$\boxed{\text{Lecture 12}}$ Seiberg-Witten theory

We have seen throughout the class the importance of first order elliptic PDEs ($d+d^*$, $d^+ + d^*$ in 4D, $\bar\partial$ in 2D).

Dirac (1928): need first order operator D on $\mathbb{R}^n$ s.t. $D^2 = \Delta \left(= -\frac{\partial^2}{\partial x_1^2} - \dots \frac{\partial^2}{\partial x_n^2}\right)$.

Assume $D = a_1 \frac{\partial}{\partial x_1} + \dots + a_n \frac{\partial}{\partial x_n}$, constant coeff in some alg. structure (to be determined).

$$D^2 = \Delta \iff \begin{cases} a_i^2 = -1 \\ a_i a_j = -a_j a_i \text{ if } i \neq j. \end{cases}$$

$\boxed{n=1}$ Can take $a_1 = i \in \mathbb{C}$,

$D = i\frac{\partial}{\partial x_1} : C^\infty(\mathbb{R};\mathbb{C})$

$\boxed{n=2}$ $a_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ (2)

$$D = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} = \begin{bmatrix} 0 & -\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} & 0 \end{bmatrix}$$

acting on $C^\infty(\mathbb{R}^2; \mathbb{C}^2)$.

Rmk holomorphic/antiholomorphic derivatives!

$\boxed{n=3}$ $a_i = \sigma_i$ Pauli matrices (basis of $su(2)$)

$$\sigma_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$D \curvearrowright C^\infty(\mathbb{R}^3, \mathbb{C}^2)$.

$\boxed{n=4}$ We can consider the 4×4 block matrices

$$\begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\sigma_i^* \\ \sigma_i & 0 \end{bmatrix} \quad i = 1, 2, 3. \quad (*)$$

$D \curvearrowright C^\infty(\mathbb{R}^4, \mathbb{C}^4)$.

Continue inductively! We obtain

Dirac operator $D : C^\infty(\mathbb{R}^n; \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}})$

Rmk in general, the "algebraic structure" the a_i live in is the Clifford algebra, and we look at functions with values in a Clifford module; we obtained the "minimal" one, but for example one can also obtain $D = d + d^* : C^\infty(\mathbb{R}^n; \Lambda^* \mathbb{R}^n)$.

We'll discuss how to use this to define the Seiberg-Witten invts ('94) of X cpt oriented

Sample application: $\exists \{X_i\}_{i \in \mathbb{N}}$ collection all homeomorphic to each other, but not pairwise diffeomorphic.

<u>Def</u> (X^4, g) Riemannian, a <u>spin^c structure</u>

is $\mathfrak{s} = (S, \rho)$ where

•) $S \to X$ is $\mathrm{rk}_{\mathbb{C}} = 4$ hermitian, <u>spinor bundle</u>

•) $\rho : TX \to su(S)$ <u>Clifford multiplication</u>

looks pointwise like (*)

<u>Rmk</u> not trivial <u>at all</u> that $\mathfrak{s}$ exists!!

$\{\text{spin}^c \text{ structures}\}/_{\text{iso}}$ is affine $/ H^2(X;\mathbb{Z})$

Define on T^*X by duality, extend via

$\rho(\alpha \wedge \beta) = \frac{1}{2}\left(\rho(\alpha)\rho(\beta) + (-1)^{|\alpha||\beta|}\rho(\beta)\rho(\alpha)\right)$.

to $\rho : \Omega^*(X) \otimes \mathbb{C} \to \mathrm{End}(S)$

<u>Then</u> $\rho(\mathrm{dvol}_g) = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix} \Rightarrow S = \underbrace{S^+ \oplus S^-}_{\mathrm{rk}_{\mathbb{C}} = 2}$

($\rho(v)$ for $v \in TX$ exchanges them).

We will consider pairs (A, Φ) where

•) $\Phi \in \Gamma(S^+)$ is a (positive) spinor

•) ∇_A is a <u>spin^c connection</u>, i.e. hermitian connection on S compatible with ρ & Levi-Civita

$$\nabla_A(\rho(X)\cdot\sigma) = \rho(\nabla X)\cdot\sigma + \rho(X)\cdot\nabla_A\sigma.$$

$$\forall X \in \Gamma(TX),\ \sigma \in \Gamma(S).$$

<u>Facts</u> •) ∇_A preserves splitting $S^+ \oplus S^-$

•) $\nabla_A - \nabla_{A'} = a \otimes \mathbb{1}_S$ for $a \in i\Omega^1(X)$ (because ρ acts irreducibly).

$\Rightarrow \mathcal{C} = \mathcal{C}(X, s) = \{(A, \Phi)\}$ = affine$/_{i\Omega^1} \oplus \Gamma(S^+)$.

configuration space. Gauge group = symmetries of (S, ρ) is

$\mathcal{G} = \{u : X \to U(1)\}$, via usual action

$u\cdot(A,\Phi) = (A - u^{-1}du,\ u\cdot\Phi)$: action

on $\mathcal{C}^* = \{(A,\Phi) \mid \Phi \neq 0\}$ is free.

↳ irreducible configurations.

$\boxed{\text{Eqn 1}}$ $\Gamma(S^+) \xrightarrow{\nabla_A} \Gamma(T^*X \otimes S^+) \xrightarrow{\rho} \Gamma(S^-)$

D_A^+ Dirac operator (1st order, elliptic)

(SW1) $D_A^+\Phi = 0$ $(\in \Gamma(S^-))$.

$\boxed{\text{Eqn 2}}$ A connection on $S^+ \rightsquigarrow A^t$ connection on

$\det S^+ = \Lambda^2 S^+$ ← hermitian line bundle,

$F_{A^t} \in i\Omega^2(X)$.

<u>Rmk</u> If $\tilde{A} - A = a \otimes 1_S \Rightarrow \tilde{A}^t - A^t = 2a$.

Given $\Phi \in \Gamma(S^+)$, we have $(\Phi\Phi^*)_0 \in i\mathfrak{su}(S^+)$ ← traceless hermitian

(if $\Phi = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, $(\Phi\Phi^*)_0 = \begin{bmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha\bar{\beta} \\ \bar{\alpha}\beta & \frac{1}{2}(|\beta|^2 - |\alpha|^2)\end{bmatrix}$)

Key point $\rho : \Lambda^+ \xrightarrow{\sim} su(S^+)$ (check)
↳ self-dual forms

Fix $\omega \in i\Omega^2$. The 2nd SW eqn (perturbed by ω) is

(SW2) $\frac{1}{2}\rho(F_{A^t}^+ - 4\omega^+) = (\Phi\Phi^*)_0 \in \Gamma(isu(S^+))$.

Def $\mathcal{M}_{(g,\omega)} = \{ \text{slns to SW1 \& 2} \} / \mathcal{G}$. (depends on s!)

Rmk These are called Seiberg-Witten monopoles
because of the topological bound:

$$\int |F_{A^t}^+| + 4\int |D_A\Phi|^2 + \int \left(|\Phi|^2 + \frac{r}{2}\right)^2 \geq \int \frac{r^2}{4} - 4\pi^2 c_1^2(s)$$

where: •) r is the scalar curvature of g

•) $c_1(s) = c_1(S^+)$.

This follows from Chern-Weil + integrating by parts + Weitzenböck formula.

Thm A If $b_2^+ \geq 1$, for generic (g, a)

$\mathcal{M}_{(g,a)}$ is a smooth closed oriented mfld of dim

$$d(X,s) = \frac{c_1^2(s) - 2\chi(X) - 3\sigma(X)}{4} \in \mathbb{Z}.$$

This uses Atiyah-Singer index thm.

Rmk Key point: generically when $b_2^+ \geq 1$, all solutions are irreducible.

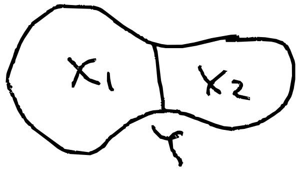
Thm B (simplest case) If $b_2^+ \geq 2$, $d(X,s) = 0$

$\Rightarrow \# \mathcal{M}_{(g,a)}$ independent of (g,a)

$\Rightarrow SW(X,s) \in \mathbb{Z}$ is diffeo invariant.

Rmk First such invts were defined by Donaldson ('87) using the $SU(2)$ ASD equs. $\{F_A^+ = 0\}/\mathcal{G}$ ← instanton moduli space, non cpt!

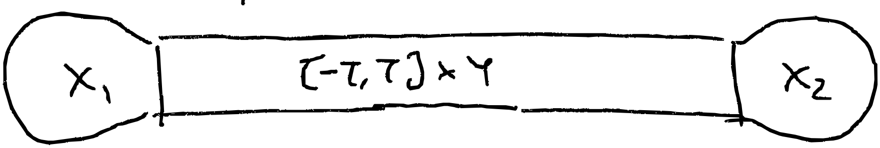
<u>Major issue</u> extremely hard to compute!

<u>Ex</u> $X =$ 

$\Rightarrow X' = X_1 \cup_\varphi X_2$ with $\varphi : Y \circlearrowleft$ diffeo

<u>Q</u> what's the effect on SW invts?

<u>Idea</u> exploit metric independence to "decouple" the two pieces via neck stretching



Now there's a 3D story of spin^c structure $S_Y \to Y$ $\mathrm{rk}_{\mathbb{C}} = 2$ hermitian; we have

$\mathcal{C}_Y = \{(B, \Psi)\}$ configuration space of Y.

On $I \times Y$, we have naturally

$$S^+ \underset{\text{via } \rho(\frac{\partial}{\partial t})}{\cong} S^- \cong S_Y .$$

So a path $(B(t), \Psi(t))$ in $\mathcal{C}_Y$ gives

to $(A, \Phi) \in \mathcal{C}$ $\left[\nabla_A = \frac{d}{dt} + \nabla_{B(t)} \right]$.

$\boxed{\text{Key computation}}$ (A, Φ) solves SW eqns

$$\Updownarrow$$

$$\begin{cases} \frac{d}{dt} B = -\left(\frac{1}{2} * F_{B^t} + \rho^{-1}(\Psi\Psi^*)_0\right) \otimes \mathbb{1}_S \\ \frac{d}{dt}\Psi = -D_B \Psi \end{cases}$$

where $D_B : \Gamma(S) \circlearrowleft$ 3D Dirac operator.

Further, these can be written as

$$\frac{d}{dt}(B, \Psi) = -\operatorname{grad} \mathcal{L}(B, \Psi)$$

where, fixing a base connection B_0, we have

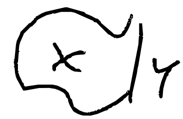
$$\mathcal{L}(B, \Psi) = -\frac{1}{8}\int_Y (B^t - B^t_0) \wedge (F_{B^t} + F_{B^t_0}) + \frac{1}{2}\int_Y \langle D_B \Psi, \Psi \rangle \, dvol$$

is the <u>Chern-Simons-Dirac functional</u>.

Employing the ideas of Floer ('80s) for (formally) doing Morse theory in certain (very specific!) ∞ dim setups (and taking into account $U(1)$-action):

<u>Kronheimer–Mrowka</u> ('07) <u>monopole Floer homology groups</u> $HM(Y, \mathfrak{s}_Y)$;

(Roughly speaking, this is the homology of a chain complex generated by soln to 3D SW eqns $= \{\mathrm{grad}(B,\Psi)=0\}/\mathcal{G}_Y$)

These are where invariants of 4-mfld 

with ∂Y take value (cf: Atiyah's TQFT framework).

These 3 mfld ints are still extremely difficult to understand / compute!

Going further down we look at $\Sigma \subset Y$.

$\boxed{\text{Key computation}}$ Translation invariant solns to $\text{grad}(B, \Psi) = 0$ on $I \times \Sigma$

$\Leftrightarrow$ solutions to (small variation) on vortex eqns on Σ.

Heuristic: one can hope to define interesting invts of 3-mflds ($\cong HM(Y)$) by studying $\text{Sym}^d(\Sigma) \cong$ vortex moduli space, in particular its symplectic geometry (again using Floer's ideas)

Ozsváth-Szabó ('04) construction of Heegaard Floer homology.

$\hookrightarrow \cong HM$, but much easier to compute!