# $\boxed{\text{Lecture 9}}$ Moment maps in symplectic geometry & gauge theory

electromagnetism : U(1)-gauge theory

= classical mechanics : symplectic geometry.

<u>Def</u> A <u>symplectic mfld</u> $(M^{2n}, \omega)$ is a smooth mfld equipped with $\omega \in \Omega^2(M)$ s.t

1) $d\omega = 0$ (closed)
2) $\omega^n \neq 0$ everywhere ($\omega$ is non degenerate skew-symmetric)

<u>Examples</u> 1) $(\mathbb{R}^{2n}, \omega_{std})$, $\omega_{std} = \sum_{j=1}^{n} dx_j \wedge dy_j$.

$\uparrow$
$(x_1, y_1, \dots x_n, y_n)$

2) More generally, $M = T^*X$ $\leadsto$ smooth mfld

has natural symplectic form $\omega_{can}$:

in coordinates $\{x_1, \dots x_n\}$ on $X$, so

$T_x^* X = \{\sum \xi_j (dx_j)_x\}$ has coordinates $\{\xi_j\}$

$\omega_{can} = \sum dx_j \wedge d\xi_j$ (well-defined!)

3) $\Sigma$ surface, $\omega$ volume form $\Rightarrow (\Sigma, \omega)$ symplectic.

4) $S^{2n}$ does not have a symplectic structure $\omega$

if $n > 1$, because $[\omega]^n$ has to be $\neq 0$.

5) $\mathbb{C}P^n$ has a natural symplectic structure $\omega_{FS}$;

(can write down explicitly)

<u>Darboux's thm</u> locally, $(M, \omega) \cong (\mathbb{R}^{2n}, \omega_{std})$

So no local invts!

<u>Rmk</u> 2) $\Rightarrow$ true pointwise, 1) $\Rightarrow$ integrability.

Recall given $H: \mathbb{R}^{2n} \to \mathbb{R}$ Hamiltonian,

the evolution eqns for $(x_1(t), \ldots y_n(t))$ are

$$\begin{cases} \dot{x}_j(t) = \partial H / \partial y_j \\ \dot{y}_j(t) = - \partial H / \partial x_j \end{cases}$$

Hamilton's eqns.

Globally on $(M, \omega)$, $\omega$ determines

$$TM \xrightarrow{\sim} T^*M$$
$$v \mapsto \omega(v, -)$$

so $H: M \to \mathbb{R} \rightsquigarrow dH \in \Omega^1(M)$

$\rightsquigarrow X_H \in \Gamma(TM)$ Hamiltonian v.f;

$(\iota_{H_X} \omega = dH)$.

Hamilton's eqns are $\dot{\underline{x}}(t) = X_H(\underline{x}(t))$,

so we are evolving along the flow $\varphi_H(t)$

of $X_H$.

<u>Key lemma</u> $\varphi_H(t)$ preserves $\omega$.

<u>Proof</u> Infinitesimally,

$$\mathcal{L}_{X_H}\omega = d(\iota_{X_H}\omega) + \iota_{X_H}(\cancel{d\omega}) = ddH = 0.$$

Cartan's magic formula

$\square$

$\Rightarrow$ $(M,\omega)$ has <u>a lot</u> of symmetries!

<u>Rmk</u> In general, a symmetry $\varphi : (M,\omega)^{\circlearrowleft}$ is called a <u>symplectomorphism</u>, these form the group $\mathrm{Symp}(M,\omega)$.

The flow of a vector field $X$ preserves $\omega$ if $\iota_X\omega$ is closed; $X$ is hamiltonian if $\iota_X\omega$ is exact $(= dH)$.

Can also do time dependent vector fields!

Let's discuss the geometric incarnation of <u>Noether's principle</u> (symmetry $\leftrightarrow$ conserved quantity). Consider an action of $G$ Lie group on $(M,\omega)$, i.e. $\rho : G \to \mathrm{Symp}(M,\omega)$ homomorphism. Let's assume:

(*) The differential at $\mathrm{id}_G$ has image in

$$\rho_* : \mathfrak{g} \to \mathrm{HamVect}(M,\omega) \subseteq \Gamma(TM)$$

$\Rightarrow \forall\, \gamma \in \mathfrak{g}$, $\rho_*(\gamma) = X_{H_\gamma}$ for some $H_\gamma \in \mathcal{C}^\infty(M)$, (well defined up to constant)

$\Rightarrow$ we obtain map $\mathfrak{g} \to \mathcal{C}^\infty(M)$ linear;

dually, $\mu : M \to \mathfrak{g}^*$ s.t.

$$\langle \mu(x), \gamma \rangle = H_\gamma(x).$$

(**) $\mu : M \to \mathfrak{g}^*$ is G-equivariant.

<u>Def</u> If these hold, we say $G \curvearrowright (M, \omega)$ is <u>Hamiltonian action</u>, and $\mu : M \to \mathfrak{g}^*$ is <u>moment map</u>.

<u>Rule</u> There's general conditions to prove $G \curvearrowright (M, \omega)$ is Hamiltonian, we'll check it by hand in interesting cases.

<u>Ex</u> $M = T^* \mathbb{R}^3 = (\mathbb{R}^6, \omega_{std})$

If $G = \mathbb{R}^3 \curvearrowright M$ action induced by translations $\mathbb{R}^3 \curvearrowright \mathbb{R}^3$, then the moment map is $\mu(\underline{x}, \underline{y}) = \underline{y} \in (\mathbb{R}^3)^* \cong \mathbb{R}^3$. This is the usual <u>momentum</u>!

If $G = SO(3) \curvearrowright M$ rotations on $\mathbb{R}^3$ (7)

$\mu(\underline{x}, \underline{y}) = \underline{x} \times \underline{y} \in so(3)^* \equiv \mathbb{R}^3$ (angular momentum)

Here we use the identification of Lie algebras $(\mathbb{R}^3, \times) \xrightarrow{\sim} (so(3), [\cdot,\cdot])$

$$\underline{a} = (a_1, a_2, a_3) \longmapsto \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

plus $so(3) \cong so(3)^*$ ($SO(3)$-equivariantly)

Then $\rho : SO(3) \to Symp(T^*\mathbb{R}^3, \omega_{can})$ has

$$\underbrace{d\rho(\underline{a})}_{\text{vector field on } T^*\mathbb{R}^3} (\underline{x}, \underline{y}) = (\underline{a} \times \underline{x}, \underline{a} \times \underline{y}).$$

This is the Hamiltonian vector field of

$H_{\underline{a}} : T^*\mathbb{R}^3 \to \mathbb{R}$ $(\underline{x}, \underline{y}) \mapsto (\underline{x} \times \underline{y}) \cdot \underline{a} =$

$$= \langle \underbrace{\underline{x} \times \underline{y}}_{\mu}, \underline{a} \rangle$$

Ex 2 $U(n) \curvearrowright \mathbb{C}^n \equiv \mathbb{R}^{2n}$

(here $\omega_{std} \equiv \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$ )

Then $\mu : \mathbb{C}^n \to \mathfrak{u}(n)^* \cong \mathfrak{u}(n)$ → use inner product $\mathrm{Tr}(AB^*)$

$$\underline{z} \longmapsto -\frac{i}{2} \underline{z}\, \underline{z}^*$$

Ex 3 For $U(1) \curvearrowright \mathbb{C}^n$ the moment map is

$$\boxed{\mu(\underline{z}) = -\frac{i}{2} |\underline{z}|^2} \in i\mathbb{R} = \mathfrak{u}(1)^*.$$

↳ will be important for us!

Rmk $\mu + c$ is also moment map $c \in i\mathbb{R}$!

Marsden-Weinstein (1974) quotients of symplectic mfld by $G \curvearrowright M$ Hamiltonian. ($G$ → cpt) Assume.

- $c \in \mathfrak{g}^*$ is $G$-invariant & regular value of $\mu$

$\Rightarrow \mu^{-1}(c) \subseteq M$ smooth mfld with $G \curvearrowright \mu^{-1}(c)$.

• ) $G \curvearrowright \mu^{-1}(c)$ freely, so $\mu^{-1}(c)/G$ is smooth mfld (of dim $M - 2\dim G$).

<u>Then</u> $\mu^{-1}(c)/G$ $(=: X/\!/G)$ admits a natural symplectic form induced by $\omega$ (it's called <u>symplectic quotient</u>)

<u>Ex</u> $U(1) \curvearrowright \mathbb{C}^n$, pick $c \neq 0$ in $i\mathbb{R}$.

$\Rightarrow \mu^{-1}(c)/U(1)$ is either $\phi$, or $\mathbb{C}P^{n-1}$

(with a scaled version of $\omega_{FS}$!!).

<u>Key symplectic linear algebra observation:</u>

$(V, \omega)$ symplectic v.s. $(\cong (\mathbb{R}^{2n}, \omega_{std}))$

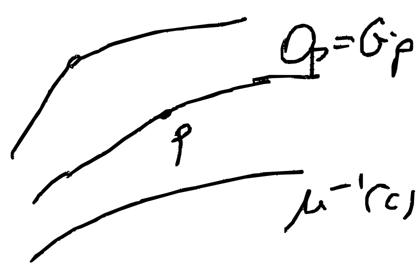
$I \subseteq V$ isotropic i.e $\omega|_I \equiv 0$

$(\Rightarrow I^\perp \supset I)$

Then $\omega$ induces natural symplectic form $\Omega$ on $I^\perp/I$ by $\Omega([v],[w]) = \omega(v,w)$.

The theorem follows by checking that $T_pO_p$ is isotropic, and

$T_p\mu^{-1}(c)\ (= \ker d\mu_p) = T_pO_p^\perp$




<u>Moment maps & gauge theory</u> key observations of Atiyah-Bott (1983) about surfaces.

$E \to \Sigma$ $\mathrm{rk}_{\mathbb{C}} = n$ hermitian bundle. U(n)-bundle.

$\mathcal{A} = \{\text{hermitian connections}\} = A_0 + \Omega^1(\mathfrak{g}_E)$

is naturally an $\infty$-dimensional symplectic manifold.

$T_A \mathcal{A} = \Omega^1(\mathfrak{g}_E)$, and

$\omega(a,b) := \int_\Sigma \mathrm{Tr}(a \wedge b)$ (constant $\Rightarrow$ closed)

$\Sigma^4$ $\Omega^2(M)$.

Now $\mathcal{G}$ = gauge grp $\curvearrowright \mathcal{A}$ via

$$u \cdot A = A - (d_A u) u^{-1}.$$

$\mathrm{Lie}(\mathcal{G}) = \Omega^0(\mathfrak{g}_E)$, and the differential

is the map $\mathrm{Lie}(\mathcal{G}) \to \Gamma(T\mathcal{A})$

$$\phi \mapsto (A \mapsto -d_A \phi \in T_A \mathcal{A} = \Omega^1(\mathfrak{g}_E))$$

<u>Lemma</u> the vector field $X$ sending $A \mapsto -d_A\phi$

is Hamiltonian for $H : \mathcal{A} \to \mathbb{R}$

$$A \mapsto \int \mathrm{Tr}(F_A \wedge \phi)$$

<u>Proof</u> $F_{A+ta} = F_A + t\, d_A a + O(t^2)$

$\Rightarrow dH_A(a) = \int \mathrm{Tr}(d_A a \wedge \phi)$

Now $\omega_A(X, a) = \int \mathrm{Tr}(-d_A\phi \wedge a)$ (by def)

$$= \int \mathrm{Tr}(\phi \wedge d_A a) \text{ (by Stokes)} \quad \square$$

Using identification $\Omega^0(\mathfrak{g}_E)^* \cong \Omega^2(\mathfrak{g}_E)$,

and gauge invariance properties of $F_A$, we have

<u>Thm</u> The action $\mathcal{G} \curvearrowright \mathcal{A}$ is Hamiltonian

with moment map

$$\mathcal{A} \to \Omega^2(\mathfrak{g}_E) \quad A \mapsto F_A$$

In particular, one expects spaces of

solutions <u>mod gauge</u> to be symplectic

manifolds in good circumstances!

<u>Rmk</u> this is all <u>formal</u>!