Collateralized Trade Pricing Made Simple

Collateralized trade pricing in the Post-Lehman world from first principals

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Sharing Thought

Abstract: This paper provides an understanding from first principals the switching of discount curve when collateral is posted: from risk free to collateral return rates. In the case of a collateral currency different than the flow currency, replication argument reveals the formula in the case of deterministic interest rates. The goal to get insight to understand the general formula given in [MFAT] from simple cases.

Keywords: OIS discounting, derivative pricing, collateralized trade pricing

Details

While discussing the new formulation of derivative pricing for bi-directional fully collateralized trades along the line of [MFAT], frequently I see the need of a somewhat more elementary treatment to help explain the results to those who are not necessarily sophisticated quants. This note hopefully would help make the subject accessible to a wider audience. All mistakes, however, shall remain solely mine.

We will follow the notations in [MFAT]: Let $r^{(i)}(t)$ be the risk-free continuous compounding zero rate for currency i at time t. We assume that one can lend and borrow at this rate without collateral. If cash collateral is posted in currency k, let $c^{(k)}(t)$ be the continuous compounding collateral return rate for at time t. Let $f_x^{(i,j)}(t)$ be the value of one unit of currency j in currency i at time t. Let $y^{(k)}(t)=r^{(k)}(t)-c^{(k)}(t)$, and $y^{(i,j)}(t)=y^{(i)}(t)-y^{(j)}(t)$.

For elementary treatment, we first assume all interest rates are deterministic. Consider a zero coupon bond in currency i maturing at time T. If no collateral is posted, by standard replication argument, the value at time t is $e^{-r^{(i)}(0,T)T}$ where $r^{(i)}(0,T)$ is the forward interest rate. To simplify notions we simply write $e^{-r^{(i)}T}$ where no confusion should arise when the period for the forward rate is omitted. The replication is to deposit $e^{-r^{(i)}T}$ in currency i with the risk-free source and wait to collect one unit of currency i at time T, at which time to pass to the holder of the zero coupon bond. When collateral posting is required, such replication won't work, especially when the collateral return rate is less than the borrowing rate of the replicator. One can however instead replicate *directly with the holder* of the zero coupon bond: Consider the collateralized zero coupon bond is valued as $V(t)$. As holder of the bond requires $V(t)$ as cash collateral, the issuer receives nothing net. Note the collateral grows at the rate of $c^{(i)}(t)$. Therefore, putting $e^{-c^{(i)}T}$ as collateral, one sees that at time T, holder of the bond will find the collateral account containing exist one unit of currency i which is exactly the payoff of the bond.

The above argument shows that in the presence of fully collateralization, the discount rate used for pricing should be the replaced by the collateral return rate.
For the case when collateral is posted in currency k , the replication would involve a foreign exchange. The issuer would need to exchange $V(t)$ into currency k so to post as collateral. It will grow at the rate of $c^{(k)}(t)$, which finally at time T we need it to be exchangeable to one unit of currency i . Since there are risk-free interest rates of the respective currencies, we can use an un-collateralized FX forward trade for the purpose. Hence:

$\frac{V(0)}{f_x^{(i,k)}(0)} e^{c^{(k)}T} F_x^{(i,k)}(T) = 1$ where $F_x^{(i,k)}(T)$ is the FX forward at time zero from an uncollateralized forward. Since we know $F_x^{(i,k)}(T) = f_x^{(i,k)}(0) e^{\left(r^{(i)} - r^{(k)}\right)T}$,we get

$V(0) = e^{\left[-c^{(k)} - r^{(i)} + r^{(k)}\right]T}$ or $V(0) = e^{-\left[r^{(i)} - y^{(k)}\right]T}$

One can interpret as a need to subtract the currency k instantaneous return when depositing risk-free the collateral, from the risk-free rate of currency i . As a result, discounting a future flow of currency i requires also knowing what the collateral currency is.

Consider now a FX forward contract to buy one unit of currency j by selling K units of currency i at time T , fully bi-directionally collateralized in currency k . Since it is a package of two collateralized zero coupon bonds, its value, in currency i is

$$V(0) = f_x^{(i,j)}(0) e^{\left(-r^{(j)} + r^{(k)} - c^{(k)}\right)T} - K e^{\left(-r^{(i)} + r^{(k)} - c^{(k)}\right)T}$$

which is the same as

$$\begin{aligned} V(0) &= f_x^{(i,j)}(0) e^{-\left[r^{(j)} - y^{(k)}\right]T} - K e^{-\left[r^{(i)} - y^{(k)}\right]T} \\ &= e^{\left(-r^{(i)} + r^{(k)} - c^{(k)}\right)T} \left(F_x^{(i,k)}(T) - K\right) \\ &= e^{-\left(r^{(i)} - y^{(k)}\right)T} \left(F_x^{(i,k)}(T) - K\right) \end{aligned}$$

This shows that the break-even FX forward rate for a collateralized FX forward is the same as the un- collateralized one. This formula has the usual interpretation of discounting the forward profit along a single currency (with collateralized adjustment).

The above is not true in general when interest rates are stochastic. To see this, apply formula (4) in [MFAT]:

$h_t^{(i)} = E^{Q^i}\left[e^{-\int_t^T r^{(i)}(s)ds} e^{\int_t^T y^{(k)}(s)ds} h^{(i)}(T) \right]$, one gets currency k collateralized currency i zero-coupon pricing as

$E^{Q^i}\left[e^{-\int_t^T r^{(i)}(s)ds} e^{\int_t^T y^{(k)}(s)ds} \right]$ which after a change of measure becomes

$E^{Q^i}\left[e^{-\int_t^T c^{(i)}(s)ds} \right] E^{T^i}\left[e^{-\int_t^T y^{(i,k)}(s)ds} \right]$, or

$$D^{(i)}(t,T) e^{-\int_t^T y^{(i,k)}(t,s)ds}$$

Where $D^{(i)}(t,T) = E^{Q^i}\left[e^{-\int_t^T c^{(i)}(s)ds} \right]$, and[1]

$e^{-\int_t^T y^{(j,i)}(t,s)ds} = E^{T^j}\left[e^{-\int_t^T y^{(j,i)}(s)ds} \right]$,

therefore the FX forward value is

$$f_x^{(i,j)}(t) \cdot D^{(j)}(t,T) e^{-\int_t^T y^{(j,k)}(t,s)ds} - K \cdot D^{(i)}(t,T) e^{-\int_t^T y^{(i,k)}(t,s)ds}$$

giving us the break-even FX forward as

$f_x^{(i,j)}(t,T) = f_x^{(i,j)}(t) \cdot \frac{D^{(j)}(t,T)}{D^{(i)}(t,T)} e^{-\int_t^T y^{(j,i)}(t,s)ds}$ which is formula (16) in [MFAT]

References

[KA] Karatzas and Shreves, *Brownian motion and Stochastic calculus,* 2nd edition, Springer-Verlag
[MFAT] Masaaki Fujii, Akihiko Takahashi, *Choice of collateral currency*, RISK Jan 2011
[VP] Vladimir Piterbarg, *Funding beyond discounting: collateral agreements and derivatives pricing*, RISK Feb 2010

[1] There seems to be a typo in formula (12) in [MFAT]

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