$\boxed{\text{Lecture 3}}$ Yang-Mills theory on Riemannian manifolds.

Recall energy density of magnetic field B in $\mathbb{R}^3$ is $\frac{1}{2}|B|^2$. This generalizes to

$E \to M$ G-bundle, A compatible connection

$\rightsquigarrow F_A \in \Omega^2(\mathfrak{g}_E)$ curvature 2-form.

$(u \in \mathcal{G} \Rightarrow F_{u \cdot A} = u F_A u^{-1})$.

To define its norm $|F_A|$

-) g Riemannian metric on M.
-) G-invariant norm $|\cdot| : \mathfrak{g} \to \mathbb{R}$ (induced by inner product).

Ex.) $\mathfrak{g} = \mathfrak{u}(n) = \{$ skew symmetric matrices $\}$.

$|\xi|^2 := L^2$-norm as vector in $\mathbb{C}^{n\times n} = -\mathrm{Tr}(\xi^2)$

is $U(n)$ invariant.

•) in general such norm exists for G cpt

by averaging.

Def The Yang-Mills functional on (M,g) is

$$YM(A) = \frac{1}{2}\int_M |F_A|^2 \, dvol_g ,$$

descends to $\mathcal{A}/\mathcal{G} \xrightarrow{YM} \mathbb{R}$

Rmk introduced by Yang & Mills (1954) for

$G = SU(2)$ to describe weak interactions;

the three components of $F_A \in \Omega^2(\mathfrak{su}(2))$

are called W^+, W^-, Z^0 gauge bosons.

<u>Q</u> what are the critical points of this functional?

The answer involves a non-linear version of <u>Hodge theory</u>, which we now review.

<u>Recall</u> M^n smooth mfld, de Rham complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \to \cdots \xrightarrow{d} \Omega^n(M)$$

with cohomology $\mathrm{Ker}\, d / \mathrm{Im}\, d \cong H^i(M; \mathbb{R})$.

<u>No preferred</u> representative in a given cohomology class!

If (M, g) is cpt, $\exists$ natural L^2-norm

$$\|\alpha\|_{L^2}^2 = \int_M |\alpha|^2 \, d\mathrm{vol}_g \qquad \alpha \in \Omega^i(M).$$

(induced by inner product $\langle \alpha, \beta \rangle_{L^2} = \int_M \langle \alpha, \beta \rangle \, d\mathrm{vol}_g$)

$\Rightarrow$ it's natural to consider the form that minimizes in a given class.

<u>Problem</u> does it exist, and is it unique?

$\rightsquigarrow$ Hodge theory.

<u>Def</u> $(\overset{\to \dim_{\mathbb{R}} = n}{V}, \langle , \rangle, o)$ oriented Euclidean v. space.

$\Rightarrow \ast : \Lambda^i V \to \Lambda^{n-i} V$ <u>Hodge star</u>

sends for oriented ON basis $e_1, \dots e_n$

$\ast(e_1 \wedge \dots \wedge e_k) = e_{k+1} \wedge \dots \wedge e_n$.

<u>Eg</u> in $(\mathbb{R}^3)^\ast$, $\ast dx_1 = dx_2 \wedge dx_3$, $\ast dx_2 = dx_3 \wedge dx_1$

<u>Lemma</u> •) $\alpha, \beta \in \Lambda^i V \Rightarrow \alpha \wedge \ast\beta = \langle \alpha, \beta \rangle \underbrace{dvol}_{= e_1 \wedge \dots \wedge e_n}$.

•) $\ast^2 : \Lambda^i V ⟲$ is $(-1)^{i(n-i)}$

On (M^n, g, o), this induces
orientation

$$* : \Omega^i(M) \to \Omega^{n-i}(M).$$

We have $d : \Omega^{p-1} \to \Omega^p$, metric independent

<u>Def</u> $d^* : \Omega^p \to \Omega^{p-1}$ is given by

$$d^* = (-1)^{n(p+1)+1} * d *$$

Assume from now on <u>M cpt</u>, so that Ω^i are inner product spaces.

<u>Key lemma</u> d^* is the (formal L^2) adjoint of d, i.e $\langle d\alpha, \beta\rangle_{L^2} = \langle \alpha, d^*\beta\rangle_{L^2}$

$$\forall \alpha \in \Omega^{p-1}, \beta \in \Omega^p.$$

<u>Proof</u> integration by parts, i.e Stokes!

$\alpha \wedge * \beta \in \Omega^{n-1}$, compute pointwise

$$d(\alpha \wedge * \beta) = d\alpha \wedge * \beta + (-1)^{p-1} \alpha \wedge d * \beta \underset{\text{check signs!}}{=}$$

$$= d\alpha \wedge * \beta - \alpha \wedge * d^* \beta = (\langle d\alpha, \beta \rangle - \langle \alpha, d^* \beta \rangle) \, dvol.$$

$\square$

<u>Rmk</u> For functions on $\mathbb{R}$ (cpty supported)

$$\int \left(\frac{d}{dx} f\right) g = - \int f \left(\frac{d}{dx} g\right),$$ so $\frac{d}{dx}$ is skew-adjoint.

Now assume $\alpha_0 \in \Omega^p$ closed minimizes L^2-norm in $[\alpha_0]$. Then $\forall \eta \in \Omega^{p-1}$

$$0 = \frac{d}{dt}\Big|_{t=0} \|\alpha_0 + t d\eta\|_{L^2}^2$$

$$= \frac{d}{dt}\Big|_{t=0} \left(\|\alpha_0\|_{L^2}^2 + 2t \langle \alpha_0, d\eta \rangle_{L^2} + t^2 \|d\eta\|_{L^2}^2 \right)$$

$$= 2 \langle \alpha, d\eta \rangle = 2 \langle d^* \alpha, \eta \rangle$$

$\Rightarrow \; d^*\alpha_0 = 0$ (i.e. α is coclosed).

So $\alpha_0 \in \ker(d+d^* : \Omega^p \to \Omega^{p+1} \oplus \Omega^{p-1})$.

Def The Hodge Laplacian on (M,g) is

$$\Delta = \underset{\substack{\| \\ dd^* + d^*d}}{(d+d^*)^2} : \Omega^p \circlearrowleft$$

Basic facts •) on Ω^0, Δ is the usual Laplace-Beltrami $\Delta = -\operatorname{div}\operatorname{grad} f$

$$\left(\begin{tikzcd} \Omega^0 \arrow[r, shift left, "d"] \arrow[r, bend left=60, "\mathrm{grad}"] & \Omega^1 \cong \Gamma(TM) \arrow[l, shift left, "d^*"] \arrow[l, bend left=60, "-\mathrm{div}"] \end{tikzcd}\right).$$

•) Δ is (formally L^2) self adjoint,

$$\langle \alpha, \Delta\beta \rangle_{L^2} = \langle \Delta\alpha, \beta \rangle_{L^2}.$$

•) Δ is non-negative, indeed

$$\langle \alpha, \Delta\alpha \rangle_{L^2} = \|(d+d^*)^2 \alpha\|_{L^2}^2 \geq 0.$$

This also shows $\ker \Delta = \ker(d+d^*)$!
= harmonic p-forms.

So we were asking for harmonic representatives!

<u>Hodge theorem</u> There is an L^2-orthogonal decomposition for (M,g) cpt

$$\Omega^p = \underbrace{d\Omega^{p-1}}_{\text{exact}} \oplus \underbrace{\mathcal{H}^p}_{\text{harmonic}} \oplus \underbrace{d^*\Omega^{p+1}}_{\text{coexact}}$$

and $\mathcal{H}^p := \ker\Delta = (\operatorname{Im}\Delta)^\perp \cong H^p(M;\mathbb{R})$

This theorem provides a deep link between geometry, topology, and linear PDEs; we'll sketch a proof in a few lectures!

<u>Rmk</u> Poincaré duality is $*: \mathcal{H}^p \to \mathcal{H}^{n-p}$.

Back to Yang-Mills on (M,g), $E \to M$,

story is very similar; let's see how

$YM(A) = \frac{1}{2}\int |F_A|^2 \, dvol_g = \frac{1}{2}\|F_A\|^2_{L^2}$ changes

<u>Lemma</u> For $a \in \Omega^1(\mathfrak{g}_E)$, we have

$$F_{A+a} = F_A + d_A a + a \wedge a$$

<u>Rmk</u> In the $U(1)$ case, this is just

$F_{A+a} = F_A + da$, same computation as earlier!

<u>Proof</u> $F_A \cdot s = d_A d_A s$. by def (d_A on $\Omega^i(E)$)

$\Rightarrow F_{A+a} \cdot s = d_{A+a} d_{A+a} s =$

$= d_{A+a}(d_A s + as) =$

$$= d_A d_A s + a \wedge d_A s + \overbrace{(d_A a) s - a \wedge d_A s}^{d_A(as)} + (a \wedge a) s.$$

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$\square$

So $YM(A+ta) = \frac{1}{2} \| F_{A+ta} \|^2_{L^2} =$

$$= \frac{1}{2} \| F_A \|^2_{L^2} + t \langle d_A a, F_A \rangle_{L^2} + O(t^2).$$

$$\Rightarrow \frac{d}{dt}\Big|_{t=0} YM(A+ta) = \langle d_A a, F_A \rangle_{L^2} =$$

$$= \langle a, d_A^* F_A \rangle_{L^2},$$

where $d_A^* : \Omega^2(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)$ adjoint of d_A.

$\Rightarrow$ critical points (called YM connections)

$\boxed{d_A^* F_A = 0}$ Yang-Mills equations,

2nd order nonlinear PDE in A.

(reduces to Ampère eqns in magnetostatics).

Bianchi : $d_A F_A = 0$

$\rightsquigarrow$ so we are looking at some non-linear version of $d + d^*$! Indeed simplest case:

<u>Prop</u> $L \to (M, g)$ hermitian line bundle on cpt M.

$\Rightarrow \exists$ Yang-Mills connection A.

<u>Proof</u> Fix A_0 connection, $F_{A_0+a} = F_{A_0} + da$.

$a \in i\Omega^1$. Want to solve,

$$0 = d^*(F_{A+a}) = d^* F_A + d^* da.$$

By Hodge theorem, $\Delta : d^*\Omega^2 \circlearrowleft$ isomorphism

$\Rightarrow d^* F_A = \Delta b$ for some $b \in d^*\Omega^2$

$= d^* db$ $\quad (d^* b = 0)$

$\Rightarrow$ choose $a = -b$. $\square$.

The case G non-abelian is much more complicated. We'll see next time interesting things happen when $\dim M = 4$!

In $\underline{\dim 2}$, $d_A^*: \Omega^2(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)$ is $-*d_A*$;

but $*F_A \in \Omega^0(\mathfrak{g}_E)$ so YM eqns say:

$*F_A$ is A-covariantly constant

Recall in trivialization τ, $d_A = d + [A^\tau, -]$;

so if $*F_A = x \in Z(\mathfrak{g}) \Rightarrow A$ is Yang-Mills.

constant, central curvature generalizes flat!
(well-defined!)

$\underline{\text{Ex}}$ $G = U(n)$ $Z(U(n)) = i\mathbb{R} \cdot I$

this determines $\pi_1(M) \to PU(n) = U(n)/\text{center}$
$U(1) \cdot \mathrm{Id}$

projective unitary representation.