Lecture 1 From magnetostatics in $\mathbb{R}^3$ to gauge theory

$U \subset \mathbb{R}^3$ open set, in absence of currents the magnetic field $B = B_1 \frac{\partial}{\partial x_1} + B_2 \frac{\partial}{\partial x_2} + B_3 \frac{\partial}{\partial x_3}$

satisfies the Maxwell eqns:

(Gauss) $\operatorname{div} B = 0$ (i.e $\nabla \cdot B = 0$, where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$)

(Ampère) $\operatorname{curl} B = 0$ (i.e $\nabla \times B = 0$)

A point particle is affected via

Lorentz force: $F = q(v \times B)$

charge velocity.

Energy density: $\frac{1}{2}|B|^2$

Natural differential form viewpoint:

Consider $\bar{B} = B_1\, dx_2 \wedge dx_3 + B_2\, dx_3 \wedge dx_1 + B_3\, dx_1 \wedge dx_2$.

two-form in $U \subseteq \mathbb{R}^3$

Then (gauss) is: $d\bar{B} = 0$, i.e $\bar{B}$ is closed.

$\Rightarrow$ locally, $\bar{B}$ is exact, i.e $\bar{B} = d\bar{A}$ for

$\bar{A} = A_1\, dx_1 + A_2\, dx_2 + A_3\, dx_3$ 1-form.

Rmk in physics, one considers

$A = \sum A_i \frac{\partial}{\partial x_i}$ vector potential, $\nabla \times A = B$.

Key point there are many choices for $\bar{A}$!

for any $f : U \to \mathbb{R}$, $\bar{A} + df$ also works.
or anything closed

($A + \text{grad} f$ in vector notation).

$\rightsquigarrow$ first instance of <u>gauge symmetry</u>!

Choosing the right gauge simplifies computations a lot! <u>Ex</u> in terms of A, (Ampère) becomes $\operatorname{grad}(\operatorname{div} A) + \Delta A = 0$

where $\Delta = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$ natural convention for Laplacian.

applied entry-wise.

So if $\operatorname{div} A = 0$ (Coulomb gauge condition)

we simply ask $\Delta A_i = 0$. $(i = 1, 2, 3)$,

i.e A_i is <u>harmonic</u>.

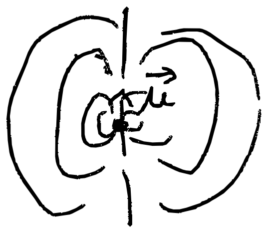
<u>Example</u>! <u>magnetic dipole</u> in $\mathbb{R}^3 \setminus \{0\}$ with

magnetic moment $\vec{\mu}$.

$$A(\vec{x}) = \frac{\mu_0}{4\pi}\cdot\frac{\vec{\mu}\times\vec{x}}{|x|^3}$$

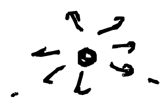
magnetic permeability.

$\left(\frac{x_i}{|x|^3} \text{ is harmonic on } \mathbb{R}^3\setminus\{0\}, \ \operatorname{div} A = 0. \right)$

$\rightsquigarrow B(\vec{x})$ looks like 

Of course, $\bar{B}$ is an exact 2-form in $\mathbb{R}^3\setminus\{0\}$.

<u>Example 2</u> <u>Dirac monopole</u> (1931) of charge g

is $B = \frac{\mu_0 g}{4\pi r^2}\hat{r}$ on $\mathbb{R}^3\setminus\{0\}$ 

$\int_{S^2_r}\bar{B} = \mu_0 g \Rightarrow$ it's non-zero class in

$H^2_{dR}(\mathbb{R}^3\setminus\{0\}) = \mathbb{R}$!

$\rightsquigarrow$ no global potential.

Key example for us (open question if it exists in nature).

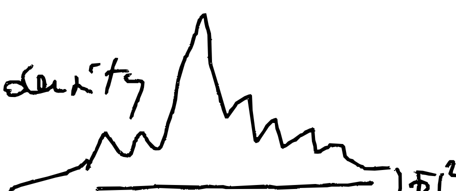
In quantum theory, particles are also described by fields. <u>Simplest case</u> $\Phi : U \to \mathbb{C}$

<u>Klein-Gordon</u> field with charge q, mass $m>0$ has energy density (set $\hbar=1$, $c=1$)

$$\mathcal{L} = \frac{1}{2}\left|\nabla \Phi - iqA\,\Phi\right|^2 + \frac{1}{2} m |\Phi|^2 .$$

$\mathbb{C}$-valued vector.

Here $|\Phi|^2$ is the probability density of the particle,



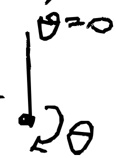
unchanged by $\Phi \rightsquigarrow e^{i\chi}\Phi$ for $\chi : U \to \mathbb{R}$. But $\mathcal{L}$ is not!

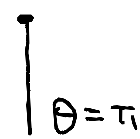
The gauge symmetry leaving $\mathcal{L}$ invariant in this case is

$$A \mapsto A + \operatorname{grad} f, \quad \Phi \mapsto e^{iqf}\,\Phi.$$

<u>Dirac</u>: this has topological meaning!

For the Dirac monopole, write

$\mathbb{R}^3 \setminus \{0\} = U^0 \cup U^\pi \quad U^0 =$ 

$U^\pi =$ 

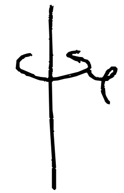
Here have $H^2_{dR} = 0 \Rightarrow B$ has primitive.

Explicitly (set $\mu_0 = 1$)

$$A^0 = -\frac{g}{4\pi}\frac{1}{r\sin\theta}(1+\cos\theta)\,\hat{\varphi}$$

$$A^\pi = \frac{g}{4\pi}\frac{1}{r\sin\theta}(1-\cos\theta)\,\hat{\varphi}$$

On $U^0 \cap U^\pi$, $\bar{A}^0 = \bar{A}^\pi + d\left(-\frac{g}{2\pi}\varphi\right)$



(well-defined closed form!).

If Φ is KG field of charge q, well-definedness

$\Rightarrow \frac{qg}{2\pi} \in \mathbb{Z} \Rightarrow q$ is <u>quantized</u>.

<u>Rmk</u> this is closely related to building

non trivial vector bundles on S^n

via clutching functions: $\mathrm{Vect}_k^{\mathbb{C}}(S^n) \overset{1:1}{\longleftrightarrow} \pi_{n-1}(GL(k,\mathbb{C}))$

<u>First goal</u>: setup the geometric language

to describe gauge theory in general

(Weyl, Yang, etc...), generalizing

magnetostatics ($= \underset{\{e^{i\theta}\}}{\underset{\shortparallel}{U(1)}}$-gauge theory).

Basic dictionary on M manifold

field Φ $\rightsquigarrow$ section of vector bundle
$$E \to M.$$

potential A $\rightsquigarrow$ connection A on E

$\nabla - iqA$ $\rightsquigarrow$ covariant derivative ∇_A on E

magnetic field B $\rightsquigarrow$ curvature F_A of A

quantization of charge $\rightsquigarrow$ characteristic classes of E.

gauge symmetry $\to$ bundle automorphisms

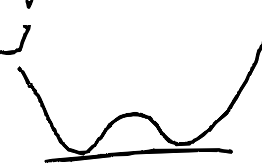
$\to$ we'll study fundamental objects which appear both in physics (e.g. standard model) but also have intrinsic geometric topological meaning.

<u>Key example</u> $A=\sum A_i\,dx_i$, $\Phi : U \to \mathbb{C}$

as before (B curvature 2-form).

The <u>abelian Yang-Mills-Higgs energy</u> (for a fixed parameter $\lambda>0$) is

$$\frac{1}{2}|B|^2+\frac{1}{2}|(\nabla-iqA)\Phi|^2+\underbrace{\frac{\lambda}{8}\left(1-|\Phi|^2\right)^2}_{\text{Higgs potential}}$$



aka <u>gauged Landau-Ginzburg</u> energy

describes the behaviour of superconducting

materials. Physically, the cases $\lambda<1$ and

$\lambda>1$ lead to different interesting phenomena;

Mathematically, $\lambda=1$ $\to$ critical coupling (e.g. alloy of

lead and 1-2% thallium) is

is the most interesting, and is in fact (10)
the reason why in Heegaard Floer homology
(key tool in contemporary low-dimensional top)
one studies the symmetric products of a
Riemann surface. We'll spend the rest of
the semester studying and explaining the
geometry, topology & analysis behind this.

<u>Geometric setup</u> M smooth mfld (usually cpt, or $\mathbb{R}^4$).
$E \to M$ vector bundle (either $\mathbb{C}$ or $\mathbb{R}$)

<u>Connection</u> A on E is defined through
<u>covariant derivative</u>

$$\nabla_A : \Omega^0(E) \to \Omega^1(E) \qquad \left(\begin{array}{c} \Omega^p(E) = E\text{-valued} \\ p\text{-forms on } M \end{array} \right)$$
$$\begin{array}{c} \parallel \\ d_A \end{array}$$

Satisfying Leibnitz rule:

$$\nabla_A(f\cdot s) = df\cdot s + f\nabla_A s \quad \forall f\in C^\infty(M),\ s\in\Omega^0(E).$$

$\{$Connection on $E\}$ is affine space$/\Omega^1(\mathrm{End}(E))$ $\underset{\displaystyle \mathfrak{gl}(E)}{\shortparallel}$

We'll think of bundles equipped with extra structures, based on some Lie group G ↙ symmetries (with Lie algebra $\mathfrak{g}$) $\rightsquigarrow$ "G-vector bundle". Ex:

-) no structure ($G = GL(n;\mathbb{C})$ or $GL(n;\mathbb{R})$).
-) E Hermitian, structure $\langle\,,\rangle$. Cor unitary
  $G = U(n) = \{A \mid A^*A = \mathrm{Id}_n\}$ unitary group
  $\mathfrak{g} = \mathfrak{u}(n) = \{$skew-Hermitian matrices $B^* = -B\}$.
-) E special unitary, structure $\langle\,,\rangle$ + $\det E \cong \mathbb{C}$ trivialization. $G = SU(n)$ ↙ $\det = 1$ in $U(n)$

$su(n) = \{$traceless skew-Hermitian$\}$.

•) oriented euclidean (real), $G = SO(n)$

special orthogonal group, $so(n) = \{$antisymmetric$\}$.

We will consider connections A compatible with the extra structure. E.g. for $\langle\,,\,\rangle$, want

$$d\langle s, s'\rangle = \langle d_A s, s'\rangle + \langle s, d_A s'\rangle \quad \forall s, s' \in \Omega^0(E),$$

These form a smaller space, affine / $\Omega^1(\mathfrak{g}_E)$

<u>Ex</u> $E \to M$ unitary,

$\{$unitary connections on $E\}$ affine / $\Omega^1(U(E))$

where $U(E) = \left\{ \begin{array}{c} E \xrightarrow{\ell} E \\ \searrow_M \swarrow \end{array} \;\middle|\; \text{fibrewise skew-hermitian} \right\}$.

<u>Rmk</u> In general, one uses the language of principal G-bundles & associated vector bundles.

<u>Locally</u>, E is trivial so

$E \overset{\tau}{\cong} U \times \mathbb{C}^n \quad (U \subseteq \mathbb{R}^n)$ so we can write

$\nabla_A = d + A^\tau, \quad A^\tau \in \Omega^1(\mathfrak{g})$ (= matrix of 1-forms)

If $\{x_i\}$ are coordinates in U, we write

$\nabla_A = \sum \nabla_i \, dx_i$ with

$\nabla_i = \frac{\partial}{\partial x_i} + A_i^\tau, \quad A_i^\tau$ function with values in $\mathfrak{g}$.