AN INTRODUCTION TO SEIBERG-WITTEN THEORY

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Lecture 1 Background on PDEs

(elliptic PDEs).

$E^n, F^\ell \to M^n$ bundles ($/\mathbb{R}$ or $\mathbb{C}$).

A differential operator (of order r, C^∞ coeff) is linear map

$$L : C^\infty(E) \to C^\infty(F)$$

E, F trivialized $\downarrow$

that looks like in a chart U

$$\mathcal{C}^\infty(U;\mathbb{R}^m) \longrightarrow \mathcal{C}^\infty(U;\mathbb{R}^\ell)$$

$$L(x) = \sum_{|\alpha|\le r} a_\alpha(x) D^\alpha \qquad x \in U$$

- $\alpha = (\alpha_1, \dots \alpha_n)$ multiindex,

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$$

- $\forall x \in U \quad a_\alpha(x) : \mathbb{R}^m \to \mathbb{R}^\ell$ linear.

(smooth function of x).

<u>Def</u> The <u>principal symbol</u> of L at x

$$P_{L,x}(\underset{\Psi}{\xi}) : \mathbb{R}^m \to \mathbb{R}^\ell$$

$\xi = (\xi_1, \ldots \xi_n)$

$$P_{L,x}(\xi) = \sum_{|\alpha| = r} a_\alpha(x) \xi^\alpha : \mathbb{R}^k \to \mathbb{R}^\ell .$$

top degree $\quad \left(\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}\right).$

<u>**Def**</u> L is <u>elliptic</u> if $\forall x \in U$,

$\xi \in \mathbb{R}^n \setminus \{0\}$,

$P_{L,x}(\xi) : \mathbb{R}^k \to \mathbb{R}^\ell$ is an <u>isomorphism</u>.

<u>Examples</u>

1) $\Delta = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2} : C^\infty(\mathbb{R}^n) \circlearrowleft$

($E = F =$ trivial $\mathbb{R}$-bundle). Laplacian.

$$\xi = (\xi_1, \dots \xi_n)$$

$$\Rightarrow P_{\Delta, x}(\xi) = -\xi_1^2 \dots - \xi_n^2$$

$$= -|\xi|^2 \qquad [-|\xi|^2]$$

(thought of as the multiplication

$\mathbb{R} \to \mathbb{R}$.)

Invertible if $\xi \neq 0$!

$\Rightarrow \Delta$ is elliptic.

2) $\Box = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} : C^\infty(\mathbb{R}^2)^{\circlearrowleft}$

D'Alembertian (wave eq).

$$P_{\Box, x}(\xi) = \xi_1^2 - \xi_2^2 : \mathbb{R} \to \mathbb{R}.$$

not iso for all $\xi \neq 0$!

$$P_{D,x}((1,1)) = 0 : \mathbb{R} \to \mathbb{R}.$$

$\Rightarrow D$ is not elliptic!

trivial $\mathbb{C}$-bundle

$$\underline{\mathbb{C}} \longrightarrow \mathbb{C}$$

3) $\bar{\partial} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$:

$$e^\infty(\underline{\mathbb{C}}) \longrightarrow e^\infty(\underline{\mathbb{C}}).$$

$$P_{\bar{\partial},x}(\xi) = \xi_1 + i\xi_2 : \mathbb{C} \longrightarrow \mathbb{C}$$

multiplication.

iso for $\xi \neq 0$!

Rmk $L : C^\infty(E) \to C^\infty(F)$ elliptic

$\Rightarrow \dim E = \dim F$

$\Rightarrow$ we'll assume this.

Intrinsic definition

$L : C^\infty(E) \to C^\infty(F)$. order r.

Then L is elliptic

$\Updownarrow$

$\forall x \in M$, $u \in C^\infty(E)$ s.t $u(x) \neq 0$.

and $\varphi \in C^\infty(M; \mathbb{R})$ with

$\varphi(x) = 0$, $d\varphi(x) = \xi \in T_x^* M$.
$\neq 0$.

$\Rightarrow \varphi \cdot \varphi \cdots \varphi$ r times.

$$L(\varphi^{r} u)(x) \neq 0.$$

$$\| \qquad \text{CORRECTION}$$

$$P_{L,x}(\xi)(u(x)) \in F_x. \qquad \text{UP TO CONSTANTS.}$$

$$\cup \qquad n!$$

$$E_x$$

Key point L elliptic operator,

solutions to $Lu=0$ are nice

Ex $u \in C^2$, $\Delta u = 0$ (i.e harmonic)

$\Rightarrow u \in C^\infty$.

Not true for $\Box$!

Right functional setup: Sobolev spaces.

Hermitian bundle $\qquad$ **M compact**

$E \to M$ Riemannian, ∇ compatible connection on E,

$k \in \mathbb{N}$. We define the L^2_k-norm

of $u \in C^\infty(E)$ as

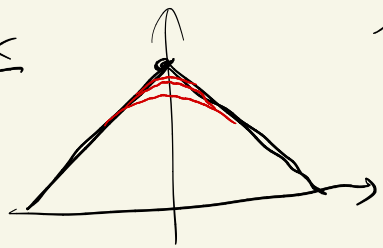
$$\|u\|^2_{L^2_k} := \int_M \left(|u|^2 + |\nabla u|^2 + \dots + |\nabla^k u|^2 \right) d\mathrm{vol}$$

<u>Def</u> <u>Sobolev space</u> $L^2_k(E)$

$=$ completion of $C^\infty(E)$ wrt $\|\cdot\|_{L^2_k}$.

$\to$ it's a Hilbert space.

Ex



$f:\mathbb{R}\to\mathbb{R}$.

$f\in L^2_1(\mathbb{R};\mathbb{R})$.

Sobolev embedding thm if $d > \frac{\dim M}{2}$

$\Rightarrow L^2_{k+d} \hookrightarrow C^k$

$\hookrightarrow \sup\{|u|, |Du| \dots |D^k u|\}$.

In particular $L^2_d \hookrightarrow C^0$ if $d > \frac{\dim M}{2}$.

Rmk $f \in L^2_1(\mathbb{R}^2)$ need not to be continuous!

<u>Rellich thm</u> $k_1 < k_2$

$$L^2_{k_2} \hookrightarrow L^2_{k_1} \text{ is } \underline{\underline{\text{compact}}}$$

(i.e bounded sets maps to a precompact set).

("analogue" of Ascoli-Arzelà's thm in Sobolev).

<u>Observation</u> M cpt,

$L : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(F)$ order r.

$\Rightarrow \exists\, C_k > 0$ s.t

$$\|L\varphi\|_{L^2_k} \leq C_k \|\varphi\|_{L^2_{k+r}} \quad \forall \varphi$$

$L\varphi$ involves r derivatives of φ

$\Rightarrow L : L^2_{k+r}(E) \to L^2_k(F)$ bounded.

Elliptic estimate assume L elliptic.

$\Rightarrow \exists\ D_k$ s.t.

$$\|\varphi\|_{L^2_{k+r}} \leq D_k \left(\|L\varphi\|_{L^2_k} + \|\varphi\|_{L^2} \right)$$

$\forall \varphi$.

Rmk This is not true for

C^k norms!!

Rmk if $L\varphi = 0$

$$\Rightarrow \|\varphi\|_{L^2_{k+r}} \leq D_k \|\varphi\|_{L^2}.$$

Key consequence M compact

L elliptic of order r.

$$L : L^2_{k+r}(E) \to L^2_k(F) \quad k \in \mathbb{Z}$$

- finite dimensional kernel ↙ consists of smooth sections.
- closed range & finite dimensional cokernel. $\cong \ker L^*$. ↙ adjoint
  ↳ elliptic operator

$$\langle Lu, v \rangle_{L^2} = \langle u, L^* v \rangle_{L^2}.$$

↳ i.e. L is Fredholm

Def $\operatorname{index} L = \dim \ker L - \dim \operatorname{coker} L$.
$\in \mathbb{Z}$.

↑

there's formulas to compute it.

<u>Rmk</u> $L^2_k \rightsquigarrow L^p_k$. works too!

<u>key example</u> Hodge operator.

$M \rightsquigarrow$ de Rham complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0.$$

has cohomology $H^i(M;\mathbb{R})$.

no preferred representative! :(

<u>Hodge star</u> $(V, \langle , \rangle, o)$

oriented euclidean $\mathbb{R}$-vector space

$\dim V = n.$

$\Rightarrow * : \Lambda^i V \to \Lambda^{n-i} V.$ Hodge star

s.t. Oriented ON basis $e_1, \dots e_n$.

$e_1 \wedge \dots \wedge e_i \mapsto e_{i+1} \wedge \dots \wedge e_n.$

<u>Notice</u> $a, b \in \Lambda^i V$

$$\underset{\substack{\cup \\ \Lambda^{n-i} V}}{a \wedge * b} = \langle a, b \rangle \underset{\substack{\cap \\ \Lambda^n V}}{dvol}.$$

$\Rightarrow \alpha \in \Omega^i(M) \ni * \alpha \in \Omega^{n-i}(M).$

$\beta \in$

$$\alpha \wedge * \beta = \langle \alpha, \beta \rangle dvol.$$

$$\Rightarrow \langle \alpha, \beta \rangle_{L^2} = \int_M \alpha \wedge * \beta .$$

$$\Omega^{p-1} \xrightarrow{d} \Omega^p$$

$$d^* := (-1)^{n(p+1)+1} * d * .$$

$$\Omega^p \xrightarrow{*} \Omega^{n-p} \xrightarrow{d} \Omega^{n-p+1} \xrightarrow{*} \Omega^{p-1} .$$

<u>Lemma</u> (Ret). d^* is the adjoint of d,

i.e $\langle d\gamma, \alpha \rangle_{L^2} = \langle \gamma, d^* \alpha \rangle$

$\forall \gamma \in \Omega^{p-1}, \ \alpha \in \Omega^p .$

Def $d+d^*: \Omega^*(M) \circlearrowleft$ Hodge operator,

order 1 diff operator!

Hodge Laplacian is

$$(d+d^*)^2 =: \Delta : \Omega^*(M) \circlearrowleft$$

$$\shortparallel$$

$$dd^* + d^*d \qquad (dd = d^*d^* = 0).$$

$\Delta : \Omega^i(M) \circlearrowleft$ order 2 diff op.

Rmk $\Delta: \Omega^0(M)$ is just

$-\mathrm{div}(\mathrm{grad} f)$, the usual

Riemannian Laplacian.

<u>Rmk</u> Δ is non-negative, $\ker\Delta = \ker(d+d^*)$.

$$\langle \alpha, \Delta\alpha \rangle_{L^2} = \langle \alpha, (d+d^*)^2 \alpha \rangle_{L^2}$$

$$= \| (d+d^*)\alpha \|^2_{L^2} \geq 0.$$

<u>Thm</u> (Hodge) There is L^2-orthogonal dec

$$\Omega^p(M) = \underbrace{d\Omega^{p-1}}_{\text{exact}} \oplus \underbrace{\mathcal{H}^p}_{\text{harmonic}} \oplus \underbrace{d^*\Omega^{p+1}}_{\text{coexact}}$$

$$\ker\Delta = \ker(d+d^*) \cong H^p(M;\mathbb{R}).$$

In particular, each class in H^p admits a unique harmonic rep.

<u>Key input</u> $d + d^* : \Omega^*(M) \circlearrowleft$ is elliptic! α k form, ξ 1-form

$$\xi \lrcorner \alpha = (-1)^{n(k+1)+1} * (\xi \wedge * \alpha).$$

(different sign convention!!).

$$P_{d+d^*,x}(\xi)(\alpha) = \xi \wedge \alpha + \xi \lrcorner \alpha$$

<u>Recall</u> $\operatorname{ind}(L) = \dim \ker - \dim \operatorname{coker}$.

$$d+d^* : \Omega^*(M) \longrightarrow \Omega^*(M)$$

is self-adjoint.

$\Downarrow$

$\Rightarrow \operatorname{ind}(d+d^*) = 0$.

<u>But</u> $d+d^* : \Omega^{even}(M) \to \Omega^{odd}(M)$.

$$\ker(d+d^*) = \bigoplus H^{even}$$

$$\operatorname{coker}(d+d^*) = \ker(d+d^* : \Omega^{odd} \to \Omega^{even})$$

$$= \bigoplus H^{odd}$$

$$\operatorname{ind}(d+d^* : \Omega^{even} \longrightarrow \Omega^{odd}) =$$

$$= \sum \dim H^{even} - \sum \dim H^{odd} = \chi(M).$$