Lecture 5 Elliptic operators & Sobolev spaces

<u>Motivation</u> need the right functional setup to solve our equations of interest. Overall strategy: prove solution in generalized sense exist in larger space (measures, distributions, etc..), then show it's actually a nice smooth function.

$E, F \to M^n$ bundles.

<u>Def</u> A <u>differential operator</u> of order $\ell \in \mathbb{N}$ is $T : \Gamma(E) \to \Gamma(F)$ linear map that in local trivializations of E, F, local coordinates $U \subseteq \mathbb{R}^n$ on M looks like

$$\mathcal{C}^\infty(U;\mathbb{R}^{\dim E}) \to \mathcal{C}^\infty(U;\mathbb{R}^{\dim F})$$ ②

$$T = \sum_{|\alpha|\le l} a_\alpha(x) D^\alpha$$ where:

- if $\alpha = (\alpha_1, \dots \alpha_n)$, $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$
- $a_\alpha(x)$ is a $\dim F \times \dim E$ matrix, smooth in x.

Ex:
- $\Delta = -\frac{\partial^2}{\partial x_1^2} - \cdots \frac{\partial^2}{\partial x_n^2}$ (Laplacian)
- $\frac{\partial^2}{\partial t^2} + \Delta$ (wave op), $\frac{\partial}{\partial t} + \Delta$ (heat op)
- $\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) : \mathcal{C}^\infty(\underset{\mathbb{C}}{U \cap} ; \mathbb{C}) \circlearrowleft$

Def T is elliptic if substituting

$$D_\alpha \rightsquigarrow \xi_1^{\alpha_1} \cdots \xi_n^{\alpha}, \quad \xi \in \mathbb{R}^n$$

$$P_{T,x}(\xi) = \sum_{|\alpha|=\ell} a_\alpha(x)\xi^\alpha : \mathbb{R}^{\dim E} \to \mathbb{R}^{\dim F}$$

is isomorphism $\forall x$, $\xi \in \mathbb{R}^n \setminus \{0\}$.

Ex (with constant coefficients) $(:\mathbb{R} \to \mathbb{R})$

- $\Delta \rightsquigarrow -\xi_1^2 - \cdots - \xi_n^2 = -|\xi|^2$ elliptic
- $\dfrac{\partial^2}{\partial x_1^2} - \dfrac{\partial^2}{\partial x_2^2} \rightsquigarrow \xi_1^2 - \xi_2^2$ not elliptic
- $\dfrac{\partial}{\partial x_1} - \dfrac{\partial^2}{\partial x_2^2} \rightsquigarrow -\xi_2^2$ not elliptic
- $\dfrac{\partial}{\partial x_1} + i\dfrac{\partial}{\partial x_2} \rightsquigarrow \xi_1 + i\xi_2$ elliptic

Intrinsically, say $\dim E = \dim F$.

Then T of order ℓ is elliptic

$\Leftrightarrow \forall x \in M$, $u \in \Gamma(E)$ s.t. $u(x) \neq 0$

$\varphi \in \mathcal{C}^\infty(M)$ s.t $\varphi(x) = 0$ and

$d\varphi = \xi \neq 0 \in T^*_x M$,

$$T(\varphi^\ell u)(x) \neq 0 \quad \Big(= P_{T,x}(\xi)(\underset{E_x}{\underbrace{u(x)}}) \in F_x\Big)$$

up to scaling.

<u>Key feature</u> if T is elliptic, solutions to $Tu = 0$ are very nice.

<u>Ex</u> if $u : \Omega \to \mathbb{C}$ is $\mathcal{C}^1$ and $\frac{\partial}{\partial \bar z} u = 0$
$\;\;\Omega \subset \mathbb{C}$

(i.e u is holomorphic) $\Rightarrow u \in \mathcal{C}^\infty$.

To discuss this in the generality needed in gauge theory, the right functional setup is that of <u>Sobolev spaces</u>.

$E \to M$, M Riemannian, fix ∇ on E hermitian.

$u \in \Gamma(E) \rightsquigarrow \nabla u \in \Gamma(T^*M \otimes E) \rightsquigarrow \nabla^2 u \in \Gamma(T^*M^{\otimes 2} \otimes E)$

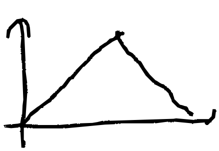
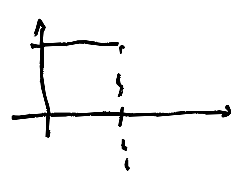
The L^2_k Sobolev norm is

$$\|u\|_{L^2_k} := \int_M |u|^2 + |\nabla u|^2 + \dots |\nabla^k u|^2 \, d\mathrm{vol}_g .$$

(induced by inner product). If $\underline{M \text{ cpt}}$, different (g, ∇) induce equivalent norms.

Def $L^2_k(E)$ = completion of $(\Gamma(E), \|\cdot\|_{L^2_k})$

Sobolev space (Hilbert).

Ex  has differential 

$u \in L^2_1(\mathbb{R})$ $\qquad$ $du \in L^2(\mathbb{R})$

(can define L^2_k in terms of distributions).

<u>For intuition</u>, think of functions on $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$. Fourier series

$$\mathcal{F} : L^2(\mathbb{T}^n) \to \ell^2(\mathbb{Z}^n)$$ square summable

$$f \mapsto \{\hat{f}(\underline{n})\}, \quad \hat{f}(\underline{n}) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(\underline{x}) e^{-i\underline{x}\cdot\underline{n}} \, d\underline{x}.$$

Fourier coefficients

is isometry $\|f\|^2_{L^2} = \sum_{\underline{n}\in\mathbb{Z}^n} |\hat{f}(\underline{n})|^2$ of Hilbert spaces.

If $f \in C^1$, $\frac{\partial}{\partial x_j} f \mapsto \{ i n_j \hat{f}(\underline{n})\}$

$$\Rightarrow \|Df\|^2_{L^2} = \sum_{\underline{n}\in\mathbb{Z}^n} |\underline{n}|^2 |\hat{f}(\underline{n})|^2.$$

So $\|f\|^2_{L^2_k} \simeq \sum_{\underline{n}\in\mathbb{Z}^n} |\hat{f}(\underline{n})|^2 (1+|\underline{n}|^2)^k$.

$\rightsquigarrow$ analysis on $L^2_k(\mathbb{T}^n)$ reduces to summability of series.

Of course, if T is diff op of order ℓ,

$u \in L^2_{k+\ell}(E) \Rightarrow Tu \in L^2_k(F)$. Indeed

$\exists\, c = c(T, g, D, k)$ s.t

$$\|Tu\|_{L^2_k} \leq c\,\|u\|_{L^2_{k+\ell}} \quad \forall u.$$

We have the key estimate.

<u>Thm A</u> (Gårding inequality) If T is <u>elliptic</u>,

$\exists\, C$ s.t

$$\|u\|_{L^2_{k+\ell}} \leq C\left(\|Tu\|_{L^2_k} + \|u\|_{L^2}\right).$$

<u>Rmk</u>) False for C^k norms!!!

•) $\|u\|_{L^2}$ is only to deal with $\mathrm{Ker}\, T$;

if $\mathrm{Ker}(T) = \{0\}$, or $u \perp \mathrm{Ker}\, T$, we have.

$$\|u\|_{L^2_{k+\ell}} \leq C\,\|Tu\|_{L^2_k}.$$

Intuition Δ on $\mathbb{T}^2$, then

$$\Delta f \overset{\mathcal{F}}{\rightsquigarrow} \{-|\underline{n}|^2 \hat{f}(\underline{n})\} \in \ell^2(\mathbb{Z}^n).$$

$$\Rightarrow \|\Delta f\|^2_{L^2_k} \simeq \sum_{\underline{n}\in\mathbb{Z}^n} |n|^4 |\hat{f}(\underline{n})|^2 (1+|n|^2)^k.$$

But up to a constant (k fixed)

$$|\underline{n}|^2 |\hat{f}(\underline{n})|^2 (1+|\underline{n}|^2)^k \gtrsim (1+|\underline{n}|^2)^{k+2}$$

unless $\underline{n}=\underline{0}$; add $|\hat{f}(0)|^2$.

($\ker\Delta = \mathbb{C} \leftrightarrow$ constant functions).

To go back to C^k norms:

Thm B (Sobolev embedding thm).

For $r > \frac{\dim M}{2}$,

$$L^2_{k+r} \hookrightarrow C^k \quad \text{(continuous embedding)}$$

i.e $\|u\|_{C^k} \lesssim \|u\|_{L^2_{k+r}}$.

Rmk This is sharp! In dim 2, $L^2_1 \not\hookrightarrow C^0$. ($k=0$.)

Ex: On $B_{1/2} \subseteq \mathbb{R}^2$, $\log\log\left(\frac{1}{|x|}\right)$ is in L^2_1

but not continuous. But $L^2_2 \hookrightarrow C^0$.

Intuition on $\mathbb{T}^n$, $k=0$. Choose f smooth

so $f(x) = \sum_{\underline{n}\in\mathbb{Z}^n} \hat{f}(\underline{n}) e^{-i\underline{n}\cdot x}$. Then

$$|f(0)| = \Big|\sum_{\underline{n}\in\mathbb{Z}^n} \hat{f}(\underline{n})\Big| \leq \sum_{\underline{n}\in\mathbb{Z}^n} |\hat{f}(\underline{n})| =$$

$$= \sum |\hat{f}(\underline{n})| \cdot \frac{(1+|\underline{n}|^2)^{r/2}}{(1+|\underline{n}|^2)^{r/2}} \overset{\text{Cauchy-Schwarz}}{\leq}$$

$$\leq \left(\sum |\hat{f}(\underline{n})|^2 (1+|\underline{n}|^2)^r\right)^{\frac{1}{2}} \cdot \underbrace{\left(\sum \frac{1}{(1+|\underline{n}|^2)^r}\right)^{\frac{1}{2}}}_{<\infty \text{ if } r>\frac{n}{2}} \approx \|f\|^2_{L^2_r}$$

For existence results:

Thm C (Rellich) For $k_1 > k_2$, the inclusion

$L^2_{k_1} \hookrightarrow L^2_{k_2}$ is compact

(i.e a bounded sequence in $L^2_{k_1}$ has a subsequence that converges in $L^2_{k_2}$)

Rmk this is the Sobolev analogue of Ascoli-Arzelà ($C^{k_1} \hookrightarrow C^{k_2}$ cpt).

Prove it on T^n (exercise!).

Finally, regarding generalized solutions.

Say $u \in C^\infty(E)$, $\varphi \in C^\infty(F)$.

Then $\langle Tu, \varphi \rangle_{L^2(F)} = \langle u, T^* \varphi \rangle_{L^2(E)}$

where $T^*: \mathcal{C}^\infty(F) \to \mathcal{C}^\infty(E)$ is the (L^2 formal) adjoint diff operator.

If $Tu = v$ $(\in \mathcal{C}^\infty(F))$, then

$$\langle u, T^*\varphi \rangle_{L^2} = \langle v, \varphi \rangle_{L^2} \quad \forall \varphi \in \mathcal{C}^\infty(F) \tag{*}$$

In this expression we are not taking derivative of u and v!!

<u>Def</u> $u, v \in L^2$. Then we say u solves $Tu = v$ <u>weakly</u> if $(*)$ holds.

<u>Thm D</u> (Elliptic regularity) T elliptic op of order ℓ. If $v \in L^2_k$, and $u \in L^2$ solves $Tu = v$ weakly $\Rightarrow u \in L^2_{k+\ell}$.

So in particular, if $u \in L^2$ solves $Tu = 0$ weakly $\Rightarrow u \in L^2_{k+l}$ $\forall k$ $\Rightarrow u \in C^\infty$ by Sobolev embedding thm!

Rmk One often needs more generally the Banach spaces L^p_k in applications, but we'll need them minimally. The proofs are much more involved, especially Thm A & Thm D (which hold for $1 < p < \infty$) & need tools such as singular integrals, Calderon-Zygmund dec, etc..