Lecture 7 The vortex equations on a surface

Geometric setup (Σ, g) oriented Riemannian surface,

$L \to \Sigma$ hermitian line bundle.

$\mathcal{C} = \{(A, \Phi) \mid A$ compatible connection on L, $\Phi \in \Gamma(L)\}$

$\mathcal{G} = \{u : \Sigma \to U(1)\} \curvearrowright \mathcal{C}$ via

$$u \cdot (A, \Phi) = (A - u^{-1}du, u \cdot \Phi)$$

We consider the Yang-Mills-Higgs functional

$$YMH(A, \Phi) = \frac{1}{2}\int |F_A|^2 + \frac{1}{2}\int |\nabla_A \Phi|^2 + \frac{\lambda}{8}\int (1 - |\Phi|^2)^2,$$

$YMH : \mathcal{C}/\mathcal{G} \to \mathbb{R}^{\geq 0}$. (fixed parameter $\lambda > 0$)

Goal topological bounds in the case of

critical coupling $\lambda = 1$; ($\lambda < 1$, $\lambda > 1$

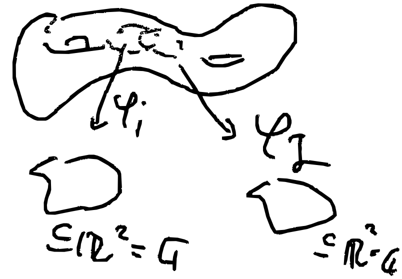
describe Type I vs Type II superconductors)

The natural top. invariant here is $c_1(L) \in H^2(\Sigma;\mathbb{Z}) \cong \mathbb{Z}$, also called the degree $d = d(L) \in \mathbb{Z}$. Facts:

- it's the only top invariant of line bundles on Σ: if $d(L) = d(L') \Rightarrow L \cong L'$.
- it's equal to the Euler number of $L_{\mathbb{R}}$, i.e. the # zeroes of a generic section $L \underset{s}{\overset{\longrightarrow}{\longleftarrow}} \Sigma$ (counted with signs).

The bounds are based on the interplay between the Riemannian and complex geometry of surfaces (simplest $\dim_{\mathbb{C}} = 1$ case of Kähler geometry), which we now discuss.

A <u>Riemann surface</u> S (= complex mfld of dim 1) is a surface Σ with given atlas for which the transition maps are



<u>biholomorphisms</u>

(= orientation preserving conformal diffeomorphism).

$\Rightarrow$ well-defined notion of $f : \Sigma \to \mathbb{C}$ <u>holomorphic</u>

These charts endow Σ with an almost complex structure

$$\begin{tikzcd} T\Sigma \arrow[rr, "J"] \arrow[dr] & & T\Sigma \arrow[dl] \\ & \Sigma & \end{tikzcd},$$

bundle map s.t. $J^2 = -\mathrm{Id}$ induced by counter clockwise 90° rotation in $\mathbb{C}$

$\left(\frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial y}, \ \frac{\partial}{\partial y} \mapsto -\frac{\partial}{\partial x} \text{ in } \mathbb{R}^2 \right)$

$$S \overset{\text{forget}}{\rightsquigarrow} (\Sigma, J)$$

In this language, $f : \Sigma \to \mathbb{C}$ is holomorphic

$\iff df : T\Sigma \to \mathbb{C}$ is $\mathbb{C}$-linear at all points

$\boxed{i \circ df = df \circ J}$ Cauchy–Riemann equations.

(If $f = u + iv$ in $\mathbb{C}$, this is just $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$)

Notice that an oriented Riemannian surface (Σ, g) also induces $J : TM^{\circ}$ a.c.s.

$$(\Sigma, g) \xrightarrow{\text{forget}} (\Sigma, J).$$

Of course, every (Σ, J) is induced by (Σ, g) (and any other is $g' = e^{2f} g$.)

(this is just linear algebra!).

More subtle is

Thm (1D Newlander–Nirenberg) Every (Σ, J) is induced by (a unique) Riemann surface structure.

This is an <u>integrability</u> result;

need to show $\forall x \in \Sigma$, $\exists$ $U \ni x$ and

$f : U \to \mathbb{C}$ diffeo onto image satisfying

CR equations. ($\Rightarrow f$ gives local chart).

<u>Idea of proof</u> (we'll prove a related

integrability result next time).

exploit holomorphic vs harmonic.

Fix g inducing J. Then $* : \Omega^1(\Sigma)$ ↺

(sends $dx \mapsto dy$, $dy \mapsto -dx$ in $\mathbb{R}^2$)

(so $* = -J^* \to$ dual action).

→ key point!

<u>Find</u> by PDE techniques $u : \overset{\cap}{V}{}^{x} \to \mathbb{R}$

s.t $\Delta u = 0$ and $du_x \neq 0$.

↳ uses g!

Then $d(*du) = 0 \Rightarrow *du$ closed

$\Rightarrow \exists\, v$ (unique up to constant) s.t

$*du = dv$, and $u+iv$ solves CR equations

(and is local diffeo) $\square$.

From now on, we'll always think $(\Sigma, g) \rightsquigarrow S$.

(i.e Riemannian surface $\Rightarrow$ Riemann surface).

A useful way to rephrase CR eqns is

via (p,q)-forms:

$J^* \curvearrowright T_x^*\Sigma$ with $(J^*)^2 = -I$

$\Rightarrow \Omega^1 \otimes \mathbb{C} := \Omega^{1,0} \oplus \Omega^{0,1}$ $(\pm i)$-eigenspaces

$\Omega^{1,0}$ in $\mathbb{R}^2$: $\operatorname{Span}(dz = dx + idy)$, $\Omega^{0,1}$: $\operatorname{Span}(d\bar{z} = dx - idy)$.

Rmk we also have $\Omega^2 \otimes \mathbb{C} = \Omega^{1,1}$ $(= \operatorname{span} dz\, d\bar{z})$

We decompose

$$d = \partial + \bar\partial : \Omega^0 \otimes \mathbb{C} \xrightarrow{d} \Omega^1 \otimes \mathbb{C} = \Omega^{1,0} \oplus \Omega^{0,1},$$

and f is holomorphic $\Leftrightarrow$ $\bar\partial f = 0$.

Rmk by maximum principle, $f : \underset{\text{cpt}}{\Sigma} \to \mathbb{C}$ hol is constant.

Back to gauge theory

Suppose now we have on (Σ, g) a hermitian line bundle L ($\cong \mathbb{C}$ locally),

$$\Rightarrow \Omega^1(L) = \Omega^{1,0}(L) \oplus \Omega^{0,1}(L).$$

If A is compatible connection, we have

$$d_A := \partial_A + \bar\partial_A : \Omega^0(L) \to \Omega^1(L) = \Omega^{1,0}(L) \oplus \Omega^{0,1}(L).$$

$\partial_A, \bar\partial_A$ 1-st order elliptic operators;

we also have the version involving $2 = (1,1)$ forms.

$$\begin{tikzcd}
 & \Omega^{1,1}(L) & \\
\Omega^{1,0}(L) \arrow[ur, "\bar\partial_A"] & & \Omega^{0,1}(L) \arrow[ul, "\partial_A"'] \\
 & \Omega^{0}(L) \arrow[ul, "\partial_A"] \arrow[ur, "\bar\partial_A"'] &
\end{tikzcd}$$

$\updownarrow *$

$\longleftrightarrow$ conjugation

All of these are inner product spaces;
the adjoints can be explicitly written down,
e.g. the adjoint of $\bar\partial_A : \Omega^0(L) \to \Omega^{0,1}(L)$
is $\bar\partial_A^* = -\bar * \,\bar\partial_A \,\bar *$, $\Omega^{1,0}(L) \to \Omega^{1,1}(L)$.

We also record:

<u>Prop</u> (1D Kähler identities) The adjoint of

$\partial_A : \Omega^0(L) \to \Omega^{1,0}(L)$ $\left(\bar\partial_A : \Omega^0(L) \to \Omega^{0,1}(L)\right)$

is $i * \bar\partial_A$ $\quad (-i * \partial_A)$

<u>Thm</u> (top bound) when $\lambda=1$, we have

$$YMH(A,\Phi) = \|\overset{\Omega^{0,1}(L)}{\bar\partial_A \Phi}\|^2_{L^2} + \frac{1}{2}\|\overbrace{*F_A}^{\in i\Omega^0} - \frac{i}{2}(|\Phi|^2-1)\|^2_{L^2}$$
$$+ \pi \deg(L).$$

<u>Proof</u> If we expand the second term,

we obtain the terms:

- $\frac{1}{2}\|F_A\|^2_{L^2}$, $\frac{1}{8}\|1-|\Phi|^2\|^2_{L^2}$ which appear in YMH ($\lambda=1$!!)
- $\int_\Sigma \frac{i}{2} F_A = \pi c_1(L)$ by Chern-Weil
- $\frac{1}{2}\langle i*F_A, |\Phi|^2\rangle_{L^2}$

But $\langle i*F_A, |\Phi|^2\rangle_{L^2} = \langle i*F_A\Phi, \Phi\rangle_{L^2}$

$= \langle i* d_A d_A \Phi, \Phi\rangle_{L^2} =$ (using $d_A = \partial_A + \bar\partial_A$)

$$\langle i * \partial_A \bar{\partial}_A \Phi, \Phi \rangle_{L^2} + \langle i * \bar{\partial}_A \partial_A \Phi, \Phi \rangle_{L^2}$$

$$= -\|\bar{\partial}_A \Phi\|^2_{L^2} + \|\partial_A \Phi\|^2_{L^2} \quad \text{by Kähler id.}$$

We conclude by $\|\underset{d_A}{\underset{\shortparallel}{\nabla_A}} \Phi\|^2_{L^2} = \|\partial_A \Phi\|^2_{L^2} + \|\bar{\partial}_A \Phi\|^2_{L^2}$

$\square$.

__Cor 2__ If $L \to (\Sigma, g)$ has $\deg L = d \geq 0$, then

$YMH(A, \Phi) \geq \pi d$ (top bound)

and bound is achieved __if and only if__

$$\begin{cases} \bar{\partial}_A \Phi = 0 & (\in \Omega^{0,1}(L)) \\ * F_A = \frac{i}{2}(|\Phi|^2 - 1) & (\in i\Omega^0) \end{cases}$$

__vortex equations__ (1st order, non-linear gauge invariant PDEs)

We'll focus on $\Phi \not\equiv 0$; if not, it's just $*F_A = -\frac{i}{2}$ constant (central) curvature eqn.

<u>Notice</u>: if solution (A, Φ) exists,

$$\deg(L) = \int_\Sigma \frac{i}{2\pi} F_A = \frac{1}{4\pi} \int_\Sigma (1 - |\Phi|^2)\, d\mathrm{vol} < \frac{\mathrm{Vol}(\Sigma)}{4\pi}.$$

Our goal for the next few lectures is to prove

<u>Thm</u> (Taubes '80, Bradlow '89) $L \to (\Sigma, g)$ with $0 \leq \deg(L) < \frac{\mathrm{Vol}(\Sigma)}{4\pi}$. Then $\exists$

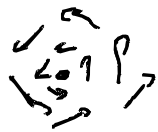
$$\underbrace{\left\{ \begin{array}{l} \text{solutions } (A, \Phi),\ \Phi \not\equiv 0 \\ \text{to vortex eqns} \end{array} \right\} \Big/ \mathcal{G}}_{\mathcal{M}} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{unordered } d\text{-uples} \\ \text{of points } x_1, \ldots x_d \in \Sigma \end{array} \right\}$$

natural bijection (to be described).

<u>Def</u> RHS is $\Sigma^d / S_d =: \mathrm{Sym}^d(\Sigma)$ <u>symmetric product</u> of Σ.

<u>Rmk</u> Taubes proved the case $\Sigma = \mathbb{R}^2 = \mathbb{C}$

with condition $|\Phi| \to 1$ at ∞.

<u>Basic vortex</u>  (justifies name)

We'll follow the proof of Garcia-Prada (92)

which also shows that $\mathcal{M}$ is naturally

a $2d$-dimensional <u>smooth</u> mfld equipped

with a <u>symplectic</u> form.

One can easily see $\mathrm{Sym}^d(\Sigma)$ is <u>topological</u>

mfld using the charts

$$\mathbb{C}^d/S_d \longleftrightarrow \mathbb{C}^d$$

$$[z_1, \dots z_d] \mapsto \text{coeff. of } (z-z_1)\cdots(z-z_d).$$

<u>Rmk</u> The "naive" variational strategies

will <u>not</u> work YMH