Lecture 2 Dirac operators.

Last time (M,g)

$$\rightsquigarrow \Delta = (d+d^*)^2 \text{ on } \Omega^*(M)$$

Hodge Laplacian.

locally: $$\Delta = -\frac{\partial^2}{\partial x_1^2} \cdots - \frac{\partial^2}{\partial x_n^2} + \text{lower order.}$$

(vector valued)

Rmk If $\underset{\text{hermitian}}{E} \to M$, $\underset{\text{compatible}}{\nabla} : C^\infty(E) \longrightarrow C^\infty(T^*M \otimes E)$

∇^* "$-\mathrm{div}$"

$$\langle \nabla s, t \rangle_{L^2} = \langle s, \nabla^* t \rangle_{L^2}$$

$\Rightarrow \nabla^* \nabla : \mathcal{C}^\infty(E)^{\circlearrowleft}$

or

Bochner (rough) Laplacian

$$= -\frac{\partial^2}{\partial x_1^2} - \cdots \frac{\partial^2}{\partial x_n^2} + \text{lower order terms.}$$

$(\nabla^* \nabla \neq \Delta \quad \text{for } E = \Lambda^* T^* M)$

in general!

$\Delta = (d + d^*)^2$ is nice because

it's the square of a first order op!

$\hookrightarrow$ Δ has a square root!

Dirac operators: square roots of

"Laplacians"

$\hookrightarrow$ Dirac 1928 - The quantum theory of the electron.

In $\mathbb{R}^n$, let's look at

$$\Delta = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}.$$

usual Laplacian.

Let's look for $D = a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n}$

s.t. $D^2 = \Delta$.

$$a_1^2 \frac{\partial^2}{\partial x_1^2} + \cdots \quad \underline{(a_1 a_2 + a_2 a_1)} \frac{\partial^2}{\partial x_1 \partial x_2} \cdots$$

$$\Rightarrow \begin{cases} a_i^2 = -1 \\ a_i a_j + a_j a_i = 0 \quad i \neq j. \end{cases}$$

$\boxed{n=1}$ we can pick $a_1 = i$ $(\in \mathbb{C})$.

$$\Rightarrow D = i \frac{\partial}{\partial x} \qquad \left(D^2 = -\frac{\partial^2}{\partial x^2} \right).$$

$\boxed{n=2}$ Pick $a_1 = i$ $a_2 = j$ $(\in \mathbb{H})$

$(ij = -ji = k)$.

$$\Rightarrow D = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y}$$

(on $\mathbb{H}$-valued functions)

$$D^2 = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}.$$

In general: Clifford algebras

inner product > 0.

Def (V, q) Euclidean space

$$Cl(V,q) = T(V) \Big/ \{ v \otimes v = -q(v,v) \}$$

$T(V) \to \bigoplus V^{\otimes n}$

implies

$$\left(\Rightarrow v \otimes w + w \otimes v = -2q(v,w) \right).$$

Properties

- $Cl(V,q)$ an associative algebra

$$Cl(V,q) \supset V, \qquad v^2 = -q(v,v)$$

$$\forall\, v \in V.$$

- $e_1, \ldots, e_n$ ON basis of V

$$\Rightarrow \begin{cases} e_i^2 = -1 \\ e_i \cdot e_j + e_j \cdot e_i = 0 \qquad i \neq j \end{cases}$$

- $Cl(V, q)$ has basis

$$\{e_{i_1} e_{i_2} \cdots e_{i_k}\} \text{ where } 1 \leq i_1 < i_2 < \cdots < i_k$$

$$\underset{e_I}{\shortparallel} \qquad\qquad i_k \leq \dim V$$

$$\Rightarrow \dim Cl(V, q) = 2^{\dim V}$$

- $T(V)$ has filtration

$$\mathbb{R} \subseteq \mathbb{R} \oplus V \subseteq \mathbb{R} \oplus V \oplus (V \otimes V) \subseteq \ldots$$

⇒ get filtration, $\mathcal{F}$ on $Cl(V,q)$ algebra.

(how many el of V you need).

$$Gr_{\mathcal{F}}\, Cl(V,q) \cong \Lambda^* V.$$

$v \otimes w + w \otimes v = \cancel{2q(v,w)}$ filtration kills this.

$$\Rightarrow Cl(V,q) \longrightarrow Gr_{\mathcal{F}}\, Cl(V,q) \cong \Lambda^* V$$

natural vector space iso

$$e_1 \wedge \dots \wedge e_k \mapsto e_1 e_2 \dots e_k$$

We can think of $Cl(V,q)$ as

$\Lambda^* V$ with a multiplication

$$\underset{\substack{\cup \\ V}}{v} \cdot \underset{\substack{\cup \\ \Lambda^* V}}{\alpha} = v \wedge \alpha + v \lrcorner \alpha \; \big(.$$

- $Cl(V, q)$ is $\mathbb{Z}/2$-graded!

↳ $\underset{\to 2}{v \otimes v} = \underset{\to 0}{-q(v, v)} \to$ homogeneous mod 2!

$\to v \cdot$ changes parity.

<u>Ex</u> $n = 2$ $\quad Cl(\underset{\dim 2}{V}, q) \cong \mathbb{H}$.

$n = 4 \quad Cl(\overset{\dim 4}{V}, q) \cong M(2; \mathbb{H})$.

2^{2n}

$(2^n)^2 = 2^{2n}$

<u>Thm</u> $\dim V = 2n$

$$\Rightarrow Cl(V,q) \otimes_{\mathbb{R}} \mathbb{C} \cong M(2^n, \mathbb{C})$$

<u>Def</u> A <u>Clifford module</u> S over $Cl(V,q)$ is a module s.t. $v \in V$ acts as skew adjoint map.

Concretely, $\exists\ V \otimes S \xrightarrow{\rho} S$ s.t.

•) $v \cdot (v \cdot s) = -q(v,v) \cdot s$.

•) $\langle v \cdot s_1, s_2 \rangle = -\langle s_1, v \cdot s_2 \rangle$.

$(v \cdot s = \rho(v) s\,.)$

Ex 1 $Cl(V,g) \curvearrowright \Lambda^* V$

(it's the ass. graded).

Ex 2 $Cl(\mathbb{R}^2) = \mathbb{H} \curvearrowright \mathbb{H} = \mathbb{C}^2$.

$$e_1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Ex 3 $Cl(\mathbb{R}^4) = M(2;\mathbb{H}) \curvearrowright \mathbb{H}^2 = \mathbb{C}^4$.

explicitly: Pauli matrices

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

e_0, e_1, e_2, e_3 ON of $\mathbb{R}^4$

$$e_0 \mapsto \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \qquad e_i \mapsto \begin{pmatrix} 0 & -\sigma_i^* \\ \sigma_i & 0 \end{pmatrix}.$$

$$\in M(4;\mathbb{C}).$$

<u>Spinor representation of $Cl(\mathbb{R}^4)$</u>

<u>Globally</u> (M,g) Riemannian mfld.

$\rightsquigarrow$ $Cl(TM,g) \to M$ bundle

of Clifford algebras.

<u>Def</u> A <u>Clifford bundle</u> is a

Clifford.

hermitian bundle $S \longrightarrow M$ of modules

over $Cl(TM, g)$

$\left(Cl(T_pM, g_p) \circlearrowright S_p \right)$.

that admits a hermitian connection ∇^S

which is compatible, i.e

$\forall s \in \mathcal{C}^\infty(S)$, X, Y vector fields.

Levi-Civita

$$\nabla^S_X(\underbrace{Y \cdot s}_{\rho(Y)s}) = (\nabla_X Y) \cdot s + Y \cdot \nabla^S_X s.$$

Rmk ∇^S is not unique!

Ex $\Lambda^* T^* M \otimes \mathbb{C}$ is a Clifford bundle.

<u>Def</u> S Clifford bundle, ∇^S compatible connection. The associated Dirac op D is

$$\begin{tikzcd}
C^\infty(S) \arrow[r, "\nabla^S"] \arrow[drr, "D"'] & C^\infty(T^*M \otimes S) \arrow[r, "\#"] & C^\infty(TM \otimes S) \arrow[d, "\rho"] \\
& & C^\infty(S)
\end{tikzcd}$$

If $e_1, \dots e_n$ is ON frame

$$Ds = \sum e_i \bullet \nabla_{e_i} s.$$

__Ex__ $\Lambda^* T^*M \otimes \mathbb{C}$ has associated Dirac operator $d + d^*$.

__Fact__ $(S, D^S) \rightsquigarrow D : C^\infty(S) \circlearrowleft$ is always a first order, elliptic self-adjoint.

__Def__ A __spin^c structure__ on M^{2n} is a Clifford bundle S which is pointwise isomorphic to the spinor rep of $\mathbb{C}l(\mathbb{R}^{2n})$.

$\underline{Ex}$ $2n=2$ $\quad Cl(\mathbb{R}^2)=\mathbb{H} \curvearrowright \underset{\sim}{\mathbb{C}^2}$

$$e_1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad e_2 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

$S \to M^2$ is $\mathrm{rank}_{\mathbb{C}}=2$

In flat $\mathbb{R}^2$

$$D = e_1 \nabla_{e_1} + e_2 \nabla_{e_2} = \begin{bmatrix} 0 & -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} & 0 \end{bmatrix}$$

(acting on $\mathbb{C}^2$-valued functions).

$\underline{Ex}$ $n=4$ $\quad S \to M^4$ $\quad \mathrm{rank}_{\mathbb{C}}=4$

$(Cl(\mathbb{R}^4) \curvearrowright \mathbb{C}^4)$

$\hookrightarrow \ldots \quad D = e_1 D_{e_1} + e_2 D_{e_2} + e_3 D_{e_3} + e_4 D_{e_4}.$

$$= \begin{bmatrix} 0 & \boxed{2\times 2} \\ \boxed{2\times 2} & 0 \end{bmatrix}$$

<u>Fact</u> (Pset)

$S \to M^4$ naturally splits as

$$S = S^+ \oplus S^- \qquad \mathrm{rk}_{\mathbb{C}} S^{\pm} = 2.$$

$\rho(v)$ exchanges $S^{\pm}$.

D decomposes as the chiral parts

$$\begin{tikzcd} e^{\infty}(S^+) \arrow[r, bend left, "D^+"] & e^{\infty}(S^-) \arrow[l, bend left, "D^-"] \end{tikzcd}$$

D self adjoint $\Rightarrow$ D^+ and D^-

are adjoint to each other.

Thm (Atiyah - Singer)

$$\mathrm{ind}_{\mathbb{C}} D^+ = \frac{\langle c_1(S^+)^2, [M] \rangle - 6(M)}{8}.$$

Thm 1 Any orientable M^4 admits a spin^c structure.

(not trivial at all! obstruction theory)

Thm 2 If $S \longrightarrow M^4$ is spin^c

structure, L hermitian line bundle

$\Rightarrow S \otimes L$ is also a spin^c structure

$(\rho \otimes \mathbb{1}_L)$.

This realizes $\{$spin^c structures on $M\}$

as affine space over

$\{\mathbb{C}$-line bundles on $M\}$

$$\updownarrow \; 1:1$$

$$H^2(X;\mathbb{Z})$$