$\boxed{\text{Lecture 10}}$ Moduli spaces in gauge theory.

<u>General principle</u> sets of gauge theoretic objects (configurations or solutions) up to symmetry have natural structures.

<u>Recap</u>: $L \to (\Sigma, g)$ hermitian line bundle.

$\mathcal{A} = \{A \text{ compatible connection}\}$

$\mathcal{C} = \{(A, \Phi) \mid A \text{ compatible connection}, \Phi \in \Gamma(L)\}$

$\cup$

$\mathcal{C}^* = \{(A, \Phi) \mid \Phi \not\equiv 0\}$

$\cup$

$\mathcal{N} := \{(A, \Phi) \mid \Phi \not\equiv 0, \bar{\partial}_A \Phi = 0\}.$

$\cup$

$$\mathcal{V} := \left\{(A, \Phi) \mid \Phi \not\equiv 0 \; \begin{cases} \bar{\partial}_A \Phi = 0 \\ *F_A = \frac{i}{2}(|\Phi|^2 - 1) \end{cases}\right\}$$

These are all acted on by

$$\mathcal{G} = \{ u : \Sigma \to U(1) \}, \quad u \cdot (A, \Phi) = (A - u^{-1}du, u \cdot \Phi)$$

$$\Rightarrow \mathcal{A}/\mathcal{G}, \quad \mathcal{C}/\mathcal{G} \supseteq \mathcal{C}^*/\mathcal{G} \supseteq \mathcal{N}/\mathcal{G} \supseteq \underbrace{\mathcal{V}/\mathcal{G}}_{}=\mathcal{M}$$ want to understand this!

Notice: the action of $\mathcal{G} \curvearrowright \mathcal{C}^*$ is free

($u^{-1}du \equiv 0 \Rightarrow u$ constant), so one hopes that

$\mathcal{C}^*/\mathcal{G}$ is an infinite dimensional smooth mfld.

To make this precise, we work with

Sobolev completions. Fix $\bar{A} \in \mathcal{A}$,

$\mathcal{A}_k := \bar{A} + L^2_k(i\Omega^1) \leftarrow L^2_k$ connections.

$\mathcal{C}_k = \{ (A, \Phi) \mid A \in \mathcal{A}_k, \Phi \in L^2_k(\Gamma(L)) \}$

L^2_k config. notions, Hilbert affine space.

For $k \geq 1$, consider

$$\mathcal{G}_{k+1} = \{ u : \Sigma \to U(1) \} \subseteq L^2_{k+1}(\Sigma, \mathbb{C})$$

(importantly, $\subseteq C^0(\Sigma, \mathbb{C})$, so u has well-defined values and preserves bundle topology!).

Then we have $\mathcal{G}_{k+1} \curvearrowright \mathcal{C}_k$ continuous

$$\Big(A - \underset{L^2_{k+1}}{\underset{\Downarrow}{u}}{}^{-1} \underbrace{du}_{\in L^2_k} \; , \; \underset{L^2_{k+1}}{\underset{\Downarrow}{u}} \cdot \Phi_{\in L^2_k} \Big),$$ in fact smooth.

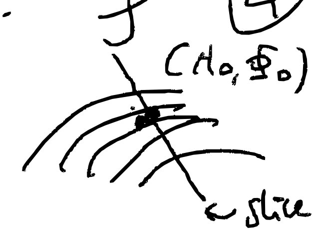
<u>Thm!</u> $\mathcal{C}^*_k / \mathcal{G}_{k+1}$ is a Hilbert manifold. $(k \geq 1)$.

The proof is analogous to: G cpt Lie group, $G \curvearrowright M$ freely $\Rightarrow$ M/G smooth mfld. Two steps

a) M/G is Hausdorff

b) build local charts for M/G

We'll work out a) in detail.

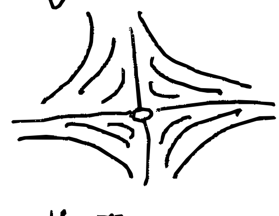


Remark b) follows by considering the analogue of Coulomb gauge fixing:

$S := L^2$ orthogonal to $T_{(A_0,\Phi_0)}(\mathcal{G}\cdot(A_0,\Phi_0))$.

a) already fails in finite dim if G not cpt! e.g. $\mathbb{R}^{>0} \curvearrowright \mathbb{R}^2 \setminus \{0,0\}$

$$t\cdot(x,y) = (tx, t^{-1}y)$$



In our setup, we want to show that

if $(A_n, \Phi_n) \to (A_\infty, \Phi_\infty)$, and for $u_n \in \mathcal{G}_{U(1)}$

$(A_n', \Phi_n') = u_n(A_n, \Phi_n) \to (A_\infty', \Phi_\infty')$

$\Rightarrow \exists\, u_\infty$ s.t. $u_\infty(A_\infty, \Phi_\infty) = (A_\infty', \Phi_\infty')$.

First, the class $[u_n] \in \pi_0 \mathcal{G}_{U(1)}$ is eventually constant, this is represented by as

$$\left[\tfrac{1}{2\pi i} u_n^{-1} du_n\right] \in H^1(\Sigma;\mathbb{Z}) \quad (= \pi_0(\mathcal{G}_{U(1)}))$$

Fix 1-forms $\{\beta_i\}$, $i=1,\dots 2g$ with of $H^1(\Sigma;\mathbb{Z})$. Then

$$\int_\Sigma \beta_i \wedge u_n^{-1} du_n = \int \beta_i \wedge (A_n - A_n') \to \int \beta_i \wedge (A_\infty - A_\infty')$$

$\Rightarrow$ can assume $u_n \in$ identity component, $u_n = e^{\xi_n}$.

Write $\xi_n = \underset{\text{constant}}{\xi_n^0} + \underset{\in \langle\text{constants}\rangle^\perp}{\xi_n^\perp}$.

$\Rightarrow d(\xi_n^\perp) = A_n - A_n'$ is Cauchy in L^2_k

$\Rightarrow \xi_n^\perp$ is Cauchy in L^2_{k+1} (Gårding)

$\Rightarrow$ converges to $\xi_\infty^\perp$. Choose subsequence s.t. $e^{\xi_n^0} \in U(1)$ converges, then

$u_n = e^{\xi_n^0} \cdot e^{\xi_n^\perp} \to u_\infty$, gauge transformation we were looking for. $\square$

$\mathcal{C}_k$ is a good space to study our PDEs! For example we have continuous (indeed smooth) map

$$\mathcal{F} : \mathcal{C}_k \longrightarrow L^2_{k-1}(\Omega^{0,1}(L))$$
$$(A, \Phi) \longmapsto \bar\partial_A \Phi .$$

Indeed, writing $A = A_0 + a$, $a \in L^2_k(i\Omega)$, it's

$$\bar\partial_{A_0} \Phi + a^{0,1} \cdot \Phi . \quad (\bar\partial_{A_0} \text{ has smooth coeff!})$$

If $k \geq 2$, this is in L^2_{k-1} by Sobolev multiplication. If $k = 1$, we use $\forall p < \infty$ $L^2_1 \hookrightarrow L^p$ (in fact, it's cpt!) ($L^2_1 \not\hookrightarrow C^0$!)

so $a^{0,1} \cdot \Phi \in L^4 \cdot L^4 \subseteq L^2$ by Cauchy-Schwarz.

Notice $\mathcal{N}_k = \{(A, \Phi) \mid \Phi \neq 0, \bar\partial_A \Phi = 0\}$

$$= \mathcal{F}^{-1}(0) \cap \mathcal{C}^*_k$$

$\hookrightarrow$ no-action.

Let's compare with the holomorphic picture.

Recall correspondence for L:

$$\{\text{hol structures } \mathcal{L}\} \xleftrightarrow{1:1} \{\text{hermitian connections } A\}$$

$$\mathcal{A}^{0,1} = \{ d^{(0,1)} : \Omega^0(L) \to \Omega^{0,1}(L) \text{ satisfying } (0,1)\text{-Leibniz}\}.$$

$\mathcal{L} \mapsto$ unique A s.t. $\bar{\partial}_A = \bar{\partial}_{\mathcal{L}}$.

This is an affine isomorphism via the map

$$L^2_k(\Omega^{0,1}) \xleftrightarrow{\sim} L^2_k(i\Omega^1)$$

$$a^{0,1} \longleftarrow a$$

($\mathbb{C}$-linear for action of $*$ on Ω^1).

The natural symmetries of $\mathcal{A}^{0,1}$ are

$$\mathcal{G}^{\mathbb{C}}_{k+1} = \{ g : \Sigma \to \mathbb{C}^* \}.$$

$g \cdot d^{(0,1)} = d^{(0,1)} - g^{-1}\bar{\partial} g$; at the level of A it's

$$A \mapsto A - (g^{-1}\bar{\partial} g) + \overline{(g^{-1}\bar{\partial} g)}$$

<u>Thm 2</u> $\mathcal{A}^{0,1}_k / \mathcal{G}^{\mathbb{C}}_{k+1} \cong H^1(\Sigma;\mathbb{R}) / H^1(\Sigma;\mathbb{Z})$

$\shortparallel$ $\qquad\qquad\qquad\qquad \uparrow$

{hol structures $\mathcal{L}$}/iso $\qquad$ $2g(\Sigma)$-dim torus.

(<u>Picard torus.</u>)

<u>Proof</u> $\pi_0(\mathcal{G}^{\mathbb{C}}_{k+1}) = H^1(\Sigma;\mathbb{Z})$, so

$$\mathcal{A}^{0,1}/\mathcal{G}^{\mathbb{C}} = \left(\mathcal{A}^{0,1}/(\mathcal{G}^{\mathbb{C}})^{id}\right) / H^1(\Sigma;\mathbb{Z}).$$

g in identity component is $g = e^f$ for $f:\Sigma\to\mathbb{C}$,

acting via $e^f\cdot A = A - \bar\partial f + \partial\bar f$.

<u>Computation</u>: if $f = u+iv$, this is

$$A - i(dv - * du)$$

$\Rightarrow$ can add any exact/coexact form

$\Rightarrow \mathcal{A}^{0,1}/(\mathcal{G}^{\mathbb{C}})^{id} \cong \mathcal{H}^1$ harmonic 1-forms

$\triangleleft$.

Cor of proof: every $A \in \mathcal{A}_k$ is $\mathcal{G}_{k+1}^{\mathbb{C}}$ equivalent to a smooth connection.

Rmk: not true for $\mathcal{G}_{k+1}$! Curvature is $\mathcal{G}_{k+1}$ invariant.

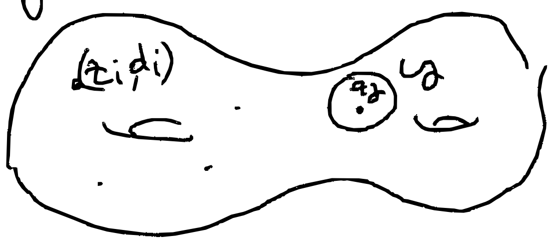
Now, we also have $\mathcal{G}_{k+1}^{\mathbb{C}} \curvearrowright \mathcal{N}_k$, and every (A, Φ) is equivalent to a smooth one: A is equivalent to A^{smooth}, and $\ker(\bar\partial_{A^{\text{smooth}}}) \subseteq C^\infty$ by elliptic reg.

Thm 3 $\mathcal{N}_k / \mathcal{G}_{k+1}^{\mathbb{C}} \overset{1:1}{\longleftrightarrow} \mathrm{Sym}^d(\Sigma)$. $\quad d$ ~ degree of L

$(A, \Phi) \longmapsto$ zeros of Φ (with multiplicity).

Rmk from this, to identify vortex moduli space we need to show that each orbit of $\mathcal{G}^a \curvearrowright \mathcal{N}$ contains a solution of the 2nd vortex eqn, unique up to $\mathcal{G}$

Next time! ↳ ⚠ not $\mathcal{G}^a$ invariant!



Proof Surjectivity:

Choose trivial bundles on $S \setminus \{z_1, \ldots\}$, $U_j \cong D^2$,

and use transition functions

$D^2 \setminus \{0\} \to \mathbb{C}^*$ $z \mapsto z^{d_j}$,

the constant section 1 on $S \setminus \{z_1, \ldots\}$ extends with zero of order d_j.

Injectivity If (A, Φ), $(A', \Phi') \in \mathcal{N}_k$ are such that Φ, Φ' have the same zeroes with multiplicities, then $\exists!$

$g \in \mathcal{G}^{\mathbb{C}}_{k+1}$ s.t $g \cdot \Phi = \Phi'$.

Then $g \cdot A = A'$ (check!) $\square$.

Rmk $\mathcal{N}/\mathcal{G}^{\mathbb{C}} \xrightarrow{\text{forget } \Phi} \mathcal{A}/\mathcal{G}^{\mathbb{C}}$ is the Abel-Jacobi map.

Going back to $\mathcal{N}_k$, we have

Thm $\mathcal{N}_k$ is a Hilbert mfld. $(k \geq 1)$

Rmk useful because we'll do symplectic quotient!

Proof We'll apply implicit function thm at (A, Φ) with $\bar{\partial}_A \Phi = 0$, $\Phi \not\equiv 0$.

$$D\mathcal{F}_{(A,\Phi)} : L^2_k(i\Omega^2) \oplus L^2_k(\Gamma(L)) \to L^2_{k-1}(\Omega^{0,1}(L))$$

$$(a, \phi) \mapsto \bar{\partial}_A \phi + a^{0,1} \Phi.$$

We need to prove surjectivity.

By $\mathcal{G}^{q}$ invariance $\left(\bar\partial_{g\cdot A}(g\Phi) = g\bar\partial_A\Phi \right)$, we can assume (A,Φ) smooth.

Note $\operatorname{Im} D\mathcal{F}_{(A,\Phi)} \supseteq \bar\partial_A\phi \Rightarrow$ finite codim $\Rightarrow$ closed; if it's not everything, $\exists\ b^{0,1} \in L^2_{k-1}(\Omega^{0,1}(L))$ L^2-orthogonal to image, i.e

$$\langle \bar\partial_A\phi + a^{0,1}\Phi,\ b^{0,1}\rangle_{L^2} = 0 \quad \forall\, a,\phi.$$

Using $(0,\phi)$, we see $\bar\partial_A^* b^{0,1} = 0$ weakly. $\Rightarrow b^{0,1}$ smooth. Now $\bar\partial_A(\bar{*}b^{0,1}) = 0$ for $\bar\partial_A : \Omega^{1,0}(L) \to \Omega^{1,1}(L)$

$\Rightarrow b^{0,1}$ is <u>not</u> identically vanishing on any open set $\Rightarrow \exists\ a^{0,1}$ s.t

$$\langle a^{0,1}\Phi,\ b^{0,1}\rangle_{L^2} \neq 0 \qquad \square.$$