Which Rate to use?

Consideration of collateral agreements in the Post-Lehman world when pricing derivatives with Black-Scholes

Tat Sang Fung, PhD

Drift 1: March 2011

Sharing Thoughts

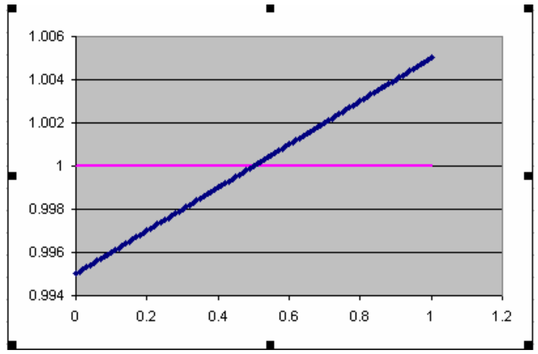
Abstract: For a desk selling stock options, when using the standard Black-Scholes (BS) framework for pricing, what kind of interest rate / funding rate should be used? There is only one variable in most BS formulas that represents interest rates. The answer is that one should use the stock repo rate, and then, either invoke an extra multiplicative factor (depending on the collateralization ratio) to adjust the option value, or equivalently, use an quanto-like adjusted dividend rate for pricing. The CVA deduced is rough linear in the collateralization ratio.

Keywords: derivative pricing, credit value adjustment, CVA

Summary: In the presence of CSA rate r_C, stock repo rates r_R, unsecured funding rates r_F, under the standard Black-Scholes setting, for European options, the BS formula should be called using the stock repo rate, stock dividend rate unadjusted, with a multiplicative adjustment factor so that the final value of the stock derivative is

$$V_\beta = \exp\left(-\left((1-\beta)r_F + \beta r_C - r_R\right)\right) BS(r_R)$$

Where β is the cash collateral pledge to value ratio. $\beta = 0$ for no collateral; $\beta = 1$ fully collateralized. This adjustment factor understandably is almost linear in practice:



Equivalently, one can view as using stock repo rates when calling Black-Scholes, but instead of the dividend rate r_D, use an adjusted dividend rate $(1-\beta)r_F + \beta r_C - r_R + r_D$, similar to what one would do when pricing a quanto option. This view is application also for American style and path depend options.

Details

This evolves from the note I made along while reading [VP]. The author wishes the somewhat more elementary treatment here help explain the results and hence making the subject accessible to a wider audience. Example in the standard Black-Scholes setting is included. All mistakes, however, shall remain solely mine.

Using notations in [VP], consider the bank's trading desk sold a stock option and hedge it dynamically using a portfolio of the underlying stock and cash. There are several interest / funding rates being relevant:

r_C : interest rate for the cash collateral posted ("CSA rate")

r_R : interest rate for desk to borrow and purchase underlying stock, while leaving the stock as collateral ("repo rate")
r_F : unsecured funding rate. Usually the rate obtained through the banks' Treasury
For notation simplicity we have skipped "(t)" though they can be functions of time, and a process in general. All of the above are short rates, continuous compounding.

As shown in [VP], the corresponding PDE is then given by

$$\frac{\partial V}{\partial t}+(r_R-r_D)\frac{\partial V}{\partial S}S+\frac{\sigma^2S^2}{2}\frac{\partial^2V}{\partial S^2}$$
$$=r_FV-(r_F-r_C)C$$

Where C is the collateral function[1], and r_D is the stock dividend. The derivation is "Hull's" style (see [Hull] page 309): since the desk has sold the option, it gets cash of amount V, it then needs to post collateral back given by C and earn r_C, leaving amount $V-C$ to lent to / borrow from the Treasury earning rate r_F. Let A be the amount of stock it needs to hold. The desk shall borrow $A\cdot S$ in cash to purchase the stocks, with funding r_R in the repo leaving the stock as collateral, while collecting the stock dividend. Hence after time dt, we have the change in the cash position given by

$$[r_C\cdot C+r_F(V-C)-r_RAS+r_DAS]dt$$

And the stock value is changed by
$A\cdot dS$
Suppose the stock follow the usual process
$dS=\mu Sdt+\sigma SdW_t$
Where μ and σ can be functions of t and S, by Ito's lemma, we have

$$dV=\left(\frac{\partial V}{\partial t}+\frac{\sigma^2S^2}{2}\frac{\partial^2V}{\partial S^2}\right)dt+\frac{\partial V}{\partial S}dS$$

[1] Note in [VP] it is suggested to think of C is either 0, or V

So putting $A=\frac{\partial V}{\partial S}$ to hedge away the risk from underlying stock, what remains is market risk free. Therefore,

$$dV=\left(\frac{\partial V}{\partial t}+\frac{\sigma^2S^2}{2}\frac{\partial^2V}{\partial S^2}\right)dt+\frac{\partial V}{\partial S}dS$$
$$=\left[r_C\cdot C+r_F(V-C)-r_R\frac{\partial V}{\partial S}S+r_D\frac{\partial V}{\partial S}S\right]dt+\frac{\partial V}{\partial S}dS$$

Equating the drift term we arrive at the PDE

$$\frac{\partial V}{\partial t}+(r_R-r_D)\frac{\partial V}{\partial S}S+\frac{\sigma^2S^2}{2}\frac{\partial^2V}{\partial S^2}$$
$$=r_FV-(r_F-r_C)C$$

Let $C=\beta V$ where β is a constant. $\beta=0$ means no collateral is to be posted. $\beta=1$ means it is fully collateralized. $\beta=110\%$ means collateral is to be post at 110% of the current market value of the derivative. The PDE simplifies to

$$\frac{\partial V}{\partial t}+(r_R-r_D)\frac{\partial V}{\partial S}S+\frac{\sigma^2S^2}{2}\frac{\partial^2V}{\partial S^2}$$
$$=[(1-\beta)r_F+\beta r_C]V \qquad (*)$$

Consider the case when all interest rates are time-deterministic. Let $BS(r)$ be the solution in the Black-Scholes world with PDE of the form

$$\frac{\partial V}{\partial t}+(r-r_D)\frac{\partial V}{\partial S}S+\frac{\sigma^2S^2}{2}\frac{\partial^2V}{\partial S^2}=rV$$

where r can be thought of as the "risk free" rate to plug into the BS formula, if one is available. Then, considering European options, one can verify that[2]

[2] Besides using the solution and verify it satisfies the PDE, a quicker way is to see that if $BS(r,q)$ solves

$$\frac{\partial V}{\partial t}+(r-q)\frac{\partial V}{\partial S}S+\frac{\sigma^2S^2}{2}\frac{\partial^2V}{\partial S^2}=rV,$$

then for

$$\frac{\partial V}{\partial t}+(a-q)\frac{\partial V}{\partial S}S+\frac{\sigma^2S^2}{2}\frac{\partial^2V}{\partial S^2}=bV, \text{ it}$$

$V_\beta = \exp(-((1-\beta)r_F + \beta r_C - r_R)(T-t))BS(r_R)$ is a solution for (*). In other words, the solution is given by "business as usual utilizing the Black-Scholes methodology using stock repo rate as the risk-free rate, and then, adjust the answer with a multiplicative factor $\exp(-((1-\beta)r_F + \beta r_C - r_R)(T-t))$. This factor is be re-written as $\exp(-((1-\beta)(r_F - r_R) + \beta(r_R - r_C))(T-t))$ given the likely case of $r_C \le r_R \le r_F$. Note that if $r_R > r_F$ we can take simply take $r_R = r_F$ since we can simply skip posting the stock collateral when borrowing in such case. When $r_C \le r_R \le r_F$, we see that for the fully collateralized case, $\beta = 1$ $V_1 \ge BS(r_R)$; and for the no collateral case, $\beta = 0$ we have $V_0 \le BS(r_R)$. The value V_β can become arbitrarily large by increasing β. If we consider V_0 represents valuation considering fully credit risk, and V_1 completely free of credit risk due to full collateralization, then along the spirit of credit value adjustment (CVA)[3], viewed from the other side perhaps, is given by $V_0 - V_1$. It seems plausible, as the credit worthiness of the desk is reflected through r_F: looking at the adjustment factor $\exp(-((1-\beta)(r_F - r_R) + \beta(r_R - r_C))(T-t))$, the higher the spread $r_F - r_R$, the bigger the CVA.

Let $t = 0$ and if we write $BS_F(r) = \exp(rT)BS(r)$ as the "forward value" of the option at maturity time, whose computation only requires a drift and not discount factors, one can also simply the expressions to get $V_1 = \exp(-r_C T)BS_F(r_R)$ and $V_0 = \exp(-r_F T)BS_F(r_R)$, we see that CVA is $[\exp(-r_C T) - \exp(-r_F T)]BS_F(r_R)$. Approximately, this is equal to $[\exp(-(r_C - r_F)T)]BS_F(r_R)$ or $[\exp(-(r_C - r_F)T)]BS(r_R)$ which is to say to obtain CVA one can roughly discount using the funding spread between fully collateralized and unsecured rates on the current option value.

Note also that in this BS case when interest rates are time-deterministic, the forward value of the stock stays the same as the one given by the BS world with stock drift given by drift $r_R - r_D$, regardless of collateral parameter β and the values of r_C and r_F. Note that this is not true in general when the interest rates are stochastic, as treated in [VP].

In the general case of stochastic rates, we can view it via the Feynman-Kay theorem (for example, see [KA] page 366), under applicable conditions, where the general solution is given by

$$V = E\left[f(S_T)\exp\left(-\int_0^T (1-\beta)r_F + \beta r_C \right)d\theta \right] (**)$$

where the stock process is given by

$$dS = (r_R - r_D)Sdt + \sigma S dW_t$$

Here we can see the financial interpretation of a clear separation between the stock process (involving stock repo rate) and the "discounting" interest rate, which the discounting curve is a weighted sum of the unsecured funding rate and the CSA rate.

---

can be see as

$$\frac{\partial V}{\partial t} + (b - q^*)\frac{\partial V}{\partial S}S + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} = bV$$

with $q^* = b - a - q$, and use the fact that $BS(a+u, q+u) = e^{-uT}BS(a,q)$ for any u and put $u = b - a$ for our case.

[3] See, for example, [S1] for an excellent introduction of CVA

Another thought to deduce is that, replacing stock by LIBOR (I think of using tradable package of buying zero coupon bond of maturity at start of the interest period and sell one of maturity at the end of the period), the formula (**) gives a justification of the recently popular method of swap valuation: projecting for forward LIBOR using one curve, discount using another (usually the one nearest "risk free" for the collateralized case, or the CSA rate in the discussion). Note, however, there may be a subtle point to know concerning curve generation. Consider interest rate curve bootstrapping for LIBOR curve using LIBOR to project for forward LIBOR, and OIS to discount. Looking at the expression (**), it seems to me that the implicitly implied assumptions could be either OIS and LIBOR are independent processes, or, LIBOR's process is to be modeled under a measure using a OIS related numeraire[4]. In each of these cases, extra care is required to ensure consistency in subsequent modeling when pricing exotic derivatives and structured products.

References

[Hull] John Hull, *Options, Futures and other derivatives*, Prentice Hall, 8th Edition

[KA] Karatzas and Shreves, *Brownian motion and Stochastic calculus,* 2nd edition, Springer-Verlag

[S1] Giovanni Cesari, John Aquilina, Niels Charpillon, Zlatko Filipovic, Gordon Lee, Ion Manda, *Modelling, pricing, and hedging counterparty credit exposure***,** Springer-Verlag

[VP] Vladimir Piterbarg, *Funding beyond discounting: collateral agreements and derivatives pricing*, RISK Feb 2010

Tat Sang Fung is a senior manager in Misys and an adjunct assistant professor at Columbia University. He holds a Ph.D. in Mathematics from Columbia University in the City of New York. He is currently based in New York and can be reached at fts@math.columbia.edu

Linkedin: http://www.linkedin.com/pub/tat-sang-fung/6/367/192

Email: fts@math.columbia.edu

[4] After using some kind of computational trick using Girsanov's theorem, for example