$\boxed{\text{Lecture 2}}$ Connections & topology.

$E \to M$ vector bundle with structure G.

$\Rightarrow \mathcal{A} = \{ A \mid A$ compatible connections on $E\}$

$\hookrightarrow$ affine space / $\Omega^1(\mathfrak{g}_E)$

<u>Gauge transformations</u> are

$$\mathcal{G} = \left\{ u \;\middle|\; \begin{array}{c} E \xrightarrow{u} E \\ \searrow_M \swarrow \end{array} \begin{array}{l} \text{bundle automorphism} \\ \text{preserving structure} \end{array} \right\}.$$

(= symmetries of E)

Of course, $\mathcal{G} \curvearrowright \Gamma(E)$ by multiplication;
it also acts on $\mathcal{A}$ by pullback, i.e.

$$\nabla_{u\cdot A} s = u \nabla_A (u^{-1} s) \quad \forall A \in \mathcal{A},\ s \in \Gamma(E)$$

$$\Rightarrow \nabla_{u\cdot A} = \nabla_A - \underbrace{(\nabla_A u) u^{-1}}_{\in \Omega^1(\mathfrak{g}_E)}$$

Here, we thing $u \in \mathcal{G} \in \Omega^0(\mathrm{End}(E))$,

and use the fact: ∇_A on E naturally

induces connection of $\mathrm{End}(E) = E \otimes E^*$

(still called ∇_A), given in a trivialization

τ (so $\nabla_A = d + \underset{\in \Omega^1(\mathfrak{g})}{A^\tau}$ on E) by

$d + [A^\tau, -]$ on $\mathrm{End}(E)$.

So locally, in the trivialization $u\tau$

$$A^{u\tau} = A^\tau - \{(d + [A^\tau, -])u\} u^{-1} =$$

$$= A^\tau - \{du + A^\tau u - u A^\tau\} u^{-1} =$$

$$= u A^\tau u^{-1} - (du) u^{-1} \in \Omega^1(\mathfrak{g}).$$

Example $L \to M$ hermitian line bundle ($G = U(1)$)

If $u = e^{if}$, $f: M \to \mathbb{R}$, then

$dv = ie^{if}(df)$

$\Rightarrow A^{vc} = A^{c} - i\,df$, same thing from magnetostatic!

<u>Rmk</u> $\mathcal{G} = \{ v : M \to U(1) \}$ gauge group.

$\Rightarrow \pi_0(\mathcal{G}) = H^1(M;\mathbb{Z})$ $\quad (U(1) = S^1 = K(\mathbb{Z},1))$

$\Rightarrow$ not all gauge transformation are $v = e^{if}$.

Natural object to study is

$\mathcal{A}/\mathcal{G}$ $\quad \underline{\text{moduli space of connections}}$,
(up to gauge)

really, we are interested in gauge-independent quantities, i.e well-defined functions

$\mathcal{L} : \mathcal{A}/\mathcal{G} \to \mathbb{R}$

These naturally arise from the geometry of the connection, in particular its <u>curvature</u>. (This intuitively measures how covariant derivatives do <u>not</u> commute:

)

Algebraically, we can obtain

$$d_A : \Omega^p(E) \to \Omega^{p+1}(E) \quad \text{(covariant exterior derivative)}$$

Combining: •) $d_A = \nabla_A : \Omega^0(E) \to \Omega^1(E)$

•) $d_A(\alpha \wedge \beta) = d_A\alpha \wedge \beta + (-1)^p \alpha \wedge d_A \beta$

$\forall \alpha \in \Omega^p(E), \ \beta \in \Omega^{p+1}(E)$.

Key point:

$$\Omega^0(E) \xrightarrow{d_A} \Omega^1(E) \xrightarrow{d_A} \Omega^2(E) \xrightarrow{d_A} \Omega^3(E) \to \dots$$

Is <u>not</u> a complex!

Checking C^∞-linearity, we have

Lemma $\exists\, F_A \in \Omega^2(\mathfrak{g}_E) \subseteq \Omega^2(\mathrm{End}(E))$ s.t

$$d_A d_A \alpha = F_A \cdot \alpha \quad \forall \alpha \in \Omega^p(E)$$

$\hookrightarrow$ wedge and $\mathrm{End}(E) \otimes E \to E$.

· F_A is the curvature of A.

Rmk (M,g) Riemannian $\Rightarrow \nabla_{LC}$ on TM

$so(n)$ connection

$\Rightarrow F_{LC} \in \Omega^2(so(TM))$ Riemann curvature tensor

4-indices, antisymmetry properties.

Locally $\nabla_A = d + A^\tau$, $A^\tau \in \Omega^1(\mathfrak{g})$

Then $F_A^\tau = dA^\tau + A^\tau \wedge A^\tau$ $(=: \Omega \in \Omega^2(\mathfrak{g}))$

wedge & matrix multiplication.

In explicit coordinates $\{x_i\}$,

$F_A^\tau = \sum F_{ij}\, dx_i \wedge dx_j$ with

$$F_{ij} = [\nabla_i, \nabla_j] = \left[\frac{\partial}{\partial x_i} + A_i^\tau, \frac{\partial}{\partial x_j} + A_j^\tau\right] =$$

$$= \frac{\partial}{\partial x_i} A_j^\tau - \frac{\partial}{\partial x_j} A_i^\tau + [A_i^\tau, A_j^\tau]$$

↳ quadratic term!

Notice that if $G = U(1) \Rightarrow A \in i\Omega^1$ and

$F_A = dA$ is simply the magnetic field $B \in i\Omega^2$!

Gauss' law $dB = 0$ is a special case of

<u>Lemma</u> (Bianchi identity) $d_A F_A = 0$ $(\in \Omega^3(\mathfrak{g}_E))$

<u>Proof</u> Locally, this is just

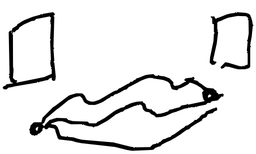
$d\Omega + [A^\tau, \Omega] = 0$, which follows from

differentiating $\Omega = dA^\tau + A^\tau \wedge A^\tau$ $\square$.

Lemma $u \in \mathcal{G}$, then $F_{u \cdot A} = u F_A u^{-1} \in \Omega^2(\mathfrak{g}_E)$.

So curvature is not gauge invariant

Notice Being flat is a gauge invariant notion! Also flat $\Rightarrow$ covariant derivatives commute $\Rightarrow$ parallel transport is htpy invariant



so we obtain well defined

$$\{\text{flat connections on } E\}/\text{gauge} \longrightarrow \{\pi_1(M) \to G\}/G$$

acting by conj.

holonomy representation $\pi_1(M) \curvearrowright E_{x_0}$.

Of course, if E is trivial ($E \cong M \times \mathbb{C}^n$ G-iso) $\Rightarrow \exists$ flat connection.

Conversely, one can use curvature to detect topological non triviality

$\rightsquigarrow$ Chern-Weil approach to <u>characteristic classes</u>.

<u>Key point</u> $G \curvearrowright \mathfrak{g}$ by conjugation, invariant functions $\mathfrak{g} \to \mathbb{R}$ or $\mathbb{C}$ applied to F_A give gauge invariant quantities of A.

<u>Ex</u> $F_A \in \Omega^2(\mathrm{End}\, E)$ ↙ $n \times n$ matrix of 2-forms / matrix valued 2-form.

$\Rightarrow \operatorname{tr} F_A \in \Omega^2_{\mathbb{C}}$ is gauge invariant!

<u>Rmk</u> $\operatorname{tr} F_A \in i\Omega^2$ if A is unitary;

Similarly, we can consider $\det F_A \in \Omega^{2n}_{\mathbb{C}}$.

We'll focus on the $G = GL_n(\mathbb{C})$ case.

<u>Fact</u> $GL_n(\mathbb{C})$-invariant polynomials on $\mathfrak{gl}_n(\mathbb{C})$ are polynomials in symmetric functions of eigenvalues; assemble for $L \in \mathfrak{gl}_n(\mathbb{C})$ as

$$\det(1+qL) = \sum q^k \operatorname{tr}(\Lambda^k L)$$

with q formal variable.

<u>Prop</u> P invariant polynomial (e.g. $P = \operatorname{tr}, \det$) of degree k, then $P(F_A) \in \Omega^{2k}_{\mathbb{C}}$ is closed & $[P(F_A)] \in H^{2k}(M;\mathbb{C})$ is independent of the connection A

$\Rightarrow$ it's a topological invariant of the bundle!

<u>Proof</u> can study $\log \det(1+qL)$ formally as power series in q.

Let's see what happens along a path $A(t)$ of connections, let's do local computation $A^2(t) = A(t) \in \Omega^1(gl_n(\mathbb{C}))$

$\Omega = dA + \underbrace{A \wedge A}_{A \cdot A}$ so differentiating in t

$$\dot{\Omega} = d\dot{A} + A\dot{A} + \dot{A}A$$

Formally (as power series in q)

$$\frac{d}{dt} \log \det(1 + q\Omega) = q \, \mathrm{tr}\left[(1+q\Omega)^{-1}\dot{\Omega}\right]$$

$$= \sum_{l=0}^{\infty} (-1)^l q^{l+1} \, \mathrm{tr}(\Omega^l (d\dot{A} + A\dot{A} + \dot{A}A)).$$

Now $\mathrm{tr}(\Omega^l (A\dot{A} + \dot{A}A)) \stackrel{\text{cyclic symmetry of trace}}{=}$

$$= \mathrm{tr}(\Omega^l A\dot{A} - A\Omega^l \dot{A}) \stackrel{\text{Bianchi identity.}}{=} \mathrm{tr}((d\Omega^l)\dot{A})$$

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$\Rightarrow \frac{d}{dt} \log \det(1+q\Omega) = d\left(\sum_{\ell=0}^{\infty}(-1)^{\ell}q^{\ell+1}\operatorname{tr}(\Omega^{\ell}\dot{A})\right)$

is <u>exact</u>. So deforming the connection changes it by an exact form.

•) locally, consider $A(t) = tA$

$\Rightarrow$ locally exact hence closed.

•) globally, $\mathcal{A}$ connected $\Rightarrow$ cohomology class independent of connection $\square$.

<u>Fact</u> $P(L) = \left(\frac{i}{2\pi}\right)^k \operatorname{tr}(\Lambda^k L)$ determines

$c_k(E) \in H^{2k}(M;\mathbb{Z}) \subseteq H^{2k}(M;\mathbb{C})$

k-th <u>chern class</u>.

<u>Rmk</u> Dirac monople $\rightsquigarrow U(1)$-bundle. corresponds to a

in $H^2(\mathbb{R}^3 \setminus \{0\};\mathbb{Z}) \cong \mathbb{Z}$ generator.

These satisfy nice axioms!

•) $c_0(E) = 1 \in H^0$, $c_i(E) = 0$ if $i > \dim_{\mathbb{C}} E$.

•) $f: N \to M$, $E \to M$ vector bundle

$\Rightarrow f^*E \to N$ pullback bundle has

$c_k(f^*E) = f^* c_k(E)$ (naturality)

•) $E, F \to M$, then

$c_k(E \oplus F) = \sum c_i(E) \cup c_{k-i}(F)$ (Whitney sum formula)

•) Normalization: $\gamma \to \mathbb{CP}^1$ tautological bundle has $c_1(\gamma) = -1 \in H^2(\mathbb{CP}^1; \mathbb{Z}) \underset{\text{natural orientation}}{\cong} \mathbb{Z}$

These uniquely determine the Chern classes!