# $\boxed{\text{Lecture 6}}$ Hodge theorem & index theory.

Let's start by finishing the proof of:

Hodge theorem $(M, g)$ cpt Riemannian. Then

$$\Omega^p = \underbrace{\mathcal{H}^p}_{\text{harmonic } p\text{-forms} = \ker \Delta} \oplus \operatorname{Im} \Delta \qquad (L^2\text{-orthogonal dec}).$$

$$(\Leftrightarrow \Omega^p = d\Omega^{p-1} \oplus \mathcal{H}^p \oplus d^*\Omega^{p+1}).$$

Prop $d + d^* : \Omega^*(M) \circlearrowleft$ is a first order elliptic op ($\Rightarrow \Delta = (d+d^*)^2$ is second order elliptic).

Proof Intrinsic def: $x \in M$, $\xi = d\varphi(x) \in T^*_x M \setminus \{0\}$ with $\varphi(x) = 0$, $\alpha \in \Omega^p$. Want to show

$$(d + d^*)(\varphi\alpha)(x) \neq 0.$$

To see this, compute:

$$d(\varphi\alpha)(x) = (d\varphi\wedge\alpha + \varphi\, d\alpha)(x) = \xi\wedge\alpha.$$

$$-{*}d{*}(\varphi\alpha)(x) = {*}d(\varphi\cdot{*}\alpha)(x) = \ldots = {*}(\xi\wedge{*}\alpha)$$

$\Rightarrow d^*(\varphi\alpha)(x) = \xi \lrcorner\, \alpha$ (where contraction is $e \lrcorner\, \omega := (-1)^{n(p+1)+1} {*}(e\wedge{*}\omega)$, usual contraction with $e^\flat$ up to sign).

$\Rightarrow$ symbol is $\xi\wedge\alpha + \xi\lrcorner\,\alpha \neq 0$.

<u>Proof of Hodge thm</u> write $\Omega^p = \mathcal{H}^p \oplus (\mathcal{H}^p)^\perp$,

want to prove $(\mathcal{H}^p)^\perp = \Delta(\Omega^p)$.

$\supseteq$) is clear: if $\alpha = \Delta\beta$, $\omega\in\mathcal{H}^p$

$$\Rightarrow \langle\alpha,\omega\rangle_{L^2} = \langle\Delta\beta,\omega\rangle_{L^2} = \langle\beta,\Delta\omega\rangle_{L^2} = 0.$$

Non trivial part is $\subseteq$!

Rmk: we don't know yet $\mathcal{H}^p \cong H^p(M;\mathbb{R})$!

If $\alpha \in (\mathcal{H}^p)^\perp$, want to show $\exists\, \phi \in \Omega^p$ s.t $\alpha = \Delta\phi$. In fact, by elliptic regularity, we only need to find a weak solution, i.e

$\beta \in L^2(\Omega^p)$ s.t

$$\langle \alpha, \varphi \rangle_{L^2} = \langle \beta, \Delta\varphi \rangle_{L^2} \quad \forall \varphi \in \Omega^p.$$

But $\beta$ exists by abstract nonsense as follows.

Define $\ell : \underset{\subseteq \Omega^p}{\operatorname{Im}\Delta} \to \mathbb{R}$

$$\Delta\varphi \mapsto \langle \alpha, \varphi \rangle_{L^2}$$

•) well-defined: if $\Delta\varphi = \Delta\varphi' \to \varphi - \varphi' \in \mathcal{H}^p$

$$\langle \alpha, \varphi \rangle_{L^2} = \langle \alpha, \varphi' \rangle_{L^2}.$$

◦) <u>bounded</u>, because if

$$\varphi = \varphi^{harm} + \varphi^{\perp} \in \mathcal{H}^p \oplus (\mathcal{H}^p)^{\perp},$$

$$|\ell(\Delta\varphi)| = |\ell(\Delta\varphi^{\perp})| = |\langle \alpha, \varphi^{\perp} \rangle_{L^2}|$$

$\leq \|\alpha\|_{L^2} \|\varphi^{\perp}\|_{L^2}$ (by Cauchy-Schwarz)

$\leq \|\alpha\|_{L^2} \|\varphi^{\perp}\|_{L^2_2}$

$\leq C \|\alpha\|_{L^2} \|\Delta\varphi^{\perp}\|_{L^2}$ (Gårding inequality).

$= C \|\alpha\|_{L^2} \|\Delta\varphi\|_{L^2}$.

By Hahn-Banach, $\ell$ extends to bounded

$\ell : \underbrace{L^2(\Omega^p)}_{\text{Hilbert space}} \longrightarrow \mathbb{R}$. (i.e $\ell \in L^2(\Omega^p)^*$).

By Riesz representation, $\ell(-) = \langle \beta, - \rangle_{L^2}$

for $\beta \in L^2(\Omega^p)$, which is weak solution

of $\Delta\beta = \alpha$ $\square$.

Elliptic operators have very interesting topological properties!

<u>Basic observation</u> $T : \mathbb{R}^n \to \mathbb{R}^m$ linear map.

By rank-nullity,

independent of $T$

$$\mathrm{ind}(T) := \dim\ker T - \dim\operatorname{coker} T = n - m$$

depends on $T$. $\left(= \mathbb{R}^m / \operatorname{Im} T\right)$

<u>Def</u> $X, Y$ Hilbert spaces (or Banach), $T : X \to Y$ bounded op is <u>Fredholm</u> if

a) $\dim\ker T < \infty$

b) $\dim\operatorname{coker} T < \infty$

c) $\operatorname{Im} T$ closed.

The <u>index</u> of $T$ is

$$\mathrm{ind}(T) := \dim\ker T - \dim\operatorname{coker} T \in \mathbb{Z}.$$

Remk in this setup, 2) $\Rightarrow$ 3) by open mapping thm.

Ex •) $R : \ell^2(\mathbb{N}) \circlearrowleft$ right shift

$(a_0, a_1, \dots) \mapsto (0, a_0, a_1, \dots)$

has $\operatorname{ind} R = 0 - 1 = -1$.

•) $L : \ell^2(\mathbb{N}) \circlearrowleft$ left shift

$(a_0, a_1, \dots) \mapsto (a_1, \dots)$

has $\operatorname{ind} L = 1 - 0 = 1$

Thm $T$ order $\ell$ elliptic op on $M$ cpt, then

$\forall k \quad T : L^2_{k+\ell} \to L^2_k$ is Fredholm,

and $\operatorname{ind}(T)$ is independent of $k$

(in fact, $\operatorname{coker} T \cong \ker T^*$ ← formally adjoint diff op)

Let's prove dim ker $L$ is finite.

**Abstract Lemma**
$$\begin{tikzcd} X \arrow[r, "L"] \arrow[dr, "k"'] & Y \\ & Z \end{tikzcd}$$
s.t

1) $k$ is cpt, 2) $\|x\|_X \le C(\|Lx\|_Y + \|kx\|_Z)$

$\Rightarrow$ ker $L$ is finite dimensional.

**Proof** If dim ker $L = \infty$, $\exists \{x_i\} \subseteq \ker L$

s.t $\underbrace{\|x_i\|_X \le 1}$, $\|x_i - x_j\|_X \ge c$. Up to subsequence,

$\{kx_i\}$ converges, but 2) implies.

$\|kx_i - kx_j\|_Z \gtrsim \|x_i - x_j\|_X \ge 1$, contradiction $\square$.

Apply lemma to
$$\begin{tikzcd} L^2_{k+l} \arrow[r, "T"] \arrow[dr, hook] & L^2_k \\ & L^2 \end{tikzcd}$$
1) is Rellich's thm

2) is Gårding ineq.

not hard!

Thm $\mathrm{Fred}(X,Y) \subseteq B(X,Y)$ is open,

Fredholm — bounded

and $\mathrm{ind}: \mathrm{Fred}(X,Y) \to \mathbb{Z}$ is locally constant.

Cor $T_t : \Gamma(E) \to \Gamma(F)$ 1-parameter family of elliptic ops on $M$ cpt $\Rightarrow$ $\mathrm{ind}(T_t) \in \mathbb{Z}$ is constant.

Ex if $T$ is self adjoint (e.g. $d + d^* : \Omega^* \to \Omega^*$)

$\mathrm{coker}\, T = \ker T^* = \ker T \Rightarrow \mathrm{ind}\, T = 0$.

More interestingly, consider

$D = d + d^* : \Omega^{\mathrm{even}} \to \Omega^{\mathrm{odd}}$. (elliptic!)

$\Rightarrow \ker = \bigoplus \mathcal{H}^{\mathrm{even}} = \bigoplus H^{\mathrm{even}}(M;\mathbb{R})$

$\mathrm{coker} = \ker(d + d^* : \Omega^{\mathrm{odd}} \to \Omega^{\mathrm{even}})$

$= \bigoplus H^{\mathrm{odd}}(M;\mathbb{R})$

$\Rightarrow \operatorname{ind}(D) = \chi(M)$

Now, choose $\alpha \in \Omega^{odd}(M)$. Then

$D + t(\alpha \wedge -) : \Omega^{ev} \to \Omega^{odd}$ is

1-parameter family of elliptic ops

$\Rightarrow \operatorname{ind}(D + (\alpha \wedge -)) = \chi(M)!$

So if $\chi(M) > 0$ (e.g. $M = S^{24}$)

$\ker(D + (\alpha \wedge -)) \neq 0$

$\rightsquigarrow$ topology forces solvability of elliptic PDEs.

<u>Atiyah-Singer index thm</u> (1963). For

$T : \Gamma(E) \to \Gamma(F)$ elliptic on $M$ cpt,

purely topological (e.g. char classes of $TM, E, F$)

explicit formula for $\operatorname{ind}(T) \in \mathbb{Z}$.

## Back to gauge theory

The (linear part) of the equations we study.

is never elliptic because of gauge invariance.

Consider for example

$$F_A^+ = 0 \in \Omega^2(\mathfrak{g}_E) \quad \text{ASD (or instanton) eqn.}$$

Locally, if $\nabla_A \overset{\sim}{=} d + \tilde{A}$ $\quad \tilde{A} \in \Omega^1(\mathfrak{g})$

$$F_A^+ = (d\tilde{A} + \tilde{A} \wedge \tilde{A})^+$$

So the highest order term is

$$\begin{tikzcd} \Omega^1(\mathfrak{g}) \arrow[r, "d"] \arrow[rr, bend right, "d^+"'] & \Omega^2(\mathfrak{g}) \arrow[r, "\pi^+"] & \Omega^+(\mathfrak{g}) \end{tikzcd}$$

and of course $d^+(\underbrace{df}_{\text{or anything closed}}) = 0$. $\Rightarrow$ not elliptic.

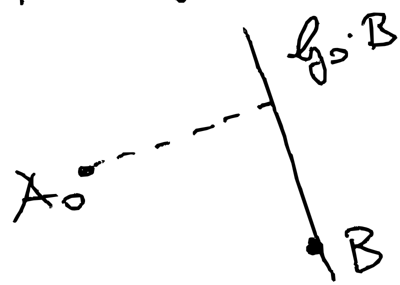
(comes from $F^+_{u \cdot A} = 0$ $\forall u \in \mathcal{G}$).

This issue is resolved by suitably <u>gauge fixing</u>. Let's do the $U(1)$ case (non-abelian case <u>highly</u> non-trivial, cf <u>Uhlenbeck's gauge fixing thm</u> (1982))

$L \to (M,g)$ cpt, fix base connection $A_0$.

$\mathcal{G}_0 = \{u = e^f, f \in i\Omega^0\}$ id. component of $\mathcal{G}$.

Then $e^f \cdot B = A - df$




It's natural to consider $\tilde{B}$ that minimizes $\|u \cdot B - A_0\|_{L^2}$.

This solves (usual computation)

$$\boxed{d^*(B - A_0) = 0}$$ <u>Coulomb gauge condition</u>

Hodge theorem $\Rightarrow \forall B, \exists! f$ up to overall constant s.t. $e^f \cdot B$ is is Coulomb gauge.

Now, the first order part of ASD eqn + Coulomb gauge fixing is

$$d^* + d^+ : i\Omega^1(M) \to i\Omega^0(M) \oplus i\Omega^+(M)$$

which is elliptic! (dimensions check: $4 = 1+3$)

One also checks $\ker = \mathcal{H}^1$, and

$$\text{coker} = \underset{\text{constants}}{\mathbb{R}} \oplus \mathcal{H}^+, \text{ self-dual harmonic 2-forms},$$

$(+1)$ eigenspace of $* : \mathcal{H}^2 \circlearrowleft$, whose dim is the $b^+(X)$ = positivity of quadratic form

$$Q_X : H^2(X;\mathbb{R}) \to \mathbb{R} \quad \alpha \mapsto (\alpha \cup \alpha)[X].$$

$\Rightarrow$ ind $= b_1 - b^+ - 1$. (top quantity!)