$\boxed{\text{Lecture 4}}$ Instantons, monopoles & vortices.

Let's study $YM(A) = \frac{1}{2}\|F_A\|^2_{L^2}$

non-abelian cpt Lie group

$SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \,\middle|\, |\alpha|^2 + |\beta|^2 = 1 \right\} \equiv S^3$ unit quaternions.

with invariant norm $\langle a, b \rangle = \mathrm{Tr}(ab^*)$ on

$su(2)$ = traceless-skew Hermitian $(\equiv \mathrm{Im}\,\mathbb{H} \cong \mathbb{R}^3)$

$= \mathrm{span}_{\mathbb{R}} \left\langle \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right\rangle$ Pauli matrices σ_i.

on $(\mathbb{R}^4, g_{flat})$ or (M^4, g) cpt. Three special things happen!

I) invariance under conformal rescaling $g \mapsto e^{2f} g$. (M cpt).

Explicitly in $\mathbb{R}^d$ $\nabla_A = d + A$, $A \in \Omega^1(su(2))$

rescaling by μ acts as

$$A \mapsto A^{(\mu)} \text{ with } A^{(\mu)}(x) = \mu A(\mu x)$$

$\Rightarrow F_{A^{(\mu)}}(x) = \mu^2 F_A(\mu x)$. For finite energy,

$$\Rightarrow YM(A^{(\mu)}) = \int |F_{A^{(\mu)}}(x)|^2 \, dvol = \mu^4 \int |F_A(\mu x)|^2 \, dvol$$

$$= \mu^{4-d} \int |F_A|^2 dx.$$

change of variables

In $\mathbb{R}^d$ we can have critical points only for $d=4$!

II) <u>Self-duality</u>: on (M^{4k}, g),

$$* : \Omega^{2k}(M) \circlearrowleft, \quad *^2 = 1$$

$$\Rightarrow \Omega^{2k} = \Omega^+ \oplus \Omega^- \quad (\pm 1)\text{-eigenspaces.}$$

Ω^+: self-dual, Ω^-: antiself-dual.

On M^4, $\Omega^+ = \operatorname{span}\left\{ \begin{array}{l} dx_1 dx_2 + dx_3 dx_4 \\ dx_1 dx_3 + dx_4 dx_2 \\ dx_1 dx_4 + dx_2 dx_3 \end{array} \right\}$.

In 4D, $\Omega^2 = \Omega^+ \oplus \Omega^-$ corresponds to the exceptional decomposition

$$so(4) = so(3) \oplus so(3) \quad \text{(as } so(4)\text{-reps)};$$

in gauge theory, this leads to

$$F_A = F_A^+ + F_A^- \in \Omega^2(\mathfrak{g}_E) = \Omega^+(\mathfrak{g}_E) \oplus \Omega^-(\mathfrak{g}_E).$$

pointwise orthogonal, so $F_A^+ \wedge F_A^- = 0$ pointwise.

III) <u>Topological bounds</u> on (M,g) cpt:

By <u>Chern-Weil</u> we have

$$\int_M \underbrace{\mathrm{Tr}(F_A \wedge F_A)}_{\Omega^4(su(2))} = 8\pi^2 c_2(E) \in H^4(M;\mathbb{R}) \cong \mathbb{R}.$$

($L \in su(2)$ is conjugate to $\begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix}$ $\lambda \in \mathbb{R}$

$\Rightarrow \mathrm{Tr}(L^2) = -2\lambda^2 = -2\det(L)$.)

The YM functional can be written as

$$YM(A) = \frac{1}{2}\int |F_A|^2 \, d\mathrm{vol} = -\frac{1}{2}\int \mathrm{Tr}(F_A \wedge \underbrace{*F_A}_{\in \Omega^2(\mathfrak{g}_E)}).$$

Hence we have

$$\int \mathrm{Tr}(F_A \wedge F_A) = \int \mathrm{Tr}(F_A^+ \wedge F_A^+) + \int \mathrm{Tr}(F_A^- \wedge F_A^-)$$

$$= \int \mathrm{Tr}(F_A^+ \wedge *F_A^+) - \int \mathrm{Tr}(F_A^- \wedge *F_A^-)$$

$$= -\|F_A^+\|^2_{L^2} + \|F_A^-\|^2_{L^2}$$

Using $YM(A) = \frac{1}{2}\left(\|F_A^+\|^2_{L^2} + \|F_A^-\|^2_{L^2}\right)$, we obtain

<u>Thm</u> (Bogomolnyi bound)

$$YM(A) = \|F_A^+\|^2_{L^2} + 4\pi^2 c_2(E).$$

purely topological quantity.

<u>Cor</u> Assume $c_2(E) \geq 0$. Then

$YM(A) \geq 4\pi^2 c_2(E)$, and minimum realized

exactly for $\boxed{F_A^+ = 0}$ <u>antiself duality eqns</u>. (ASD)

This is a <u>first order</u>, gauge & conformal invariant equation.

Of course, it implies Yang-Mills $d_A^* F_A = 0$.

(Indeed, $0 \underset{\text{Bianchi}}{=} d_A F_A \underset{\text{ASD}}{=} d_A(-* F_A)$).

<u>BPST 1975</u> there are interesting finite energy ASD connections in $\mathbb{R}^4$!

<u>Rmk</u> not possible in abelian (linear) theory! (L^2 harmonic form vanishes)

Use action $SO(4) \curvearrowright \mathbb{R}^4$, $SO(4) \curvearrowright so(3) \cong su(2)$

($SU(2) \xrightarrow{2:1} SO(3)$), impose equivariance on A

⇒ ASD reduces to explicitly solvable first order ODE.

Identify $\mathbb{R}^4 \equiv \mathbb{H}$, $su(2) = \mathrm{Im}\,\mathbb{H}$ then

$$A(x) = \mathrm{Im}\left(\frac{x\,d\bar{x}}{1+|x|^2}\right) \in \Omega^1(su(2))$$

$$F_A(x) = \frac{dx \wedge d\bar{x}}{(1+|x|^2)^2} \in \Omega^-(su(2))$$

⇒ BPST (or basic) instanton

with $YM(A) = 4\pi^2$ so "$c_2 = 1$".

To interpret this topologically, if A is any connection with $|F_A| \to 0$ sufficiently fast,

⇒ A is asymptotically flat

$$\Rightarrow A \simeq -(du)u^{-1} \text{ for } u : \underset{\underset{S^3}{\wr}}{\mathbb{R}^4 \setminus B_R} \to \overset{\overset{S^3}{\shortparallel}}{SU(2)}$$

$\Rightarrow$ well defined $\deg_\infty(A) \in \mathbb{Z}$.

(corresponding to $\pi_3(S^3) = \mathbb{Z}$) and

$$YM(A) = \|F_A^+\|_{L^2}^2 + 4\pi^2 \deg_\infty(A).$$

This is an example of a <u>topological soliton</u>

<u>Other source</u>: (M,g) Riemannian, $E \to M$ G-bundle.

$\mathcal{C} = \{(A, \Phi) \mid A$ compatible connection, Φ section$\}$.

"Higgs field"

For fixed $\lambda > 0$

$$YMH(A,\Phi) := \frac{1}{2}\int |F_A|^2 + \frac{1}{2}|\nabla_A \Phi|^2 + \frac{\lambda}{8}\int (1-|\Phi|^2)^2$$

<u>Yang-Mills-Higgs functional</u>, invariant

for $\mathcal{G} \curvearrowright \mathcal{C}$. $\left(\nabla_{u\cdot A}(u\cdot\Phi) = u\cdot\nabla_A\Phi\right)$

by def of $u\cdot A$

(8)

Rule name comes from $G = SU(2)$, introduced to explain mass of bosons in weak interaction. (Higgs mechanism). When $G = U(1)$, this is describes superconductivity (Ginzburg-Landau)

The same rescaling argument in $\mathbb{R}^d$

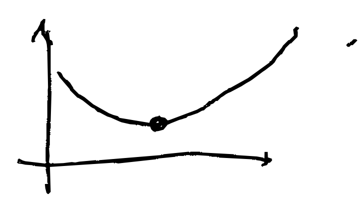
($\Phi \rightsquigarrow \Phi^{(\mu)}$ with $\Phi^{(\mu)}(x) = \Phi(\mu x)$ scalar,

so $\nabla_{A^{(\mu)}} \Phi^{(\mu)}(x) = \mu \nabla_A \Phi(\mu x)$) shows,

writing $YM(A, \Phi) = E_4 + E_2 + E_0$

$$YM(A^{(\mu)}, \Phi^{(\mu)}) = \mu^{4-d} E_4 + \mu^{2-d} E_2 + \mu^{-d} E_0$$

$\Rightarrow$ we can have interesting critical points only when $d = 2, 3$.



(Derrick's argument).

Cases of special topological interest:

•) $d=3$, $G=SO(3)$ monopoles ('t Hooft - Polyakov)
↳ $E \to M^3$ $\mathrm{rk}_{\mathbb{R}}=3$ orthogonal bundle.

•) $d=2$ $G=U(1)$ vortices (Abrikosov).

Let's look at monopoles in $\mathbb{R}^3$.

In this case, it's natural to ask $|\Phi| \to 1$ sufficiently fast.

Again, critical points satisfy 2nd order PDE;

Special properties as instantons in $\mathbb{R}^4$ arise setting $\lambda=0$, with

$$YMH(A,\Phi) = \frac{1}{2}\|F_A\|^2_{L^2} + \frac{1}{2}\|\nabla_A \Phi\|^2_{L^2}$$

with boundary condition $|\Phi| \to 1$

$\Phi(x) \in so(3)$, 3D, so we have

$\Phi/|\Phi| : S^2_R \to S^2 \Rightarrow$ well-defined $\deg_\infty(\Phi)$.

In this situation, we have then the Bogomolnyi bound

$$YMH(A,\Phi) = \frac{1}{2}\int |F_A - *D_A\Phi|^2 + 4\pi \deg_\infty(\Phi)$$

$\Rightarrow$ in each topological class the bound is realized by the Bogomolnyi eqns

$$\boxed{F_A - *D_A\Phi = 0} \qquad (\in \Omega^2(so(3))).$$

Solutions are called <u>Bogomolnyi monopoles</u>

<u>why monopoles?</u> In the $U(1)$-case, they are simply $B = \operatorname{grad}\Phi$, with

$\Phi(x) = \text{const} \cdot \frac{1}{|x|}$ is the Dirac monopole!

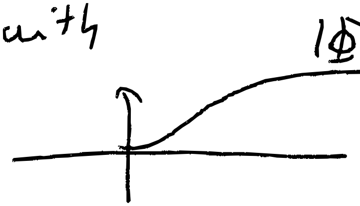
The non-abelian case has interesting smooth finite energy solutions!

<u>BPS monopole 1975</u>: found imposing equivariance. Set:

•) $\hat{r} = \frac{x}{|x|}$ unit radial vector with entries $\frac{x_i}{|x|}$

•) $so(3) \cong \mathbb{R}^3$, $\{e_j\}$ ON basis $\left(e_j = \frac{i}{\sqrt{2}} \sigma_j\right)$.

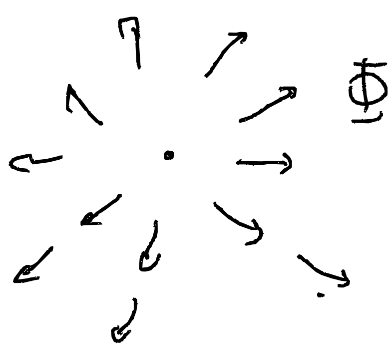
Then solution(*) is (A, Φ) with



$$\Phi(x) = \left(\frac{1}{\tanh r} - \frac{1}{r}\right) \hat{r} \cdot \vec{e}$$

$$A(x) = \left(\frac{1}{\sinh r} - \frac{1}{r}\right) (\hat{r} \times \vec{e}) \cdot d\vec{x}$$

These are smooth at $\vec{0}$, and Φ is radial with only $\vec{0}$ at $\vec{0}$.



(*) probably off by factors of 2 ⚠

Finally, any section ϕ has a parallel part defined as $\langle \phi, \Phi \rangle_{so(3)} \frac{\Phi}{|\Phi|^2}$.

outside the sing'n. (and transverse).

One computes that the parallel part of $\star F_A = \nabla_A \Phi$ (1-form with values in $so(3)$) is

$$\left(\frac{1}{r^2} - \frac{1}{\sinh^2 r}\right)(\hat{r} \cdot \vec{e}) \cdot \hat{r} \cdot d\vec{x},$$

asymptotic to Dirac monopole $\frac{\hat{r}}{r^2}$.

The story for vortices at $\lambda = 1$ in $\mathbb{R}^2$ is analogous, top bound based on $\deg_\infty \in \pi_1(S^1) = \mathbb{Z}$. We'll study them in detail on a surface (Σ, g)!