circ \mathcal{L}_\Pi^d)(\mathcal{B})$ since $\mathcal{B}$ is invariant under shifts

$= \phi^* (T_\Pi (\mathcal{B} \otimes \mathcal{L}_\Pi^d))$

$= \phi^* (\mathcal{B})$ by (9)

$= \mathcal{B}_X$

So ρ preserves $\mathcal{B}_X$, and it commutes with $\mathcal{L}_X$ by (11) thus it preserves each

(19) piece of $\mathcal{C}$.

Similarly, using that $S_\pi(\mathcal{B}\otimes\mathcal{L}_\pi^i) = \mathcal{B}\otimes\mathcal{L}_\pi^{i-m}$ (one can write

$\mathcal{B}\otimes\mathcal{L}_\pi^i = {}^\perp\langle \mathcal{B}\otimes\mathcal{L}_\pi^{i-m+1}, \dots, \mathcal{B}\otimes\mathcal{L}_\pi^{i-1}\rangle$

$\Rightarrow S_\pi(\mathcal{B}\otimes\mathcal{L}_\pi^i) = \langle \mathcal{B}\otimes\mathcal{L}_\pi^{i-m+1}, \dots, \mathcal{B}\otimes\mathcal{L}_\pi^{i-1}\rangle^\perp = \mathcal{B}\otimes\mathcal{L}_\pi^{i-m}$) one gets:

$$\sigma(\mathcal{B}_X) = \sigma(\phi^*(\mathcal{B})) = \phi^*\circ\mathcal{L}_\pi^m\circ S_\pi(\mathcal{B})$$
$$= \phi^*(\mathcal{B}\otimes\mathcal{L}_\pi^{m-m}) = \mathcal{B}_X$$

Finally σ commutes with $\mathcal{L}_X$ by (11) and (*) applied to S_X, therefore it preserves each piece of $\mathcal{C}$. $\blacksquare$

B. Rotation functors

Let Y be a smooth projective variety with rectangular Lefschetz decomposition $D^b(Y) = \langle \mathcal{B}_Y, \mathcal{B}_Y\otimes\mathcal{L}_Y, \dots, \mathcal{B}_Y\otimes\mathcal{L}_Y^{s-1}\rangle$.

Then $O_{\mathcal{B}} = \mathbb{L}_{\mathcal{B}}\circ\mathcal{L}_Y$

Lemma: $O_{\mathcal{B}}^i = \mathbb{L}_{\langle\mathcal{B},\dots,\mathcal{B}\otimes\mathcal{L}_Y^{i-1}\rangle}\circ\mathcal{L}_Y^i \quad \forall\, 0\leq i\leq s$.

Proof: $O_{\mathcal{B}}^i = (\mathbb{L}_{\mathcal{B}}\circ\mathcal{L}_Y)\circ(\mathbb{L}_{\mathcal{B}}\circ\mathcal{L}_Y)\circ\dots\circ(\mathbb{L}_{\mathcal{B}}\circ\mathcal{L}_Y)$

$$= \mathbb{L}_{\mathcal{B}}\circ(\mathcal{L}_Y\circ\mathbb{L}_{\mathcal{B}}\circ\mathcal{L}_Y^{-1})\circ(\mathcal{L}_Y^2\circ\mathbb{L}_{\mathcal{B}}\circ\mathcal{L}_Y^{-2})\circ\dots\circ(\mathcal{L}_Y^{i-1}\circ\mathbb{L}_{\mathcal{B}}\circ\mathcal{L}_Y^{i-1})\circ\mathcal{L}_Y^i$$

$$= \mathbb{L}_{\mathcal{B}}\circ\mathbb{L}_{\mathcal{B}\otimes\mathcal{L}_Y}\circ\dots\circ\mathbb{L}_{\mathcal{B}\otimes\mathcal{L}_Y^{i-1}}\circ\mathcal{L}_Y^i = \mathbb{L}_{\langle\mathcal{B},\dots,\mathcal{B}\otimes\mathcal{L}_Y^{i-1}\rangle}\circ\mathcal{L}_Y^i$$

since for any autoequivalence T of $D^b(Y)$, $T\circ\mathbb{L}_{\mathcal{B}}\circ T^{-1} = \mathbb{L}_{T(\mathcal{B})}$:

if $\beta: \mathcal{B}\hookrightarrow D^b(Y)$ then $T\circ\beta: T(\mathcal{B})\hookrightarrow D^b(Y)$ and $(T\circ\beta)^! = \beta^!\circ T^{-1}$

$\Rightarrow \quad T\circ\beta\circ\beta^!\circ T^{-1} \longrightarrow id \longrightarrow \mathbb{L}_{T(\mathcal{B})}$

On the other hand $T(\beta\circ\beta^! \to id \to \mathbb{L}_{\mathcal{B}})T^{-1} = T\circ\beta\circ\beta^!\circ T^{-1} \to id \to T\circ\mathbb{L}_{\mathcal{B}}\circ T^{-1}$. $\blacksquare$

Corollary: $O_\pi^i(\langle\mathcal{B}\otimes\mathcal{L}_\pi^{-i}, \dots, \mathcal{B}\otimes\mathcal{L}_\pi^{-1}\rangle) = 0$, $O_\pi^m = 0$, where $O_\pi = O_{\mathcal{B}}$.

Proof: $\mathcal{L}^i \otimes (\langle B\otimes \mathcal{L}_\Pi^{-i}, \dots, B\otimes \mathcal{L}_\Pi^{-1}\rangle) = \langle B, \dots, B\otimes \mathcal{L}_\Pi^{i-1}\rangle$

and $\mathbb{L}_{\langle B,\dots,B\otimes \mathcal{L}_Y^{i-1}\rangle}(\langle B,\dots,B\otimes \mathcal{L}_Y^{i-1}\rangle) = 0$. ∎

We will denote $O_\Pi = O_B$, $O_X = O_{B_X}$.

Lemma: O_X commutes with ρ and σ.

Proof: We have already seen that ρ and σ commute with $\mathcal{L}_X$, so we only have to show they commute with $\mathbb{L}_{B_X}$.

We have seen that ρ preserves B_X, hence $\beta\circ\beta^!\circ\rho = \rho\circ\beta\circ\beta^!$. Thus:

$$\begin{array}{ccccc} \beta\circ\beta^!\circ\rho & \longrightarrow & \rho & \longrightarrow & \mathbb{L}_B\circ\rho \\ \| & & \| & & \vdots\wr \\ \rho\circ\beta\circ\beta^! & \longrightarrow & \rho & \longrightarrow & \rho\circ\mathbb{L}_B \end{array}$$

and ρ commutes with $\mathbb{L}_B$. Similarly σ commutes with $\mathbb{L}_B$. ∎

Lemma: For all $0\leq i\leq d-1$ there is a morphism $\phi^*\circ O_\Pi^i \xrightarrow{\gamma_i} O_X^i\circ\phi^*$ such that $\gamma_i|_{\langle B\otimes\mathcal{L}_\Pi^{d-m},\dots,B\otimes\mathcal{L}_\Pi^{d-i-1}\rangle}$ is an isomorphism.

Proposition: For all $0\leq i\leq d$ there is a distinguished triangle:

$$\phi^*\circ O_\Pi^i\circ\phi \xrightarrow{\varepsilon_{\phi^*,\phi}\circ\gamma^i} O_X^i \longrightarrow T_X\circ\mathcal{L}_X^i$$

Corollary: $O_X^d|_{\mathcal{A}_X} \simeq \rho|_{\mathcal{A}_X}$.

Proof: We have the distinguished triangle:

$$\phi^*\circ O_\Pi^d\circ\phi|_{\mathcal{A}_X} \longrightarrow O_X^d|_{\mathcal{A}_X} \longrightarrow \tilde{T}_X\circ\mathcal{L}_X^d|_{\mathcal{A}_X} \quad (*)$$

(13)

Since $\mathcal{A}_X = \{F, \phi(F)\in\langle B\otimes\mathcal{L}_\Pi^{-d},\dots,B\otimes\mathcal{L}_\Pi^{-1}\rangle\}$ then $\phi^*\circ O_\Pi^d\circ\phi(\mathcal{A}_X) = \phi^*\circ O_\Pi^d(\langle B\otimes\mathcal{L}_\Pi^{-d},\dots,B\otimes\mathcal{L}_\Pi^{-1}\rangle) = 0$ by the property of rotation functors we have seen.

Thus $(*)$ is in fact $0\to O_X^d|_{\mathcal{A}_X} \longrightarrow \rho|_{\mathcal{A}_X}$ and $O_X^d|_{\mathcal{A}_X}\simeq\rho|_{\mathcal{A}_X}$ ∎.

(21) Lemma: $S_{\mathcal{A}_X}^{-1} \simeq \mathcal{O}_X^{m-d} \circ \rho \circ \sigma^{-1}$!

Proof: We have seen $S_{\mathcal{A}_X}^{-1} = \mathbb{L}_{\langle B_X, \dots, B_X \otimes \mathcal{L}_X^{m-d-1} \rangle} \circ S_X^{-1}$.

But by definition of σ, $S_X^{-1} = \mathcal{L}_X^{m} \circ T_X \circ \sigma^{-1}$.

$$\Rightarrow S_{\mathcal{A}_X}^{-1} = \mathbb{L}_{\langle B_X, \dots, B_X \otimes \mathcal{L}_X^{m-d-1} \rangle} \circ \mathcal{L}_X^{m} \circ T_X \circ \sigma^{-1}$$

$$\simeq \mathbb{L}_{\langle B_X, \dots, B_X \otimes \mathcal{L}_X^{m-d-1}} \circ \mathcal{L}_X^{m-d} \circ \mathcal{L}_X^{d} \circ T_X \circ \sigma^{-1}$$

$$= \mathcal{O}_X^{m-d} \circ \rho \circ \sigma^{-1} \quad \blacksquare$$

Corollary: $S_{\mathcal{A}_X}^{-d/c} = \rho^{m/c} \circ \sigma^{-d/c}$ and $S_{\mathcal{A}_X}^{d/c} = \rho^{-m/c} \circ \sigma^{d/c}$

Proof: First, ρ and σ commute, since $\mathcal{L}_X$ and T_X commute by (11) and S_X commutes with any autoequivalence of $D^b(X)$ by $(*)$.

We have also seen that ρ and σ commute with $\mathcal{O}_X$.

Thus, if $c = \gcd(d, m)$ one gets:

$$S_{\mathcal{A}_X}^{-d/c} = \mathcal{O}_X^{(m-d)\,d/c} \circ \rho^{+d/c} \circ \sigma^{-d/c}$$

$$= \rho^{(m-d)/c} \circ \rho^{d/c} \circ \sigma^{-d/c} \quad \text{since } \rho|_{\mathcal{A}_X} = \mathcal{O}_X^{d}|_{\mathcal{A}_X}$$

$$= \rho^{m/c} \circ \sigma^{-d/c}$$

Since σ and ρ commute this yields $S_{\mathcal{A}_X}^{d/c} = \rho^{-m/c} \circ \sigma^{d/c}$ $\blacksquare$.