Lecture 8 Geometry of holomorphic line bundles.

Σ Riemann surface, $L \to \Sigma$ $\mathbb{C}$-line bundle (no metrics here!) so locally $L|_{U_i} \xrightarrow{\tau_i} U_i \times \mathbb{C}$, transition functions $g_{ij} : U_i \cap U_j \to GL(1,\mathbb{C}) = \mathbb{C}^*$.

$$\left(\begin{tikzcd} U_i \cap U_j \times \mathbb{C} \arrow[rr, bend right, "g_{ij}"'] & L|_{U_i \cap U_j} \arrow[l, "\tau_i", "\sim"'] \arrow[r, "\tau_j", "\sim"'] & U_i \cap U_j \times \mathbb{C} \end{tikzcd} \right)$$

Def A holomorphic line bundle $\mathcal{L} \to \Sigma$ (with underlying smooth bundle L) is a collection of trivializations $\{(U_i, \tau_i)\}$ s.t. the transition functions g_{ij} are holomorphic.

We have a well defined operator

$$\bar\partial_{\mathcal{L}} : \Omega^0(L) \to \Omega^{0,1}(L).$$

If the section $s \in \Omega^0(L)$ is given in trivializations by s_i, s_j, $s_j = g_{ij} s_i$

so $\bar\partial s_j = \underbrace{\cancel{(\bar\partial g_{ij})}}_{\text{holomorphic}} \cdot s_i + g_{ij} \bar\partial s_i$.

The operator $\bar\partial_L$ is the $(0,1)$ version of a covariant derivative

$$\bar\partial_L(fs) = \bar\partial f \cdot s + f \cdot \bar\partial_L s \quad \forall f \in \Omega^0_{\mathbb{C}},\ s \in \Omega^0(L)$$

($(0,1)$-Leibniz rule).

<u>Def</u> For $L \to S$, set

$$\mathcal{A}^{0,1} = \{ d^{(0,1)} : \Omega^0(L) \to \Omega^{0,1}(L) \text{ satisfying } (0,1)\text{-Leibniz rule} \}.$$

$\mathcal{A}^{0,1}$ is affine space / $\underset{\Omega^{0,1}(\mathrm{End}_{\mathbb{C}}(L,L))}{\Omega^{0,1}}$.

Key integrability result:

Thm (1D Koszul–Malgrange '58) Every $d^{(0,1)} \in \mathcal{A}^{(0,1)}$ is $\bar\partial_{\mathcal{L}}$ for some holomorphic structure $\mathcal{L}$ on L.

Rmk True for higher rank $E \to S$ too! Our proof readily adapts.

Riemann surface

To prove this, we need to find trivialization defining a holomorphic structure!

Concretely, want: $\forall x \in S$, $\exists U \ni x$ nbhd & section s of $L|_U$ s.t. 1) $d^{(0,1)} s \equiv 0$

2) $s \neq 0$ everywhere on U.

Then 2) gives $L_U \cong U \times \mathbb{C}$, 1) implies transitions are holomorphic.

Now, fix σ any section on U nowhere vanishing, so that

$d^{(0,1)} \sigma = \theta \sigma$ for some $\theta \in \Omega^{0,1}$ (4)

For <u>any</u> $g : U \to \mathbb{C}^*$, $g\sigma$ is again nowhere vanishing, and

$$d^{(0,1)}(g\sigma) = (\bar\partial g)\sigma + g \cdot d^{(0,1)}\sigma =$$
$$= (\bar\partial g)\sigma + g\theta\sigma.$$

So $s = g\sigma$ solves the problem provided

$$\boxed{g^{-1}\bar\partial g = -\theta} \quad \in \Omega^{0,1}.$$

We'll show that we can solve this PDE in g $\forall \theta \in \Omega^{0,1}$, possibly in a smaller nbhd.

(we follow the proof of Atiyah-Bott '83)

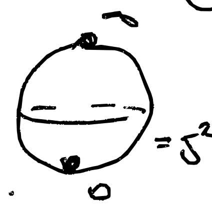
<u>Strategy</u>: 1) linear problem on $\mathbb{P}^1$ → Riemann sphere

2) non-linear problem on $\mathbb{P}^1$

3) local non linear problem.

Here $\mathbb{P}^1$ is given by two charts

$\mathbb{C}_z = \mathbb{P}^1 \setminus \{\infty\}$, $\mathbb{C}_w = \mathbb{P}^1 \setminus \{0\}$



related by $z \mapsto \frac{1}{w}$ on $\mathbb{C}_z^* \xrightarrow{\sim} \mathbb{C}_w^*$;

we can fix the standard round metric.

$\boxed{\text{Step 1}}$ <u>Lemma</u> On $\underline{\mathbb{C}} \to \mathbb{P}^1$ trivial bundle, the

$\bar\partial : L^2_{k+1}(\Omega^0_{\mathbb{C}}) \to L^2_k(\Omega^{0,1})$ has

$\ker \cong \mathbb{C} =$ constants, $\operatorname{coker} = \{0\}$; in particular,

$\bar\partial|_{\langle \text{constants} \rangle^\perp}$ is isomorphism

<u>Proof</u> by ellipticity, can work smooth stuff.

$\ker \bar\partial =$ holomorphic functions $=$ constants.

$$\operatorname{coker} \bar\partial = \ker\left(\underset{\substack{\| \\ -*\bar\partial *}}{\bar\partial^*} : \Omega^{0,1} \to \Omega^0_{\mathbb{C}}\right)$$

with $\bar\partial : \Omega^{1,0} \to \Omega^{1,1}$

So $\hat{\pi}$ identifies coker $\bar\partial$ with

$\ker(\bar\partial : \Omega^{1,0} \to \Omega^{1,1})$ = holomorphic 1-forms

(stuff locally $f(z)\,dz$, f holomorphic)

(Serre duality!). But there's no hol

1-forms on $\mathbb{P}^1$ (check using $dz = -\frac{1}{w^2}\,dw$
+ power series expansion) $\square$.

To deal with the non-linearity (coming from <u>multiplication</u>), we use

<u>Thm E</u> (Sobolev multiplication thm). $k \geq j \geq 0$, and $k > \frac{\dim M}{2}$. Then multiplication defines continuous map

$$L^2_k \times L^2_j \to L^2_j.$$

(in particular, L^2_k is Banach algebra).

$\boxed{\text{Step 2}}$ **Prop** On $\mathbb{P}^1$, if $\theta \in \Omega^{0,1}$ has sufficiently small L^2_1-norm, then $\exists\, g : \mathbb{P}^1 \to \mathbb{C}^*$ smooth s.t. $-\theta = g^{-1}\bar{\partial} g$.

Proof Set $\mathcal{U}_2 = \left\{ \begin{array}{l} f \neq -1 \text{ everywhere} \\ f \perp \text{constants.} \end{array} \right\} \subseteq L^2_2(\langle \text{constants} \rangle^{\perp})$

well defined open set ($L^2_2 \hookrightarrow C^0$ in dim 2),

consider the map (we'll set $g = 1+f$ later)

$\Phi : \mathcal{U}_2 \to L^2_1(\Omega^{0,1})$ $f \mapsto (1+f)^{-1}\bar{\partial}(1+f)$

which is well-defined by Sobolev multiplication, and in fact smooth map

(Here F differentiable in Banach setup means

$F(x_0+h) = F(x_0) + Lh + o(\|h\|)$

with L bounded operator)

with derivative at $f_0 \equiv 0$ given by

$$D_{f_0} P : L^2_2(\langle \text{constants} \rangle^\perp) \longrightarrow L^2_1(\Omega^{0,1}),$$
$$f \longmapsto \bar\partial f$$

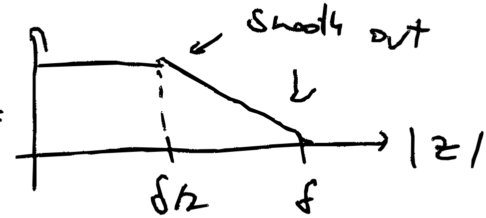
which is iso by Lemma! Inverse function thm $\Rightarrow$ P local diffeo onto nbhd of $P(f_0) = 0$. So if $\|\theta\|_{L^2_1} < \varepsilon$, $\exists f \in \mathcal{U}_2 \subseteq L^2_2$ s.t. $(1+f)^{-1}\bar\partial(1+f) = -\theta$; set $g = 1+f \in L^2_2$.

To conclude g is smooth: $g \in L^2_2$, θ smooth (by assumption) $\Rightarrow \theta \in L^2_k \ \forall k$. Then

$$\bar\partial g = -g\theta \in L^2_2 \underset{\text{elliptic reg}}{\Rightarrow} g \in L^2_3$$

$\Rightarrow g\theta \in L^2_3 \Rightarrow g \in L^2_4 \Rightarrow \ldots$ (this is called elliptic bootstrapping) $\square$.

$\boxed{\text{Step 3}}$ To conclude local solvability of $g^{-1}\bar\partial g = -\theta$, we can first assume $\theta(0)=0$ (change $g \rightsquigarrow g(1+c\bar z)$ for suitable $c\in\mathbb{C}$)

Consider $\rho_\delta(|z|) =$ 

,

and let $\phi_\delta = \rho_\delta \cdot \theta \in \Omega^{0,1}(U)$, thought of as $(0,1)$-form on $\mathbb{P}^1$. Key estimate:

$\|\phi_\delta\|_{L^3_1} \to 0$ for $\delta \to 0$, so by Prop we solve $g^{-1}\bar\partial g = \phi_\delta$, which is $\equiv -\theta$ on $|z| < \delta/2$. $\square$.

Recap we showed for $L \to S$

$$\{\text{hol structures on } L\} \xrightarrow{1:1} \{d^{(0,1)} : \Omega^0(L) \to \Omega^{0,1}(L)\}$$

$$\mathcal{L} \longmapsto \bar\partial_{\mathcal{L}} \qquad\qquad \Omega^{0,1}(L) = \mathcal{A}^{0,1}.$$

Now, fix hermitian metric h on L, then we also have

$$\mathcal{A} = \{\text{hermitian connections on } L\} \longrightarrow \mathcal{A}^{0,1}$$

$$A \longmapsto \bar\partial_A$$

where $\bar\partial_A$ is $(0,1)$-part of $d_A : \Omega^0(L) \to \Omega^1(L)$

Prop This is also a bijection! i.e given $d^{0,1} \in \mathcal{A}^{0,1}$, $\exists!\ A \in \mathcal{A}$ s.t $\bar\partial_A = d^{0,1}$.

Proof we write an explicit formula for A!

Fix a trivialization $L|_U \cong U \times \mathbb{C}$

$\vec{e}_0 \equiv 1$ constant

given by a section s with $d^{(0,1)}s = 0$.
This does not need to preserve the hermitian structure! Write a general connection
$\nabla_A = d + A$ with $A \in \Omega^1 \otimes \mathbb{C}$.

Now, $\bar\partial_A = d^{0,1} \Rightarrow \bar\partial_A e_0 = 0 \Rightarrow A \in \Omega^{1,0}$.

Imposing hermiticity for h:

$d\, h(e_0, e_0) = h(d_A e_0, e_0) + h(e_0, d_A e_0)$,

so setting $H = h(e_0, e_0)$ $(: U \to \mathbb{R})$

$dH = HA + H\bar{A}$, taking $(1,0)$-part

$\partial H = HA \Rightarrow A = H^{-1}\partial H$.

$\square$.

Putting everything together, for $L \to S$ hermitian $\exists$ correspondence

$$\{\text{hol structures}\} \xrightarrow{1:1} \{\text{hermitian connections}\}$$

$$\mathcal{L} \mapsto \text{unique } A \text{ s.t. } \bar\partial_A = \bar\partial_{\mathcal{L}}$$

(this is called <u>Chern connection</u>).

<u>Useful consequence</u> a solution $\Phi \in \Gamma(L)$ to the equation $\bar\partial_A \Phi = 0$ locally looks like a holomorphic function $f : U \to \mathbb{C}$ (in suitable trivialization) $\Rightarrow$ either $\Phi \equiv 0$, or zeroes isolated with well-defined multiplicity >0

($f(z) = a_k z^k + a_{k+1} z^{k+1} + \ldots \quad a_k \neq 0$).

by <u>unique continuation</u>; topologically,

$\#\{\text{zeroes of } \Phi\} = \deg L$.