Lecture 4 Seiberg-Witten invariants

$\mathfrak{s}$ spin^c structure on M^4 $(b_1 = 0)$

A spin^c connection Φ spinor $\in C^\infty(S^+)$

$$\underbrace{\begin{cases} D_A^+ \Phi = 0 \\ \frac{1}{2}\rho(F_A^+) - (\Phi\Phi^*)_0 = 0 \end{cases}}$$

$$\mathcal{F}(A, \Phi) = 0.$$

$$\| \mathcal{F}(A, \Phi) \|_{L^2}^2 \geq 0$$

$$\shortparallel$$

$$\frac{1}{4}\int_X |F_A^+|^2 + \int_X |D_A \Phi|^2 + \frac{1}{4}\int_X \left(|\Phi|^2 + \frac{s}{2}\right)^2$$

$$-\int_X \frac{s^2}{16} - \frac{1}{4}\underbrace{\int F_{A^t}\wedge F_{A^t}}_{\hat{=}\, c_1^2(s)}$$

by Chern-Weil

looks functionals arising in superconductivity, Higgs mechanism.

<u>Witten</u> - From superconductivity & 4-manifolds to weak interactions.

<u>Today's goal</u> $(X, s) \rightsquigarrow SW(X,s)\in\mathbb{Z}$.

"counts" solutions to SW eqns.

we say
$\mathcal{M}(X,\mathfrak{s})$ is compact.

<u>Recall</u> (A, Φ) is reducible if $\Phi = 0$.

$$
\begin{array}{ccc}
B(X,s) = \mathcal{C}(X,s)/\mathcal{G} & \supseteq & B^*(X,s) \\
\cup\!| & & \uparrow \\
\mathcal{M}(X,s) & & \text{irreducible configurations.}
\end{array}
$$

<u>Thm</u> (The Sobolev completion of) $B^*(X,s)$ is a Hilbert manifold, homotopy equivalent to $\mathbb{CP}^\infty$

$$\left(H^*(B^*(X,s)) \cong \mathbb{Z}[\overset{\text{deg } 2}{U}] \right.$$

<u>Proof</u> Fix A_0 base. Then (A, Φ)

can be put in Coulomb gauge

$(d^*(A^t - A_0^t) = 0)$ by $u = e^{\xi}$.

$(u \cdot A = A - u^{-1} du)$.

But if u works, then $u \cdot \underbrace{e^{i\phi}}_{\text{constant}}$

works too!

$$\mathcal{G} = \{u : M \to S^1\}$$

$$\cup$$

$$\mathcal{G}^0 = \{u \mid u(\underset{\text{basepoint.}}{y_0}) = 1\}$$

$\mathcal{G}/\mathcal{G}^0 = S^1$ constant gauge transf.

$$\mathcal{B}^* = \mathcal{C}^* \Big/ \mathcal{G} = $$

$= \{A, \Phi) \mid \Phi \neq 0\}$

$$= \left(\mathcal{C}^* / \mathcal{G}^0\right) \Big/ \left(\mathcal{G} / \mathcal{G}^0\right) = \left(\mathcal{C}^* / \mathcal{G}^0\right) \Big/ S^1.$$

The coulomb gauge identifies

$$\mathcal{C}^* / \mathcal{G}^0 \cong \{(A, \Phi) \mid d^*(A^t - A_0^t) = 0$$
$$\Phi \neq 0\}.$$

$(\forall (A, \Phi), \exists!\, u \in \mathcal{G}^0$ s.t. $u \cdot A$ is in Coulomb gauge$)$.

$$\mathcal{C}^* / \mathcal{G}^0 \cong \ker d^* \times \left(\Gamma(S^+) \setminus \{0\}\right)$$

$\mathbb{A}$-dim affine space $\mathbb{A}$-dim affine space $\setminus$ pt.

$\Rightarrow \mathcal{C}^*/\mathcal{G}^0$ is contractible!

$\circlearrowright_{S^1}$ is free (configs are irreducible).

$$\Rightarrow \mathcal{B}^* = \left(\mathcal{C}^*/\mathcal{G}^0\right)\Big/ S^1 \simeq BS^1 \simeq \mathbb{C}P^{\infty}.$$

$\square$

$\mathcal{B} \supset \mathcal{B}^* \simeq \mathbb{C}P^{\infty}$

$\cup$

$\mathcal{M}(X,\mathfrak{s}) \circlearrowleft S^1$ not free!

because of reducible solutions!

$\Rightarrow$ hard to get a well defined count!

<u>Ex</u> $f : \mathbb{C} \to \mathbb{R}$

$$f(z) = |z|^4 + \varepsilon |z|^2$$

$$f^{-1}(0) = \begin{cases} \bullet & \text{if } \varepsilon > 0 \\ \odot & \text{if } \varepsilon < 0 \end{cases}$$

$\Rightarrow$ reducibles are <u>bad</u>!

<u>Reducibles</u>

$$\begin{cases} D_A^+ \Phi = 0 & \in \mathcal{C}^\infty(S^+) \\ \frac{1}{2}\rho(F_A^+) = (\Phi\Phi^*)_0 & \in \mathcal{C}^\infty(isu(S^+)), \end{cases}$$

$(A, \Phi) = (A, 0)$ solves

$\Leftrightarrow \quad F_{A^t}^+ = 0$ abelian ASD equation.

$(A = A_0 + a$

has solutions in general!

(Hodge theory.

$\left(F_{A^t}^+ = \underset{\overset{\shortparallel}{0}}{F_{A_0^t}^+} + 2da^+ \right)$

The key idea is to perturb the equations! by $\omega^+ \in i\Omega^+$.

$$\begin{cases} D_A^+ \Phi = 0 \\ \frac{1}{2}\rho(F_{A^t} - 4\omega^+) = (\Phi\Phi^*)_0 \end{cases}$$

$\rightsquigarrow \mathcal{M}_{\omega^+}(X, s)$ perturbed moduli space.

Thm If $b_2^+ \geq 1$, for a generic choice of ω^+, $\mathcal{M}_{\omega^+}(X,s)$ does not contain reducibles, i.e $\mathcal{M}_{\omega^+}(X,s) \subseteq \mathcal{B}^*(X,s) \simeq \mathbb{CP}^\infty$.

$$Q_X : H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z}) \to \mathbb{Z}.$$

$$\begin{array}{ccc} Q_X : H^2(X;\mathbb{R}) & \otimes & H^2(X;\mathbb{R}) \to \mathbb{R} \\ \Psi & & \Psi \\ [\alpha] & & [\beta] \end{array}$$

$$Q_X([\alpha],[\beta]) = \int_X \alpha \wedge \beta$$

de Rham interpretation.

$H^2(X;\mathbb{R}) \cong \mathcal{H}^2$ harmonic 2-forms.

$\circlearrowleft_{*}$

$\mathcal{H}^2 = \mathcal{H}^+ \oplus \mathcal{H}^-$

$\hookrightarrow$ self-dual harmonic forms.

$\left(\begin{array}{c} (d+d^*)(k) = 0 \\ *k = k \end{array} \right).$

$k \in \mathcal{H}^+.$

$$Q_X(k,k) = \int_X k \wedge k = \int k \wedge *k$$

$$= \int |k|^2 d\mathrm{vol} = \|k\|^2_{L^2} > 0 \quad \text{if } k \neq 0.$$

$Q_X|_{\mathcal{H}^\pm}$ is $\pm$ definite

$\underline{\text{Cor}}$ $b^+(M) = \dim \mathcal{H}^+$

$\underline{\text{Proof}}$

$$\begin{cases} D_A^+ \Phi = 0 \\ \frac{1}{2}\rho(F_{A^t}^+ - 4\omega^+) = (\Phi\Phi^*)_0 \end{cases}$$

reducibles $(A, 0)$ solutions are solutions to

$$\boxed{F_{A^t}^+ = 4\omega^+}$$

Pick A that solves this.

$\forall u \in \mathcal{H}^+$ $\left(\begin{array}{c} (d+d^*)u = 0 \\ *u = u \end{array} \right)$

$$4\int_X \omega^+ \wedge k = \int_X F^+_{A^t} \wedge k =$$

$$= \int F^+_{A^t} \wedge k + \underbrace{\int \overset{\Omega^-}{\overset{\in}{F^-_{A^t}}} \wedge \overset{\Omega^+}{\overset{\in}{k}}}_{=\,0} =$$

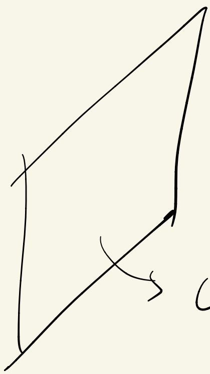
$$= \int F_{A^t} \wedge k \underset{\text{Chern-Weil}}{=} \frac{2\pi}{i} c_1(S^+) \cup [k]$$

$$\Rightarrow \int \omega^+ \wedge k = \frac{\pi}{2i} c_1(S^+) \cup [k], \quad \forall k \in \mathcal{H}^+.$$

linear system of eqns.

$\Rightarrow \omega^+$ lies in a codimension

$\dim \mathcal{H}^+ = b^+$ subspace of $i\Omega^+$.



$b^+ \geq 1$

ω^+ needs to lie in a proper subspace of $i\Omega^+$.

<u>Thm</u> If $b^+ \geq 1$, for a generic choice of $\omega^+ \in i\Omega^+$,

$$\mathcal{M}_\omega(X,s) \subseteq B^*(X,s)$$

is a compact smooth mfld of dimension

$$d(S) = \frac{c_1^2(S^+) - 2\chi(M) - 3\sigma(M)}{4}.$$

<u>Idea</u> $f : X^m \to \mathbb{R}^n$

we say 0 is <u>regular value</u> if

$\forall p \in f^{-1}(0)$, $df : T_p X \to T_0\mathbb{R}^n = \mathbb{R}^n$

is surjective.

<u>Fact 1</u> if 0 is a regular value,

$f^{-1}(0)$ is a smooth mfld (Inverse function thm)

of $\dim = m - n \overset{\text{rank-nullity}}{=} \dim \ker df - \dim \operatorname{coker} df$

$= \operatorname{ind}(df)$.

<u>Fact 2</u> we can "wiggle" f to have a regular value at 0. (Sard's thm)

This holds for us too!

<u>Fact 1</u> we think as SW + gauge fixing as our f.

$$\dim M_{\omega^+}(X,s) = \mathrm{ind}(\text{linearization})$$

$$= \underbrace{2\,\mathrm{ind}_{\mathbb{C}}(D_A^+)} + \underbrace{\mathrm{ind}(d^+ + d^*)}_{\text{Pset.}}$$

$$= \frac{c^2 - \sigma(X)}{4} + b_1 - (1 + b^+)$$

$$= \cdots = d(s)$$

<u>Fact 2</u> key input : <u>unique continuation property of $\not{D}_A^+$</u>.

$$\not{D}_A^+ \Phi = 0, \quad \text{and} \quad \Phi \equiv 0 \text{ on } \overset{\text{open}}{U} \subseteq M$$

$\Rightarrow \Phi \equiv 0$ everywhere !

<u>Thm</u> Under the assumptions above,

$\mathcal{M}_{\omega^+}(X, s)$ is <u>orientable</u>.

An orientation of $\mathcal{H}^+$ induces naturally an orientation on $\mathcal{M}_{\omega^+}(X, s)$.

To conclude, we have

$$\mathcal{M}_{\omega^+}(X, s) \subseteq \mathcal{B}^*(X, s) \simeq \mathbb{C}P^\infty$$

smooth, oriented.

S^4

$H^* = \mathbb{Z}[U]$.

Def If $b_2^+ \geq 1$, ω^+ generic, $\dim \mathcal{M}_\omega = d(s) = 2n \geq 0$

$$SW_{(g,\omega^+)}(X, s) = \langle U^n, [\mathcal{M}_\omega(X, s)] \rangle$$

otherwise, $SW_{(g,\omega^+)}(X, s) := 0$.

Rmk if $d(s) = 0$

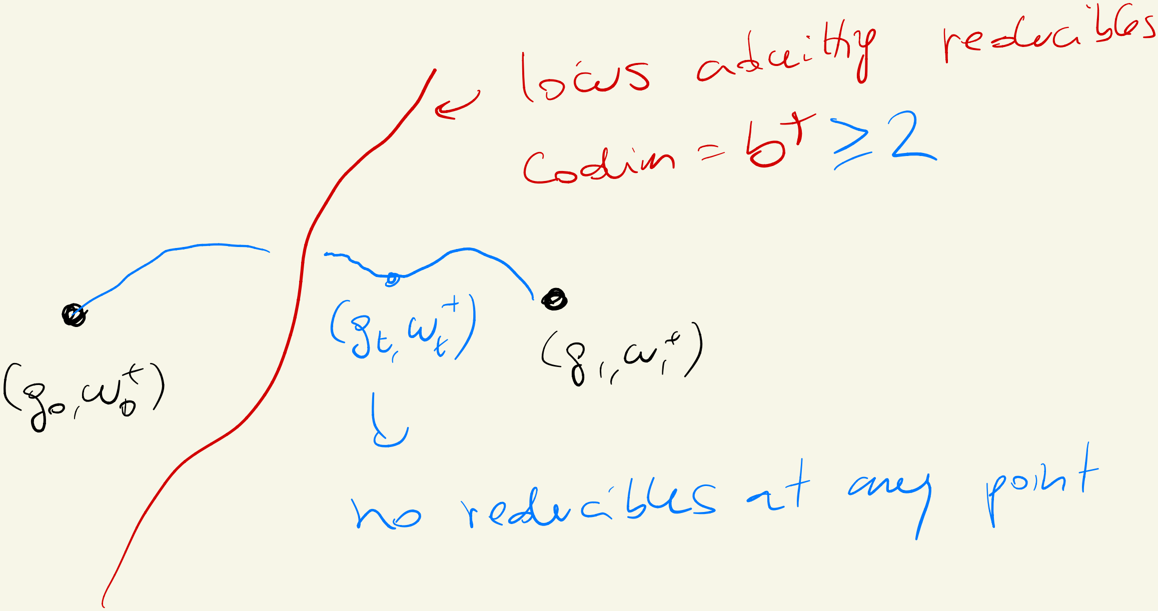
$$\Rightarrow SW_{(g,\omega^+)}(X, s) = \# \mathcal{M}_{\omega^+}(X, s).$$

Thm If $b_2^+ \geq 2$, $SW_{(g,\omega^+)}(X,s)$ is independent of the choice of (g,ω^+).

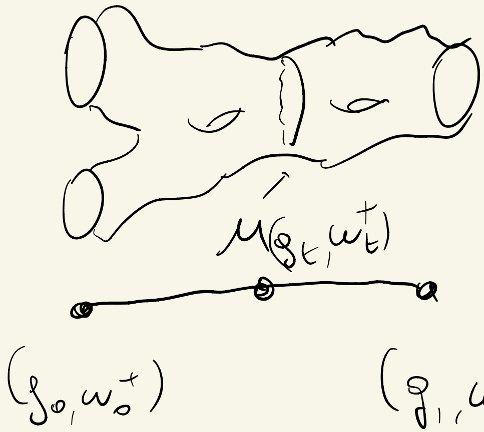
Def $SW(X,s) \in \mathbb{Z}$.

Idea of proof

$(g_i,\omega_i^+) \rightsquigarrow M_{(g_i,\omega_i^+)}(X,s)$.



$$\Rightarrow \bigcup_{t\in[0,1]} \mathcal{M}_{(g_t,\omega_t^+)}(X,s) \to \text{orientable cobordism between } \mathcal{M}_{(g_0,\omega_0^+)} \text{ \& } \mathcal{M}_{(g_1,\omega_1^+)}.$$



$$\Rightarrow [\mathcal{M}_{(g_0,\omega_0^+)}] = [\mathcal{M}_{(g_1,\omega_1^+)}]$$

$$\in H_*(\mathbb{CP}^\infty).$$

Rmk 1 We'll see $b_2^+ = 1$ case next time.

Rmk 2 If (X, g) PSC $(s > 0)$

$\Rightarrow$ no irreducible solutions for small perturbations ω^+

$\Rightarrow \mathcal{M}_{(g,\omega^+)}(X, s) = \phi$

$\Rightarrow SW(X, s) \equiv 0$.

Simple type conjecture

If $b_2^+ \geq 2$, $b_1 = 0$, $d(s) > 0$

$\Rightarrow SW(X, s) = 0$.