Lecture 11 Vortices & symmetric products

(Σ, g) cpt Riemannian surface, $L \to \Sigma$ hermitian line bundle of degree $d = d(L) \geq 0$.

$$\mathcal{M} = \left\{ (A, \Phi) \mid \Phi \not\equiv 0, \begin{cases} \bar\partial_A \Phi = 0 \\ * F_A = \frac{i}{2}(|\Phi|^2 - 1) \end{cases} \right\} / \mathcal{G}$$

vortex moduli space.

Recall $\deg(L) = \int_\Sigma \frac{i}{2\pi} F_A = \frac{1}{4\pi} \int_\Sigma (1 - |\Phi|^2) \, d\mathrm{vol} < \frac{\mathrm{Area}(\Sigma)}{4\pi}$.

Thm (Taubes, Bradlow, García-Prada)

Assume $d < \mathrm{Area}(\Sigma)/4\pi$. Then

$\mathcal{M}$ admits a natural structure of smooth symplectic mfld of $\dim = 2d$

$$\mathcal{M} \xrightarrow{1:1} \mathrm{Sym}^d(\Sigma) \quad [(A, \Phi)] \to \{\text{zeroes of } \Phi\}$$

is a homeomorphism.

This will follow by comparing two geometric views:

<u>Symplectic</u>: $\mathcal{N} = \{ \Phi \neq 0, \bar{\partial}_A \Phi = 0 \} \subseteq \mathcal{C}^*$

is a <u>symplectic</u> submanifold (∞-dim)

↳ (will elaborate more later!)

$\mathcal{G} \curvearrowright \mathcal{N}$ is hamiltonian action with moment map

$$\mu : \mathcal{N} \to i\Omega^0 \quad (A, \Phi) \mapsto *F_A - \frac{i}{2}|\Phi|^2 + \frac{i}{2}.$$

$$\Rightarrow \mathcal{M} = \mu^{-1}(0)/\mathcal{G} = \mathcal{N}/\!/\mathcal{G} \quad \text{symplectic quotient!}$$

<u>Complex</u>: $\mathcal{N}/\mathcal{G}^{\mathbb{C}} \overset{\text{last time}}{\overset{!}{=}} \mathrm{Sym}^d(\Sigma)$.

<u>Upshot</u> $\forall\ (A, \Phi) \in \mathcal{N}$, need to show

$\exists\ g \in \mathcal{G}^{\mathbb{C}}$ s.t. $\mu(g \cdot (A, \Phi)) = 0$

(unique up to $u \in \mathcal{G}$), i.e

the value $\mu=0$ is attained in each (3)

orbit. Let's focus on existence first (harder!)

This will be based on special properties

in the case of X Kähler, i.e. X cplx

mfld (with induced $J: TX^{\circlearrowleft}$ a.c.s)

equipped with ω symplectic compatible:

$g(v,w) := \omega(v, Jw)$ is Riemannian metric.

Ex.) $(\mathbb{C}^n, \omega_{std})$, $(\mathbb{CP}^n, \omega_{FS})$

•) (X,ω) Kähler $V \subseteq X$ cplx submanifold

$\Rightarrow (V, \omega|_V)$ is Kähler.

•) $\mathcal{N} \subset \mathcal{C}^*$ is Kähler submanifold

(in suitable Sobolev completions).

The key computation (in finite dim)

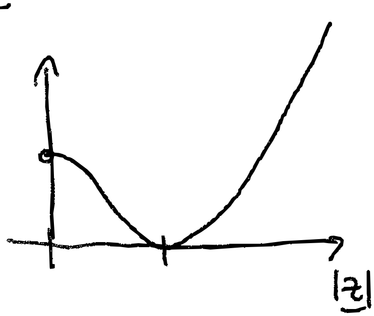
Thm (Fraenkel '59, simplest case $G = \mathbb{C}^*$)

(X, ω) Kähler, $\mathbb{C}^* \curvearrowright X$ freely by biholomorphisms s.t. the restriction $U(1) \curvearrowright X$ is Hamiltonian action (with moment map $\mu : X \to i\mathbb{R} = \mathrm{Lie}(U(1))$).

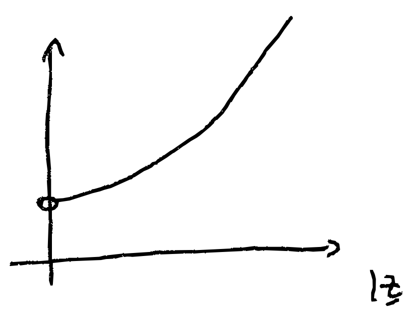
Consider $F : X \to \mathbb{R}^{\geq 0}$ $x \mapsto |\mu(x)|^2$.

Then the critical points of F are critical points of $F|_{\mathbb{C}^*\text{-orbits}}$, and $\mu = 0$ at them.

Ex $\mathbb{C}^* \curvearrowright \mathbb{C}^n \setminus \{0\}$, $\mu = -\frac{i}{2}|z|^2 + ic$ $c \in \mathbb{R}$.



$c > 0$



$c < 0$

Rmk setup $\Rightarrow$ $U(1)$ acts by isometries

<u>Proof</u> Let's identify $i\mathbb{R} \equiv \mathbb{R}$, so $\mu : X \to \mathbb{R}$.

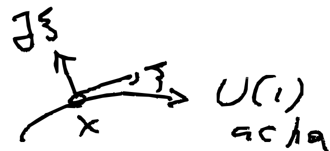
$$\langle \operatorname{grad} F, w \rangle_x = dF_x(w) = 2\, \mu_x \, d\mu_x(w).$$

By def, $\mu(x) = \langle \mu(x), 1 \rangle$ is the Hamiltonian function for the vector field ξ generating $U(1) \curvearrowright X$, i.e.

$$d\mu = \omega(\xi, -)$$

$$\Rightarrow d\mu_x(w) = \omega(\xi, w) = \omega(J\xi, Jw) =$$

$$= \langle J\xi, w \rangle$$



$$\Rightarrow \operatorname{grad} F_x = 2\mu_x \cdot J\xi \in T_x X$$

$\Rightarrow$ grad F is tangent to $\mathbb{C}^*$ orbits. Action free $\Rightarrow \xi \neq 0$ everywhere $\Rightarrow$ at critical points $\mu = 0$. $\square$

To apply in our setup, notice that

if $\mu(A,\Phi)=0 \Rightarrow \|\Phi\|^2_{L^2} = \text{Area}(\Sigma) - 4\pi d$

is determined. So we can consider

$$\overline{\mathcal{N}} := \{ \bar\partial_A \Phi = 0,\ \|\Phi\|^2_{L^2} = \text{Area}(\Sigma) - 4\pi d \} / \underset{\text{constants}}{U(1)}$$

(non empty by assumption!)

$\overline{\mathcal{G}}^{\mathbb{C}} = \mathcal{G}^{\mathbb{C}} / \underset{\text{constants}}{\mathbb{C}^*} \curvearrowright \overline{\mathcal{N}}$ where

$\bar g \cdot (A,\Phi) := g\cdot(A,\Phi)$ where $g \in \mathcal{G}^{\mathbb{C}}$ is s.t

$\|g\Phi\|_{L^2} = \|\Phi\|_{L^2}$ (unique, up to $U(1)$)

<u>Rmk</u> from a <u>symplectic</u> viewpoint, we have

$U(1) \subset \mathcal{G}$ and this is equivalent to:

$$\mathcal{M} = \mathcal{N}/\!\!/\mathcal{G} = \underbrace{(\mathcal{N}/\!\!/U(1))}_{= \overline{\mathcal{N}}}/\!\!/(\mathcal{G}/U(1))$$

<u>Goal</u> Fix $(A_0, \Phi_0) \in \mathcal{N}$, show that $\|\mu\|^2_{L^2}$ attains a minimum at (A_{min}, Φ_{min}) $\in \mathcal{O} := \overline{\mathcal{G}^{\mathbb{C}}} \cdot (A_0, \Phi_0)$ $(\Rightarrow \mu(A_{min}, \Phi_{min}) = 0)$

This we can do with a <u>direct</u> variational approach: Set

$$C = \inf_{\mathcal{O}} \|\mu(A, \Phi)\|^2_{L^2},$$

and consider $(A_n, \Phi_n) (= \bar{g}_n (A_0, \Phi_0), \bar{g}_n \in \mathcal{G}^{\mathbb{C}})$ s.t.

$$\|\mu(A_n, \Phi_n)\|^2_{L^2} \searrow C.$$

<u>Want</u> up to subsequence & $\mathcal{G}$

$(A_n, \Phi_n) \to (A_{min}, \Phi_{min}) \in \mathcal{O}$ realizing inf.

<u>Notice</u> $\|\mu(A_n, \Phi_n)\|^2_{L^2} \leq C$. $\forall n$.

$\boxed{\text{Step 1}}$ $\|F_{A_n}\|^2_{L^2}$ is bounded.

<u>Proof</u> This follows from key identity

$$\frac{1}{2}\int|F_A|^2+\frac{1}{2}\int|\nabla_A\Phi|^2+\frac{1}{8}\int(1-|\Phi|^2)^2=$$

$$=\cancel{\|\bar{\partial}_A\Phi\|^2_{L^2}}+\frac{1}{2}\|\mu(A,\Phi)\|^2_{L^2}+\pi d.$$

in our setting.

$\boxed{\text{Step 2}}$ up to $\mathcal{G}$, $\{A_n\}$ is bounded in L^2_1.

(ie $A_n=A_0+a_n$, $\|a_n\|_{L^2_1}<C$ $\forall n$).

<u>Proof</u> we can make transformations

$A_n\to u_n\cdot A_n$, $u_n\in\mathcal{G}$ so that

•) A_n is in Coulomb gauge w.r.t to A_0

$d^*a_n=0$ (just need $\mathcal{G}_{id}$)

•) $a_n^{harm} \in \mathcal{H}^1$ is in fixed fundamental domain

for $H^1(\Sigma;\mathbb{R})/H^1(\Sigma;\mathbb{Z})$.

Then F_{A_n} bounded $\Rightarrow$ da_n bounded

$\Rightarrow$ L^2_1-bounds by Gårding ineq.

$\boxed{\text{Step 3}}$ $\{\Phi_n\}$ is bounded in L^2_1

Proof By key identity, $\{\Phi_n\}$ is bounded in L^4.

Now $\bar\partial_{A_n} = \bar\partial_{A_0} + a_n^{0,1}$, so

$$\|\Phi_n\|_{L^2_1} \le C(\|\bar\partial_{A_0}\Phi_n\|_{L^2} + \|\Phi_n\|^2_{L^2}) \quad \text{(Gårding)}$$

$$= C(\|a_n^{0,1}\cdot\Phi_n\|_{L^2} + \|\Phi_n\|^2_{L^2})$$

fixed!

This is bounded because $L^2_1 \hookrightarrow L^4$

by Sobolev embedding in dim 2.

$\boxed{\text{Step 4}}$ Abstract nonsense moment:

<u>Prop</u> H cplx separable Hilbert space, $\{x_i\} \subseteq H$ bounded sequence.

$\Rightarrow$ after taking subsequence,

$x_i \rightharpoonup x \in H$ <u>weak</u> convergence, i.e

$\langle x_i, v\rangle_H \to \langle x, v\rangle_H \quad \forall\, v \in H$.

<u>Ex</u> In $\ell^2(\mathbb{N})$, $e_n = (0, \dots 0, 1, 0, \dots)$
↳ nth place

$e_n \rightharpoonup 0$.

<u>Consequence</u> $\exists (A, \Phi) \in \mathcal{C}_1 \leftarrow L^2_1$-configurations

s.t $(A_n, \Phi_n) \rightharpoonup (A, \Phi)$ (weak L^2_1)

This is our candidate minimum in $\mathcal{B}$!!

$\boxed{\text{Step 5}}$ $\|\mu(A, \Phi)\|_{L^2}$ is $c = \inf$

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This is because the embedding $L^2_1 \hookrightarrow L^4$ is actually cpt (cf. Rellich), and cpt ops send weakly convergent sequences to convergent ones.

To conclude existence proof.

$\boxed{\text{Step 6}}$ (A, Φ) is in the orbit

$\overline{\mathcal{G}_2^{\mathbb{C}}} \cdot (A_0, \Phi_0)$.

Idea. write $(A_n, \Phi_n) = \bar{g}_n \cdot (A_0, \Phi_0)$, show $\bar{g}_n$ converges in L^2_2 as in proof of Hausdorff

$\boxed{\text{Step 7}}$ (A, Φ) is smooth $\in \overline{\mathcal{G}^{\mathbb{C}}} \cdot (A_0, \Phi_0)$.

Idea elliptic bootstrap in Coulomb gauge (need serious L^p theory for this step!!).

To prove uniqueness up to action of $\mathcal{G}$

notice $\mathcal{G}^{\mathbb{C}}/\mathcal{G} \cong \{ e^h \mid h : \Sigma \to \mathbb{R} \}$,

and we computed last time

$e^h \cdot (A, \Phi) = (A + i * dh, e^h \Phi)$. If

$$\mu(e^h(A,\Phi)) = *F_A + * i\,\underbrace{d*dh}_{-\Delta h} - \frac{i}{2} e^{2h} |\Phi|^2 + \frac{i}{2} = 0$$

and $\mu(A,\Phi) = 0$, subtracting we have

$$\Delta h = (1 - e^{2h}) \underbrace{|\Phi|^2}$$ has isolated zeroes

Now, $\| e^h \cdot \Phi \|_{L^2} = \| \Phi \|_{L^2} \Rightarrow$ if $h \not\equiv 0$,

$h(\bar{x}) > 0$ at maximum $\bar{x}$ of h. But

$\Delta h(\bar{x}) \geq 0$ (in $\mathbb{R}^2$, $\Delta = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$ 

)

$\Rightarrow$ contradiction if $\Phi(\bar{x}) \neq 0$. If $\Phi(\bar{x}) = 0$,

use $\Delta h \leq 0$ near $\bar{x}$ + strong maximum principle $\square$.