REPRESENTATION THEORY AND CATEGORIFICATION SEMINAR

Fan Zhou
fz2326@columbia.edu

The color scheme for these pnotes is a green value of 0.420. This is a set of notes for the seminar run in Fall of 2023 by Cailan Li, Alvaro Martinez, and myself; the website for the seminar is here.

Contents

1. 09/07 – The nilBrauer Algebra and Stratification Theory (Fan Zhou)

1.1. **graded triangular bases and stratification theory.** Let $\mathsf{D}^{\mathsf{b}}\mathsf{Mod}\, A$ denote the bounded derived category, where A is an algebra which may not necessarily be unital. Indeed, our algebra will have infinitely many orthogonal idempotents, called the "local units". We stress from the onset the analogy to weight theory from Lie algebras (more precisely, idempotented enveloping algebras). One should roughly think of the local units as "weight operators", so that acting on a module by them picks out the weight space corresponding to that unit. Another strange feature of our algebra is that it is graded, and is infinite-dimensional in the sense that $\mathbb{C}[x]$ is; it is made manageable by the grading. Warning: this theory is typically stated in lowest weight notation rather than highest weight.

We now precisely state the setup. Let I index the local units (orthogonal and homogeneous) of A, $\mathfrak{h} \subseteq I$ label the "distinguished idempotents", Θ be a (lower-finite)[1] poset of "weights", and $\varpi \colon \mathfrak{h} \longrightarrow \Theta$ be a map (with finite fibers) from distinguished idempotents to the weight poset. Diagrammatically this is:

$$\begin{array}{ccc} I & & \\ \uparrow & & \\ \mathfrak{h} & \xrightarrow{\varpi} & \Theta \end{array}$$

An example of this setup (which we will not delve into) is the partition algebra, for which Θ is the poset of natural numbers, and $\mathfrak{h}$ is the set of partitions. We will let symbols like $i, j \in I$ and $\alpha, \beta \in \mathfrak{h}$ and $\theta, \phi, \psi \in \Theta$.

Let A have a "(graded) triangular basis" in the sense of Brundan, which is to say that

Definition 1.1. A is "graded triangular based" if there are sets $\mathrm{X}(i, \alpha) \subseteq 1^i A 1^\alpha$, $\mathrm{H}(\alpha, \beta) \subseteq 1^\alpha A 1^\beta$, $\mathrm{Y}(\beta, j) \subseteq 1^\beta A 1^j$ such that

(1) products of these things in these sets give a basis for A;
(2) $\mathrm{X}(\alpha, \alpha) = \mathrm{Y}(\alpha, \alpha) = \{1^\alpha\}$;
(3) for $\alpha \neq \beta$,

$$\begin{aligned} \mathrm{X}(\alpha, \beta) \neq \emptyset &\implies \varpi(\alpha) > \varpi(\beta), \\ \mathrm{H}(\alpha, \beta) \neq \emptyset &\implies \varpi(\alpha) = \varpi(\beta), \\ \mathrm{Y}(\alpha, \beta) \neq \emptyset &\implies \varpi(\alpha) < \varpi(\beta); \end{aligned}$$

(4) for each $i \in I - \mathfrak{h}$, there are only finitely many $\alpha \in \mathfrak{h}$ such that $\mathrm{X}(i, \alpha) \cup \mathrm{Y}(\alpha, i) \neq \emptyset$.

I do not fully understand the last axiom, but as I understand it you can ignore it most of the time.

Let $A^{\geq \theta}$ be defined as

$$A^{\geq \theta} := A \big/ \langle e^\phi : \phi \not\geq \theta \rangle,$$

and let e.g. $A^{\leq \theta}, A^{> \theta}$ be defined similarly. Let

$$e^\theta := \sum_{\alpha \in \varpi^{-1}\theta} 1^\alpha,$$

and let

$$A^\theta = e^\theta A^{\geq \theta} e^\theta.$$

This plays the role of Cartan in the sense that modules over it are induced to form Vermas, and in this analogy I suppose $A^{\geq \theta}$ is sort of like $\mathfrak{b}^-$ (this analogy is even looser because it depends on θ). Note that

$$A^{> \theta} = A^{\geq \theta} \big/ A^{\geq \theta} e^\theta A^{\geq \theta};$$

hence, by general theory, one expects a recollement $\mathsf{D\,Mod}\, A^{>\theta} \longrightarrow \mathsf{D\,Mod}\, A^{\geq \theta} \longrightarrow \mathsf{D\,Mod}\, A^\theta$. We write $M^\theta := e^\theta M$, in analogy with weight space notation.

Let Λ_θ label the simple modules $L_\lambda(\theta)$ of A^θ as well as their projective covers $P_\lambda(\theta)$ and injective hulls $Q_\lambda(\theta)$, and let $\Lambda = \bigsqcup_\theta \Lambda_\theta$ be all of them bunched together. Then one shows that Λ labels all simples of A. Frequently it is the case (for example if $A^\theta/\mathfrak{j} \cong \prod_{\alpha \in \varpi^{-1}(\theta)} \mathbb{k}$) that Λ can be identified with $\mathfrak{h}$; in this case we will instead use $\lambda, \mu \in \Lambda$ for the distinguished idempotents. Here's an artist's rendition:

$$\begin{array}{ccc} I & & \Lambda \\ \uparrow & \nearrow\!\!\!\cdots & \downarrow \\ \mathfrak{h} & \xrightarrow{\varpi} & \Theta \end{array}$$

Apparently frequently (and it is the case for nilBrauer) there is moreover a "split triangular decomposition", which is to say that there are A^-, A^0, A^+ locally unital[2] algebras such that

(1) $A^\flat = A^- A^0$ and $A^\sharp = A^0 A^+$ are subalgebras;

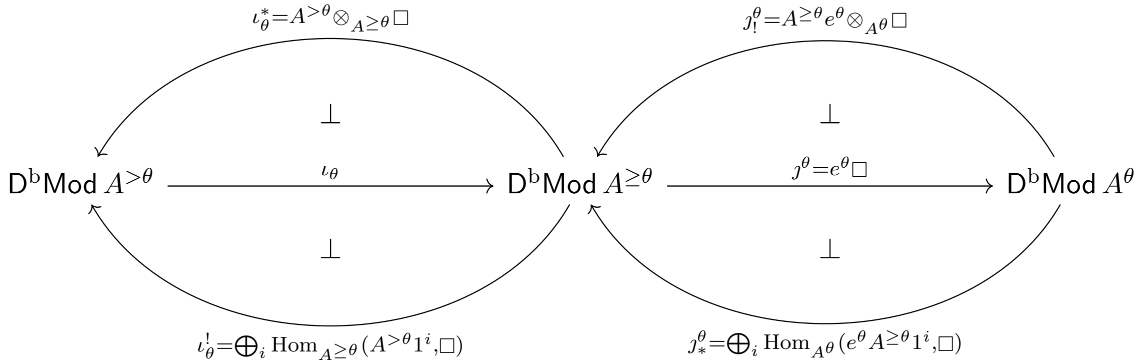
[1] perhaps this could be weakened to locally finite

[2] not sure about this condition since e.g. it seems you want A^- to not have 1^i

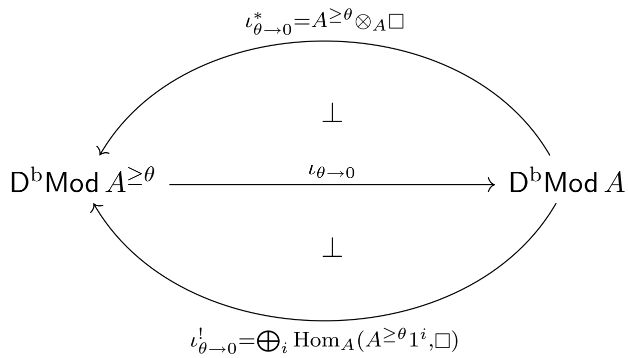
(2) the multiplication map $A^- \otimes_{\mathbb{K}} A^0 \otimes_{\mathbb{K}} A^+ \longrightarrow A$ is a linear isomorphism;
(3) $1^i A^\flat 1^i = 1^i A^\sharp 1^i = \Bbbk 1^i$ for each i and

$$1^i A^0 1^j \neq \emptyset \implies \varpi(i) = \varpi(j),$$
$$1^j A^- 1^i, 1^i A^+ 1^j \neq \emptyset \implies \varpi(i) > \varpi(j).$$

1.2. **recollement.** Recall the following recollement diagrams from Brundan's "Graded Triangular Bases".



and



Define the "big/small (co)Vermas" Δ and $\overline{\Delta}$ by:

$$\Delta_\lambda = \jmath_!^\theta P_\lambda(\theta), \qquad \overline{\Delta}_\lambda = \jmath_!^\theta L_\lambda(\theta),$$
$$\nabla_\lambda = \jmath_*^\theta Q_\lambda(\theta), \qquad \overline{\nabla}_\lambda = \jmath_*^\theta L_\lambda(\theta).$$

Let me write a couple of these functors in another slightly more familiar form.

$$\jmath^\theta = \operatorname{Hom}_A(A^{\geq\theta} e^\theta, \square)$$

is evident. Now let there be a duality coming from an antiautomorphism of A. On the level of A-modules this is denoted $\square^\dagger$, while on the level of A^θ-modules we will denote this as $\square^*$[3]. Then

$$\iota_{\theta\to 0}^* M = (\iota_{\theta\to 0}^! M^\dagger)^\dagger = \left(\bigoplus_i \operatorname{Hom}_A(A^{\geq\theta} 1^i, M^\dagger) \right)^\dagger,$$

where the left $A^{\geq\theta}$-action on the Hom comes from the right action on $\bigoplus_i A^{\geq\theta} 1^i$. Then

$$\jmath^\theta \iota_{\theta\to 0}^* M = e^\theta \left(\bigoplus_i \operatorname{Hom}_A(A^{\geq\theta} 1^i, M^\dagger) \right)^\dagger = \left(\bigoplus_i e^\theta \operatorname{Hom}_A(A^{\geq\theta} 1^i, M^\dagger) \right)^*$$
$$= \left(\bigoplus_i \operatorname{Hom}_A(A^{\geq\theta} 1^i e^\theta, M^\dagger) \right)^* = \operatorname{Hom}_A\left(\bigoplus_\alpha A^{\geq\theta} 1^\alpha, M^\dagger \right)^* = \operatorname{Hom}_A(A^{\geq\theta} e^\theta, M^\dagger)^*.$$

Note well that there is a left A^θ-action on this coming the from right $e^\theta A^{\geq\theta} e^\theta$-action on $A^{\geq\theta} e^\theta$; it is with respect to this action that the outermost dual is taken.

[3] It is probably worth saying that $M^\dagger := \bigoplus_{i\in I} \bigoplus_{k\in\mathbb{Z}} (1^k M_{-k})^*$.

Now suppose there is an identification of $\mathfrak{h}$ with Λ; this is for example the case with nilBrauer, where in fact $I = \mathfrak{h} = \Lambda = \Theta = \mathbb{N}$. The decomposition $\sum_{\lambda\in\theta} 1^\lambda$ of the unit e^θ of A^θ gives a decomposition into projectives

$$A^\theta e^\theta = \bigoplus_{\lambda\in\theta} P_\lambda(\theta)^{\oplus \overline{\dim L_\lambda(\theta)}},$$

where the bar is included for the graded case. For shorthand we will let $l_\lambda(\theta) = \dim L_\lambda(\theta)$. Then

$$A^{\geq\theta} e^\theta = \bigoplus_{\lambda\in\theta} \Delta_\lambda^{\overline{l_\lambda(\theta)}}.$$

Then, from the above discussion, one has (recall $\jmath_!^\theta$ is exact)

Lemma 1.2.

$$\jmath_!^\theta = \bigoplus_{\lambda\in\theta} \Delta_\lambda^{\overline{l_\lambda(\theta)}} \otimes_{A^\theta} \square$$

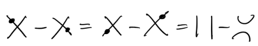
as well as

$$\jmath^\theta \iota_{\theta\to 0}^* = \mathsf{RHom}_A\left(\bigoplus_{\lambda\in\theta} \Delta_\lambda^{\overline{l_\lambda(\theta)}}, \square^\dagger\right)^*.$$

Again note well where the A^θ-action on this comes from – it comes from all the big Vermas bunched together, and if you took RHom from each individual big Verma then you lose this action (as far as I can tell).

1.3. **the nilBrauer algebra.** The nilBrauer algebra is a diagrammatic algebra, generated by vertical lines, caps, cups, crossings, and dots, subject to the following relations and degrees:

diagrams

Note the dependence on a parameter $t = 0, 1$ (that these must be the values can be seen by closing the relation  with a cap and a cup). This algebra has $I = \mathfrak{h} = \Lambda = \Theta = \mathbb{N}$, i.e. the local units are labelled by natural numbers. When one considers the quotient by units $\mathrm{N}\mathcal{B}^{\geq\theta}$, none of the relations are changed except the one involving dot-sliding, which becomes the relation for the nilHecke.

Consider the "Schur q-functions", defined by

$$\sum_n q_n z^{-n} = \left(\sum_n e_n z^{-n}\right)\left(\sum_n h_n z^{-n}\right),$$

which then satisfies $q(z)q(-z) = 1$, which translates to

$$q_{2k} = (-1)^{k-1}\frac{1}{2}q_k^2 + \sum_{i=1}^{k-1}(-1)^{i-1} q_k q_{2k-i}$$

for $k \geq 1$ and $q_0 = 1$. The subalgebra of the ring of symmetric functions generated by these q is denoted Γ; it is some fact that Γ is freely generated by q_{odd}, so for our purposes we will treat Γ as $\mathbb{C}[q_1, q_3, q_5, \cdots]$.

The relevance of this Γ is that nilBrauer receives an action of Γ. Indeed, the bubbles

diagram

behave as q_{2k+1}, and we can tack these diagrams to the right (or left, depending on your preference) of any nilBrauer diagram.

In particular, when one forms $\mathrm{N}\mathcal{B}^\theta$, one obtains the nilHecke on θ strands over the algebra Γ.

1.4. **ι-quantum group.** The so-called "ι-quantum group" we will be concerned with is the subalgebra of $\mathcal{U}_q(\mathfrak{sl}_2)$ generated by $B = F + qK^{-1}E$. This will have certain bases, the "PBW basis" Δ_n and the "canonical basis" P_n, and its $\mathbb{Q}(q)$-linear dual has bases dual to the previous two, $\overline{\Delta}_n, L_n$, in that order. Then there is the following 'character formula':

$$L_n = \sum_{k=0}^{\infty} (-1)^k \frac{q^{-k(2\delta_{n\not\equiv t}+1)}}{(1-q^{-4})(1-q^{-8})\cdots(1-q^{-4k})} \overline{\Delta}_{n+2k}.$$

It turns out this is literally a character formula: if you define the character of a module as

$$\chi_M := \sum (\dim M^\theta) e^\theta,$$

then the above formula is a character formula, much like the Weyl character formula. Notable features: it is an infinite sum; the small Vermas rather than big Vermas are involved; and the dependence on t is not too complicated.

1.5. **general machinery.** There is a general fact that in appropriate recollement situations, the identity functor on the big category ($\mathsf{D}^{\mathrm{b}}\mathsf{Mod}\, A$ in this case) admits a filtration (appropriately defined) whose graded pieces can be described in terms of the recollement functors:

$$\mathrm{gr}^\theta \mathcal{I}\mathrm{d} = \iota_{\theta\to 0} \jmath^\theta_! \jmath^\theta \iota^*_{\theta\to 0}.$$

From the above we then know this is the same as

$$= \bigoplus_{\lambda\in\theta} \Delta_\lambda^{\overline{l_\lambda(\theta)}} \otimes_{A^\theta} \mathsf{RHom}_A \left(\bigoplus_{\mu\in\theta} \Delta_\mu^{\overline{l_\mu(\theta)}}, \square^\dagger \right)^* .$$

If we can put an additional linear ordering on Θ, for example via some sort of a length function (as is the case for usual category $\mathcal{O}$), then there is then a spectral sequence whose first page are (direct sums of) terms of the form above, and which converges to (the graded pieces of) the cohomology of the input. In the case of category $\mathcal{O}$ one inputs the finite-dimensional simples to recover the BGG resolution; this works thanks to the fine control on the Ext groups $\mathrm{Ext}^\bullet(\Delta, L)$ afforded by Kostant (in the guise of $\mathfrak{n}^+$-cohomology). Note how the above generalizes the spectral sequence of standard objects appearing in Dhillon's work (the setting there is in an actual highest weight category). And in the nilBrauer setting we do in fact have a linear ordering on Θ (per block, i.e. within even/odd), so we can say

Theorem 1.3. For any $M \in \mathsf{D}(\mathsf{Mod}^\vartheta\, \mathrm{N}\mathcal{B})$ (here $\vartheta = 0$ or 1, namely even or odd, denotes the homological block M lies in), one has

$$\mathrm{gr}^k \mathcal{I}\mathrm{d}_\vartheta\, M = \Delta_{\vartheta+2k}^{\overline{l_{\vartheta+2k}(\vartheta+2k)}} \otimes_{\mathrm{N}\mathcal{B}^{\vartheta+2k}} \mathsf{RHom}^\bullet_{\mathrm{N}\mathcal{B}}(\Delta_{\vartheta+2k}^{\overline{l_{\vartheta+2k}(\vartheta+2k)}}, M^\dagger)^*;$$

and there is a spectral sequence

$$E_1^{p,q} = H^{p+q}\, \mathrm{gr}^{-p} \mathcal{I}\mathrm{d}_\vartheta\, M \implies \mathrm{gr}^{-p} H^{p+q} M = E_\infty^{p,q}$$

which in this case reads as, for $M \in \mathsf{Mod}^\vartheta \mathrm{N}\mathcal{B}$,

$$E_1^{p,q} = \Delta_{\vartheta-2p}^{\overline{l_{\vartheta-2p}(\vartheta-2p)}} \otimes_{\mathrm{N}\mathcal{B}^{\vartheta-2p}} \mathrm{Ext}^{-(p+q)}_{\mathrm{N}\mathcal{B}}(\Delta_{\vartheta-2p}^{\overline{l_{\vartheta-2p}(\vartheta-2p)}}, M^\dagger)^* \implies \mathrm{gr}^{-p} H^{p+q} M = E_\infty^{p,q}.$$

So it remains to compute these Ext groups for $M = L$ a simple as well as the A^θ-action on them.

2. 09/14 – Stratification/recollement of (stable infinity) categories and the "reconstruction philosophy" (Fan Zhou)

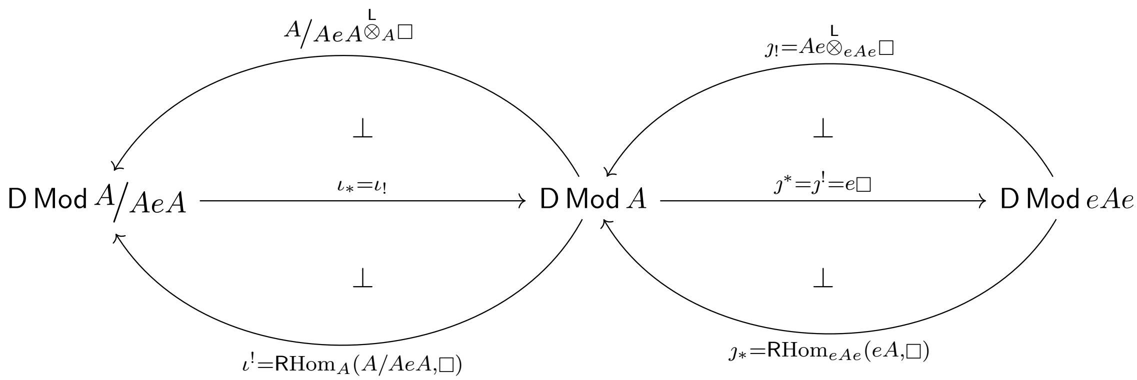
2.1. **recollement.** I found two definitions in the literature for recollements of stable infinity-categories; I presume they must be equivalent (at least for presentable stable categories).

Definition 2.1. A recollement of stable ∞-categories is
$$\mathcal{C}^{<} \xrightarrow{\iota_*=\iota_!} \mathcal{C}^{\leq} \xrightarrow{\jmath^!=\jmath^*} \mathcal{C}^{=}$$
where each functor has left (ι^*, $\jmath_!$) and right ($\iota^!$, $\jmath_*$) adjoints such that there are equalities
$$\operatorname{Img}\jmath_! = \operatorname{Ker}\iota^*, \qquad \operatorname{Img}\iota_* = \operatorname{Ker}\jmath^!, \qquad \operatorname{Img}\jmath_* = \operatorname{Ker}\iota^!$$
among full subcategories of $\mathcal{C}^{\leq}$. You probably should also first require that $\iota_*, \jmath_!, \jmath_*$ are all full embeddings.

The other definition requires first that the counits and units for the adjunctions are isomorphisms, that $\operatorname{Img}\iota_* = \operatorname{Ker}\jmath^!$, and that the usual exact triangles ($\jmath_!\jmath^! \to \mathrm{Id} \to \iota_*\iota^* \to [+1]$ and its cousin $\iota_!\iota^! \to \mathrm{Id} \to \jmath_*\jmath^* \to [+1]$) are indeed exact (in the stable definition).

I'm told this first arose in geometry; something about closed subschemes and their open complements; the geometers here know better than I.

The algebraic analogue to this is the classical work of Cline-Parshall-Scott, in a sequence of works setting the foundations for what we today consider as 'highest weight theory'. The claim/setup/scenery for this is that, given an algebra A with an idempotent e, one has a recollement



Remark. It is not guaranteed that $\jmath_!$ and $\jmath_*$ are exact on the underived level, but with some additional structure on A (which you may think is either reasonable or unreasonable depending on your religion), for instance the presence of a PBW basis theorem, you can make it so that they are.

In this algebraic setup, one would intuitively say that the 'subcategory' is the modules over A/AeA, and the 'stratum' is modules over eAe.

The closed-open recollement can be carried a little further by considering a 'stratification' of a scheme X, which is to say the assignment of closed subschemes $Z_{\leq\lambda}$ for a poset Λ, with the condition that $\bigcup_\lambda Z_{\leq\lambda} = X$ and $Z_{\leq\lambda} \cap Z_{\leq\mu} = \bigcup_{\nu\leq\mu,\lambda} Z_{\leq\nu}$. Letting $Z_{<\lambda} = \bigcup_{\mu<\lambda} Z_{\leq\mu}$ and $U_{=\lambda} = Z_{\leq\lambda} \setminus Z_{<\lambda}$, one then has recollements $\mathsf{Sh}(Z_{<\lambda}) \longrightarrow \mathsf{Sh}(Z_{\leq\lambda}) \longrightarrow \mathsf{Sh}(U_{=\lambda})$ for each λ.

2.2. **the setup of AMR.** There is a paper by Ayala-MazelGee-Rozenblyum at here. This is the closest thing to a literature reference I could find. They phrase things in the language of "noncommutative stacks", which by definition are presentable stable ∞-categories. I do not know what presentable means, but I am told that derived categories are presentable, which is good enough to me. Strangely (to me), they require "closed substacks" to have a tower of *two* right adjoints to the inclusion. Their definition of a stratification is:

Definition 2.2. A "stratification" of a presentable stable ∞-category $\mathcal{X}$ is a functor
$$\Lambda \longrightarrow \text{poset of closed substacks ordered by inclusion}$$
such that

(1) $\mathcal{X} = \bigcup_\lambda \mathcal{Z}_\lambda$,
(2) for any λ, μ, there exists a following factorization:

$$\begin{array}{ccc} \bigcup_{\nu\le\mu,\lambda} \mathcal{Z}_\nu & \hookrightarrow & \mathcal{Z}_\lambda \\ \uparrow & & \uparrow \\ \mathcal{Z}_\mu & \hookrightarrow & \mathcal{X} \end{array}$$

2.3. **our setup.** We use highest weight notation in this section on general machinery, and will return to lowest weight notation when we return to the setting of nilBrauer or other specific diagram algebras. Let $\mathsf{St\infty Cat}$ denote the category of stable infinity-categories, and let Λ be a poset with a unique final object (our convention on arrows is that final means maximal).

Definition 2.3. A "filtration" on $\mathcal{C} \in \mathsf{St\infty Cat}$ is a functor

$$\mathcal{F}\colon \Lambda \longrightarrow \mathsf{St\infty Cat}$$

such that all arrows go to fully faithful embeddings and the final object goes to $\mathcal{C}$. For $\lambda \in \Lambda$, we denote $\mathcal{C}^{\le\lambda} = \mathcal{F}(\lambda)$ and $\mathcal{C}^{<\lambda}$ as the smallest stable subcategory containing all $\mathcal{C}^{\le\mu}$ for all $\mu < \lambda$. Let $\mathcal{C}^{=\lambda} = \mathcal{C}^{\le\lambda}/\mathcal{C}^{<\lambda}$ be the Verdier quotient.

Our convention is to write $\mu \to \lambda$ for $\mu < \lambda$. Let us name

$$\begin{aligned} \bar\iota_\lambda &\colon \mathcal{C}^{\le\lambda} \longrightarrow \mathcal{C}, \\ \iota_\mu^{<\lambda} &\colon \mathcal{C}^{\le\mu} \longrightarrow \mathcal{C}^{<\lambda}, \\ \iota_\mu^{\lambda} &\colon \mathcal{C}^{\le\mu} \longrightarrow \mathcal{C}^{\le\lambda}, \\ \iota_\lambda &\colon \mathcal{C}^{<\lambda} \longrightarrow \mathcal{C}^{\le\lambda}, \\ \jmath^\lambda &\colon \mathcal{C}^{\le\lambda} \longrightarrow \mathcal{C}^{=\lambda}; \end{aligned}$$

for our purposes let's require both the inclusions of each $\mathcal{C}^{\le\lambda}$ as well as the Verdier quotient functors to have both adjoints. This is for example the case with recollement situations. Let the adjoints, like in the recollement case, be named $\bar\iota_\lambda^*$ and $\jmath_!^\lambda$ for the left adjoints.

In another direction, one can filter an object of an infinity-category by a poset Λ:

Definition 2.4. A "Λ-filtered object of $\mathcal{C}$" is a functor

$$\mathcal{X} \in \mathsf{Fun}(\Lambda, \mathcal{C}).$$

Letting Λ^0 be the 0-skeleton of Λ, the "associated graded" of this filtered object is

$$\begin{aligned} \mathrm{gr}\colon \mathsf{Fun}(\Lambda,\mathcal{C}) &\longrightarrow \mathsf{Fun}(\Lambda^0,\mathcal{C}) \\ \mathcal{X} &\longmapsto \mathrm{gr}\,\mathcal{X} \end{aligned}$$

defined by

$$(\mathrm{gr}\,\mathcal{X})(\lambda) = \mathrm{gr}^\lambda\,\mathcal{X} = \mathrm{Fib}\Big(\mathcal{X}(\lambda) \longrightarrow \lim_{\mu\leftarrow\lambda} \mathcal{X}(\mu)\Big),$$

assuming such limits exist.

An object $X \in \mathcal{C}$ is said to have a Λ-filtration if after adding an initial/minimal element $\lambda_{-\infty}$ to Λ there is a functor $\mathcal{X}\colon \Lambda \cup \{\lambda_{-\infty}\} \longrightarrow \mathcal{C}$ such that $\mathcal{X}(\lambda_{-\infty}) = X$.

Remark. Note that the definition of a filtered object here is sort of opposite to one might expect; rather than inclusions, one simply has maps $\mathcal{X}(\mu) \longrightarrow \mathcal{X}(\lambda)$ for each $\mu \to \lambda$ (i.e. $\mu \le \lambda$), where the smallest element $\lambda_{-\infty}$ corresponds to the actual object X, so that X has maps to all the filtered pieces rather than the filtered pieces 'including' into X; and rather than being a quotient, the associated graded pieces are sort of kernels. Of course in the derived category these things are the same up to shift anyway, but roughly speaking this is why there is a minus sign on the gr in Theorem 3.1.

Then the general nonsense claim alluded to above is that

Theorem 2.5 (folklore)**.** Let $\mathcal{C}$ admit a filtration by Λ such that the inclusions and quotients have both adjoints. Then the identity functor $\mathrm{Id}_{\mathcal{C}} \in \mathsf{End}\,\mathcal{C}$ admits a Λ^{op}-filtration with terms

$$\mathcal{I}\mathrm{d}(\lambda) = \bar{\iota}_\lambda \bar{\iota}_\lambda^*,$$

and the associated graded can be computed as

$$\mathrm{gr}^\lambda \mathcal{I}\mathrm{d} = \bar{\iota}_\lambda \jmath_!^\lambda \jmath^\lambda \bar{\iota}_\lambda^*,$$

provided that either Λ is down-finite (namely that for any λ the set $\{\mu : \mu \leq \lambda\}$ is finite), or Λ is locally-finite and "eventually disjoint-totally-ordered" (which means that for any λ, there exists $\mu \leq \lambda$ such that $\{\nu : \nu \leq \mu\}$ is totally ordered).

If there is a "dimension/length function" $\ell\colon \Lambda \longrightarrow \mathbb{Z}^{\mathrm{op}}$ and $\mathcal{C}$ moreover has a t-structure, then there is a spectral sequence

$$E_1^{p,q} = \bigoplus_{\lambda \in \ell^{-1}(-p)} \pi^{p+q}(\mathrm{gr}^\lambda \mathcal{I}\mathrm{d}) \implies E_\infty^{p,q} = \mathrm{gr}^{-p} \pi^{p+q}(\mathrm{Id}_{\mathcal{C}}).$$

Proof. Omitted from these notes, included in the talk. ■

Applied to the algebraic recollement setup of last week (see last week's notes) (note well that the poset in that case is named Θ), this affords us that you would have a (functorial!) spectral sequence looking like

$$E_1^{p,q} = \bigoplus_{\theta \in \ell^{-1}(-p)} \left(\bigoplus_{\lambda \in \theta} \Delta_\lambda^{\overline{l_\lambda(\theta)}} \otimes_{A^\theta} \mathrm{Ext}_A^{-(p+q)} \left(\bigoplus_{\lambda \in \theta} \Delta_\lambda^{\overline{l_\lambda(\theta)}}, M^\dagger \right)^* \right) \implies E_\infty^{p,q} = \mathrm{gr}^{-p} H^{p+q}(M).$$

One example of this is category $\mathcal{O}$. In that case the algebraic recollement setup doesn't fit on the nose (because the algebra controlling a block of category $\mathcal{O}$ is famously difficult to handle), but for instance one would instead consider the (derived) category of objects with weight at most (or less than, respectively) λ, with Verdier quotient (derived) vector spaces, corresponding to the fact that the Cartan has very simple semisimple representation theory. Then the Ext groups are simply multiplicity vector spaces, there is only one type of Verma module as opposed to a big and a small, and this gives the standard BGG resolution.

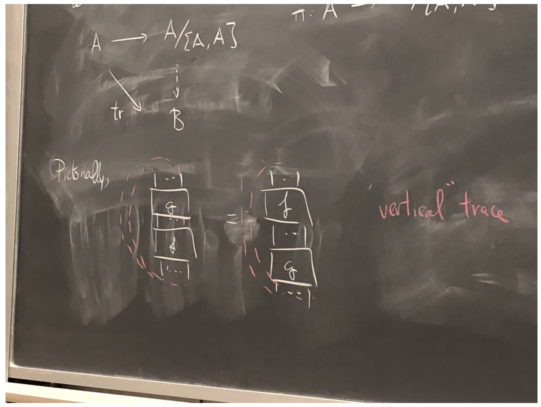
The category $\mathcal{O}$ example can be reworded to fit a more geometric narrative. Consider the flag variety and its stratification via Bruhat cells, and consider the (derived) categories of $\mathcal{D}$-modules on the appropriate Z, U. 'Recall' that $\mathcal{D}$-modules have a six-functor formalism; one can prove that recollement is satisfied in this setup. I think in this case the analogue of the Verma modules in the spectral sequence above is the 'transfer modules', though maybe duality needs to be involved at some point.

Remark. I'm not sure what happens in the above setup if you just consider sheaves instead.

The nilBrauer is another example of this setup, perhaps more closely. The poset describing the weights of nilBrauer could be either considered as a single totally ordered set or two disjoint totally ordered sets (even vs. odd); both considerations give valid triangular based setups. The spectral sequences they give will be equivalent also, it is just that the totally ordered poset will give a less fine description of the input M, whereas the 'filtration' given by the spectral sequence coming from the disjoint copy of two totally ordered sets will correspond to the direct sum decomposition into blocks (namely even vs. odd).

3. 09/21 – The dg trace (Alvaro Martinez)

3.1. **Traces and categorification.** Let A be a $\Bbbk$-algebra. A map $\mathrm{tra}\colon A \longrightarrow B$ is a trace if $\mathrm{tra}(xy) = \mathrm{tra}(yx)$. Every trace factors through the 'cocenter', $A/[A,A]$. The map through which it factors, $\pi\colon A \longrightarrow A/[A,A]$, is the "universal trace" of A. Pictorially, this would be called a "vertical trace", denoted like this:

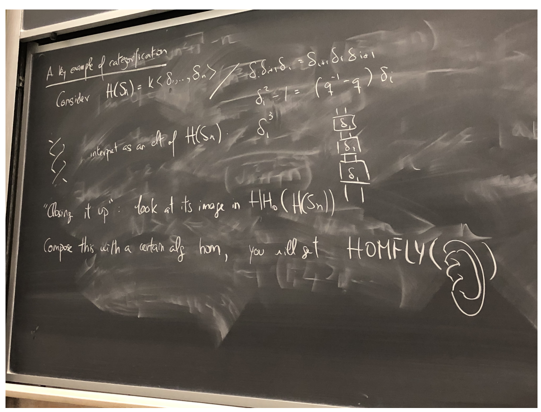


Remark: Sometimes, instead of taking the cocenter $A/[A,A]$ which is a quotient of things like $ab - ba$, it is desirable to keep track of it in a complex. 'Homology bad, complex good'. To this end consider

$$\begin{aligned} \cdots \longrightarrow A \otimes A \otimes A &\longrightarrow A \otimes A \longrightarrow A \\ a \otimes b \otimes c &\longmapsto ab \otimes c - a \otimes bc + ca \otimes b \\ a \otimes b &\longmapsto ab - ba \end{aligned}$$

This is the "cyclic bar complex", and the n-th homology of it is the "Hochschild homology", denoted HH_n.

3.2. **A key example of categorification.** Consider the Hecke $\mathcal{H}(S_n)$ where we call the transpositions δ_i and the relation is $\delta_i^2 - 1 = (q^{-1} - q)\delta_i$, as well as the usual triple crossing relation. Pictorially this is considered as:



If you take such an element and 'close it up', namely look at its image in $\mathrm{HH}_0(\mathcal{H}(S_n))$, if you compose this with a certain algebra homomorphism you will get the homfly polynomial of the closed up braid. This is classical.

The categorified story of this is the story of Soergel. Consider the category SBim_n of Soergel bimodules in rank n. For example, for $n = 2$, this is the direct sum, direct summand, and shift completion of the category of R-bimodules where $R = \Bbbk[x_1, x_2]$, where each x_i is of degree 2. This category is generated by $R = \Bbbk[x_1, x_2]$ and $B_s = R \otimes_{R^{S_2}} R(1)$. Then for instance one commutes that $B_s^{\otimes 2} = R \otimes_{R^s} R \otimes_R R \otimes_{R^s} R(2) = R \otimes_{R^s} R(0) \oplus R \otimes_{R^s} R(2) = B_s(1) \oplus B_s(-1) = qB_s + q^{-1}B_s$. This witnesses the categorification of the Hecke algebra.

On the level of Grothendieck groups, one has $b_s = \delta_s + q$ and $\delta_s = b_s - q$. Using the philosophy that 'differences should be complexes', one represents δ_s in the Soergel story by using the "Rouquier complex", which is

$$\begin{aligned} \Delta_s \quad = \quad \underline{B_s} &\longrightarrow R(1) \\ f \otimes g &\longmapsto fg \end{aligned}$$

and

$$\nabla_s \quad = \quad R(-1) \longrightarrow \underline{B_s}$$
$$1 \longmapsto \frac{1}{2}((x_1 - x_2) \otimes 1 + 1 \otimes (x_1 - x_2));$$

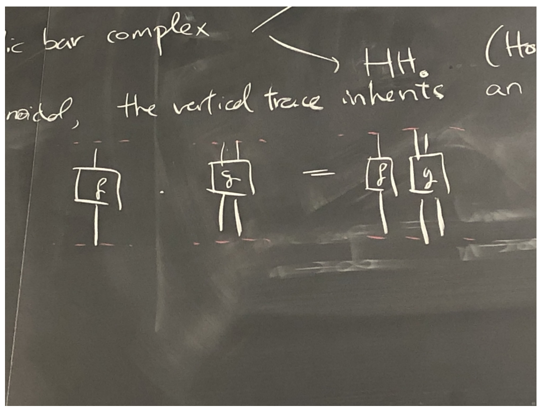
here the underline denotes homological degree zero.

Theorem 3.1 (Rouquier)**.** These complexes satisfy the braid relations up to homotopy.

This story takes place in the dg monoidal category of bounded complexes in SBim_n. Then one would wish to associate to δ_1^3 the categorified version which is $\Delta_s^{\otimes 3}$. Now the question is: How does one close up a category?

There are two ways to do this.

(1) "Vertical trace". Interpret $\mathcal{D} = \mathsf{Ch}^{\mathrm{b}}(\mathsf{SBim}_n)$ as a big nonunital dg $\Bbbk$-algebra by taking its path algebra $\bigoplus_{X,X'} \mathrm{Hom}_{\mathcal{D}}(X, X')$. Then take its cyclic bar complex, and obtain the Hochschild homology. The higher ones are called the "derived vertical trace". Since $\mathcal{D}$ is monoidal, the vertical trace inherits an algebra structure given by horizontal concatenation:



Remark: Fun fact, $\mathrm{HH}_0(\mathcal{U}_Q(\mathfrak{sl}_n)) \cong K_0(\mathcal{U}_Q(\mathfrak{sl}_n)) \cong U_q(\mathfrak{sl}_n)$. Here $\mathcal{U}_Q(\mathfrak{sl}_n)$ refers to the 2-category which is the KLR categorification of the full enveloping algebra.

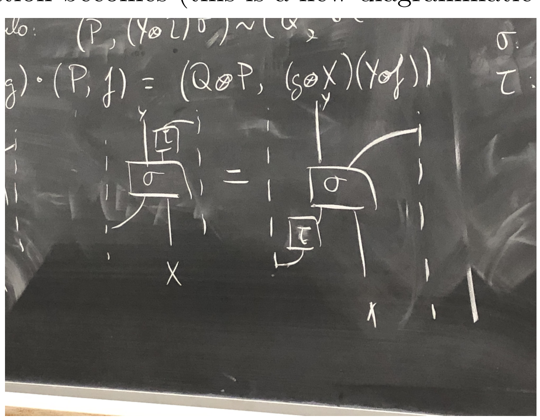
(2) "Horizontal trace". This is the main topic of today. Let $\mathcal{C}$ be a $\Bbbk$-linear mnoidal category, maybe dg. Let $\mathrm{Tra}(\mathcal{C})$ be the category whose objects are those of $\mathcal{C}$ and whose morphisms are pairs $(P \in \mathcal{C}, f)\colon X \longrightarrow Y$ such that $f\colon P \otimes X \longrightarrow Y \otimes P$, modded out by the condition that (here $\sigma\colon P \otimes X \longrightarrow Y \otimes Q$ and $\tau\colon Q \longrightarrow P$)

$$(P, (\mathrm{id}_Y \otimes \tau)\sigma) \sim (Q, \sigma(\tau \otimes \mathrm{id}_X)).$$

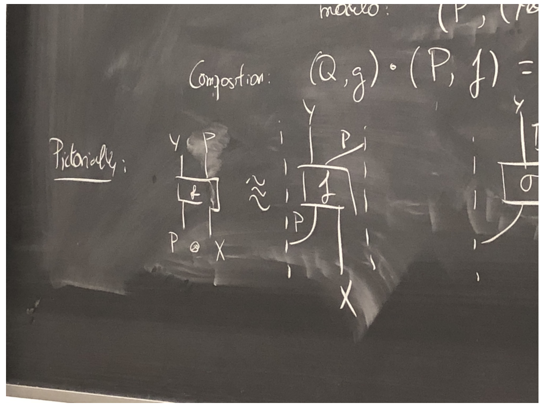
The composition is given as

$$(Q, g) \circ (P, f) = (Q \otimes P, (g \otimes \mathrm{id}_X)(\mathrm{id}_Y \otimes f)).$$

Pictorially, the modded relation becomes (this is a new diagrammatic in the category $\mathrm{Tra}_0(\mathcal{C})$)



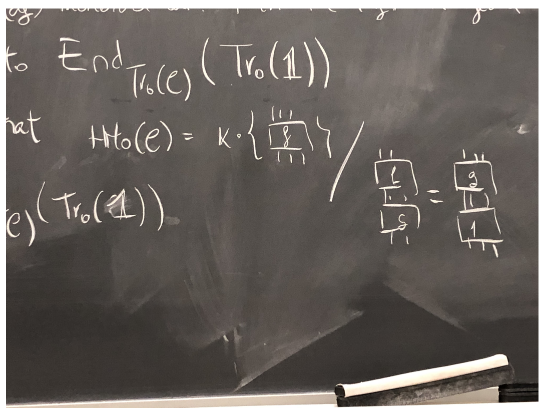
and one should think of a morphism f as



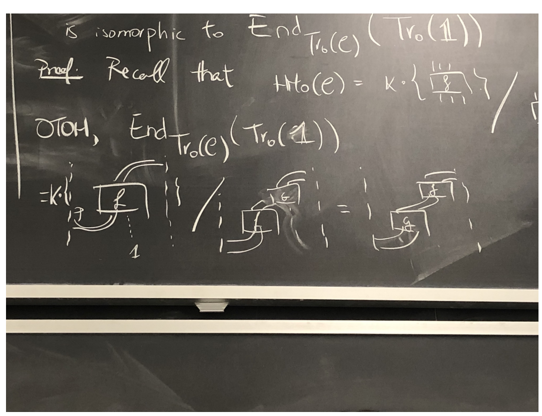
This second one is supposed to be better. The reason is:

Proposition 3.2. Let $\mathcal{C}$ be a (dg) monoidal category. The the (dg) $\Bbbk$-algebra $\mathrm{HH}_0(\mathcal{C})$, which is the vertical trace, is isomorphic to $\mathrm{End}_{\mathrm{Tra}_0(\mathcal{C})}(\mathbf{1})$.

Proof. Recall that $\mathrm{HH}_0(\mathcal{C})$ is pictorially thought of as, via the $A/[A, A]$ thing,



The proof is pictorial:



Basically because you are looking at $\mathbf{1}$, you can ignore the vertical lines in the diagrammatic of $\mathrm{Tra}_0(\mathcal{C})$. The $ab - ba$ condition becomes the horizontal commutativity of $\mathrm{Tra}_0(\mathcal{C})$. ■

One example of how this is used is in the categorification of HOMFLY. On the classical level one has

$$\begin{array}{ccc} \mathcal{H}_n & \longrightarrow & \mathcal{H}_n/[\mathcal{H}_n, \mathcal{H}_n] \\ & \overset{\text{HOMFLY}}{\searrow} & \downarrow \\ & & \mathbb{C}(q)[a^{\pm 1}] \end{array}$$

and on the categorified level this becomes

$$\begin{array}{ccc} \mathsf{K}^{\mathrm{b}}(\mathsf{SBim}_n) & \longrightarrow & \mathrm{Tra}_0(\mathsf{K}^{\mathrm{b}}(\mathsf{SBim}_n) \\ & \overset{\text{HOMFLY homology}}{\searrow} & \downarrow \\ & & \mathsf{Vec}_{\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}} \end{array}$$

3.3. **The vertical trace of SBim_n.** Recall the Rouquier complexes Δ_w, ∇_w.

Proposition 3.3. (1) The Δ_w generate $\mathcal{D} = \mathsf{Ch}^{\mathrm{b}}(\mathsf{SBim}_n)$ with respect to cones, shifts, and homotopy equivalences.
(2) $\mathrm{Hom}_{\mathcal{D}}(\Delta_v, \nabla_w) \simeq 0 \simeq \mathrm{Hom}_{\mathcal{D}}(\nabla_w, \Delta_v)$ unless $v \leq w$. (This is 'semi-orthogonality'; note the resemblance to highest weight stories.)

As a consequence, the cyclic bar complex of $\mathcal{D}$ deformation retracts to $\bigoplus_{w\in W} \mathrm{CycBar}^\bullet(\mathrm{End}_{\mathcal{D}}(\Delta_w))$. This is supposed to be computable – $\mathrm{End}\,\Delta_w$ is supposed to basically be R (maybe with a tensor with an exterior algebra or something).

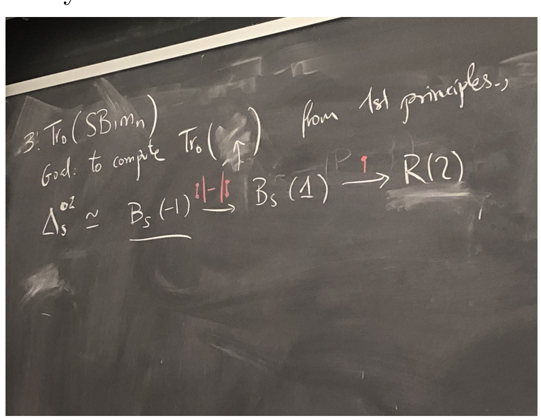
Theorem 3.4. As (dg) $\Bbbk$-algebras, $\mathrm{HH}_\bullet(\mathcal{D}) \cong R \otimes \Lambda \rtimes W$, where $R = \Bbbk[x_1, \cdots, x_n]$ of degrees $\deg x_i = (2, 0)$ (first entry is internal, second is homological) and $\Lambda = \Bbbk\langle\theta_1, \cdots, \theta_n\rangle$ anticommute of degrees $\deg \theta_i = (2, -1)$ and W elements are of degree $(0, 0)$.

Ideas to proving this: turns out $\mathrm{End}_{\mathcal{D}}\,\Delta_w \simeq {}^wR$, where ${}^wR$ is action-twisted in the sense that for $a \in R$ and $b \in {}^wR$ we have $a \cdot b = a(w^{-1}x_1, \cdots, w^{-1}x_n)b$. Same thing happens for the action of $\mathrm{HH}_\bullet(R)$ on $\mathrm{HH}_\bullet(\mathrm{End}(\Delta_w))$. One uses the vector space equivalence $\mathrm{End}\,\Delta_w \simeq \mathrm{End}\,\mathbf{1}$, and somehow gets that w acts as $(-1)^{\ell(w)}w$. Now we draw a square and move on due to time.

3.4. **Horizontal trace of Soergel bimodules.** The goal is to compute the trace $\mathrm{Tra}_0(\delta_1^2)$ from first principles (in the Karoubi completion). I guess $n = 2$. One computes that

$$\Delta_s \otimes \Delta_s \simeq \underline{B_s(-1)} \longrightarrow B_s(1) \longrightarrow R(2),$$

where the maps are diagrammatically



In order to study the horizontal trace category better we must use a little different notation to avoid confusion. Let $\mathrm{Tra}_0(\mathbf{1})$ denote $\mathbf{1}$ considered as an object of the horizontal trace category. As we have seen earlier,

$$\mathrm{End}_{\mathrm{Tra}_0(\mathcal{C})}(\mathsf{Tra}_0(\mathbf{1})) = \mathrm{HH}_0(\mathcal{D}).$$

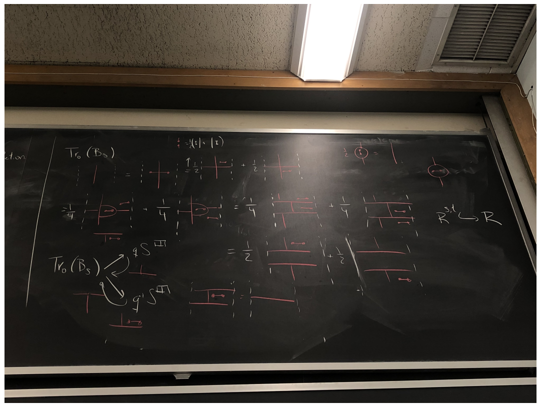
But the degree 0 part of the latter is precisely $\Bbbk S_n$, and idempotents must be degree 0, so the Karoubi breakdown of $\mathsf{Tra}_0(\mathbf{1})$ is just given by Young idempotents/partitions. He formally denotes this using Schur functors:

$$\mathsf{Tra}_0(\mathbf{1}) = \bigoplus_{\lambda \vdash \mathbf{n}} \mathcal{S}^{\lambda}(\mathsf{Tra}_{\mathbf{0}}(\mathbf{1})).$$

For instance, he writes that for $n = 2$

$$\mathsf{Tra}_0\,\mathbf{1} = \mathcal{S}^{\square\square} \oplus \mathcal{S}^{\substack{\square\\\square}}.$$

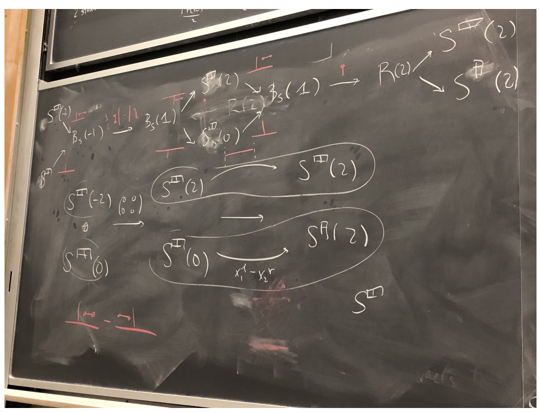
Next he computes $\mathsf{Tra}_0(B_s)$. This is a diagrammatic mess.



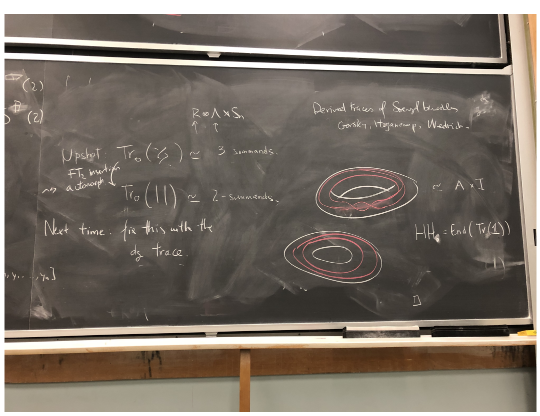
The takeaway is that we have diagrammatic projectors/inclusions for the breakdown

$$\mathsf{Tra}_0(B_s) = q\mathcal{S}^{\square\square} \oplus q^{-1}\mathcal{S}^{\square\square}.$$

Here's another computation for the breakdown of $\mathsf{Tra}_0(\Delta_s^{\otimes 2})$.



Lastly there is some knot invariants reason for why the answers so far are unsatisfactory. I didn't tex this in the moment and now (several days later) no longer remember exactly what was said, so I'll attach a boardshot instead:



4. 09/28 – The dg trace, Part II (Alvaro Martinez)

This is a continuation from last time. Goal today: to understand the derived horizontal trace of a dg category $\mathcal{C}$ (later we will think about $\mathcal{C} = \mathsf{K}^{\mathrm{b}}(\mathsf{SBim}_n)$).

4.1. "Bimodule" language for categories. Recall that a right exact functor $\mathcal{F}\colon {}^{A}\mathsf{Mod} \longrightarrow {}^{B}\mathsf{Mod}$ is given by tensoring by a bimodule $B \circlearrowright M \circlearrowleft A$, also denoted ${}_{B}M_{A}$. Note that composition is the tensor product of bimodules.

Proposition 4.1. The following are equivalent:

- A natural transformation $\alpha_X \colon {}_{B}\mathcal{F}(X) \longrightarrow {}_{B}\mathcal{G}(X)$;
- A (A, A) bimodule homomorphism $A \longrightarrow {}_{A}\operatorname{Hom}_{B}({}_{B}\mathcal{F}(A)_{A}, {}_{B}\mathcal{G}(A)_{A})_{A}$;
- A (B, A)-homomorphism $\alpha_A \colon {}_{B}\mathcal{F}(A)_{A} \longrightarrow {}_{B}\mathcal{G}(A)_{A}$.

The proof of this is probably you just look at it and convince yourself.

4.2. "Algebraic" language for categories.

Definition 4.2. A dg category is a $\mathbb{k}$-linear $\mathbb{Z}$-graded category enriched over $\mathsf{Ch}(\mathbb{k})$.

Let $\mathcal{C}$ be a dg category. Let $X, Y \in \mathcal{C}$. He writes (my eyes are hurting) $Y\mathcal{C}X = \operatorname{Hom}_{\mathcal{C}}(X, Y)$, and similarly $\mathcal{C}X = \bigoplus_{Y} Y\mathcal{C}X$ and $Y\mathcal{C} = \bigoplus_{X} Y\mathcal{C}X$, so that $\mathcal{C} = \bigoplus_{X,Y} Y\mathcal{C}X$. Note the similarity to non-unital algebra philosophy/language.

A dg functor can be thought of as an "algebraic homomorphism" whose action on Hom spaces is a degree 0 chain map. In this language, $\mathcal{F}\colon \mathcal{C} \longrightarrow \mathcal{D}$ can be thought of as $Y\mathcal{F}X \coloneqq \mathcal{F}(Y)\mathcal{D}\mathcal{F}(X)$ as a $\mathcal{C}$-bimodule. Note that now it is imperative to note the difference between the notations $\mathcal{F}X$ and $\mathcal{F}(X)$.

In this language, composition is again the tensor product ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}} \otimes_{\mathcal{C}} {}_{\mathcal{C}}\mathcal{N}_{\mathcal{C}}$, where for all $X, Z \in \mathcal{C}$,

$$Z({}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}} \otimes_{\mathcal{C}} {}_{\mathcal{C}}\mathcal{N}_{\mathcal{C}})X = \bigoplus_{Y} (Z\mathcal{M}Y \otimes_{\mathcal{C}} Y\mathcal{N}X)/\sim,$$

where the $\sim$ is relations such as $(mf) \otimes n \sim m \otimes (fn)$.

What about natural transformations, what should they be?

Definition 4.3. Let $\mathcal{F}, \mathcal{G}\colon \mathcal{C} \longrightarrow \mathcal{D}$ between dg categories. A "degree k natural transformation" is an assignment

$$X \longmapsto \alpha_X \in \operatorname{Hom}^{k}_{\mathcal{D}}(\mathcal{F}(X), \mathcal{G}(X))$$

such that the following diagram commutes up to a sign (which is $(-1)^{k|f|}$):

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(Y) \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ \mathcal{G}(X) & \xrightarrow{\mathcal{G}(y)} & \mathcal{G}(Y) \end{array}$$

Observe that a pair of functors $(\mathcal{F}, \mathcal{G})$ determines a $\mathcal{C}$-bimodule $\mathcal{B} = \mathcal{B}(\mathcal{F}, \mathcal{G})$ defined by

$$Y\mathcal{B}X \coloneqq \mathcal{G}(Y)\mathcal{D}\mathcal{F}(X).$$

Definition 4.4. A "natural transformation" is a map of $\mathcal{C}$-bimodules

$$\alpha\colon \mathcal{C} \longrightarrow \mathcal{B}(\mathcal{F}, \mathcal{G}).$$

Normally one would ask $\mathcal{G}(f) \circ \alpha_X - \alpha_Y \circ \mathcal{F}(f) = 0$, but here we weaken this (homology versus complex) to asking that

$$\mathcal{G}(f) \circ \alpha_X - \alpha_Y \circ \mathcal{F}(f) = \mathrm{d}_{\mathcal{D}}(\alpha_f)$$

for some α_f. ...

Apparently

Fact 4.5. There exists a complex $B(\mathcal{C})$ such that a homotopy coherent natural transformation is the same as a $\mathcal{C}$-bimodule map

$$\alpha\colon B(\mathcal{C}) \longrightarrow \mathcal{B}(\mathcal{F}, \mathcal{G}).$$

4.3. **The two-sided bar complex.** Let $\mathcal{I}$ be the dg subcategory of $\mathcal{C}$ generated by the identity morphisms (i.e. $X\mathcal{I}X = \Bbbk \operatorname{Id}_X$). Then $\mathcal{C} \otimes_{\mathcal{I}} \mathcal{C} \otimes_{\mathcal{I}} \mathcal{C} \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} \mathcal{C}$ consists of elements $f_0 \otimes f_1 \otimes \cdots \otimes f_{r+1}$, where the f_i's are composable. Consider the bicomplex

$$\cdots \longrightarrow \mathcal{C} \otimes_{\mathcal{I}} \mathcal{C} \otimes_{\mathcal{I}} \mathcal{C} \xrightarrow{\mathrm{d_{bar}}} \mathcal{C} \otimes_{\mathcal{I}} \mathcal{C},$$

where the horizontal arrows $\mathrm{d_{bar}}$ are given by the usual alternating sums of squishes. Then $B(\mathcal{C})$ is the total complex of this bicomplex, where the differential is (here $\mathrm{d}_{\oplus}$ denotes the internal differential)

$$\mathrm{d} = \mathrm{d_{bar}} + \mathrm{d}_{\oplus}.$$

The $\mathcal{C}$-bimodule structure on this is

$$g \cdot (f_1 \otimes \cdots \otimes f_{r+1}) \cdot g' = (-1)^{r|g|}(gf_1 \otimes \cdots \otimes f_{r+1}g').$$

Now a homotopy coherent natural transformation is obtained from $\alpha\colon B(\mathcal{C}) \longrightarrow \mathcal{B}(\mathcal{F}, \mathcal{G})$, e.g.

$$\alpha_X = \alpha(\operatorname{Id}_X \otimes \operatorname{Id}_X),$$
$$\alpha_f = \alpha(\operatorname{Id}_X \otimes f \otimes \operatorname{Id}_X),$$
$$\vdots$$

4.4. **The cyclic bar complex.** Let

$$C(\mathcal{C}) = B(\mathcal{C})\big/[\mathcal{C}, B(\mathcal{C})],$$

where the denominator consists of $fm - (-1)^{|f||m|}mf$. In other words,

$$C(\mathcal{C}) = \bigoplus_{i \geq 0} \bigoplus_{X_0, \cdots, X_{r+1}} \big(X_0\mathcal{C}X_1 \otimes X_1\mathcal{C}X_2 \otimes \cdots \otimes X_{r+1}\mathcal{C}X_0\big)[-r]/\sim,$$

where the sign on the shift might be wrong and the $\sim$ is

$$f_0 \otimes f_1 \otimes \cdots \otimes f_{r+1} \sim (-1)^{\text{some sign}} \operatorname{Id} \otimes f_1 \otimes \cdots \otimes f_{r+1} \circ f_0.$$

At this point I should warn the reader that he uses $\|$ instead of $\otimes$ for elements. We can then define Hochschild homology as

Definition 4.6.

$$\mathrm{HH}_0(\mathcal{C}) = H^0(C(\mathcal{C})),$$

and more generally

$$\mathrm{HH}_\bullet(\mathcal{C}) = H^\bullet(C(\mathcal{C})).$$

Remark: Write $[X]$ for the image of $\operatorname{Id}_X \| \operatorname{Id}_X$ in $\mathrm{HH}_\bullet(\mathcal{C})$. Then $[\operatorname{Cone}(X \xrightarrow{f} Y)] = [Y] - [X]$, where $[X[-1]] = -[X]$.

4.5. **Monoidal dg categories.**

Definition 4.7. A monoidal dg category $\mathcal{C}$ is a triple $(CC, \boxtimes, \mathbf{1})$ where the associators and unitors are degree 0 closed morphisms.

There is a coalgebra structure on $B(\mathcal{C})$ coming from 'splitting up', e.g.

$$f_0\|f_1\|f_2 \longmapsto (f_0\|\operatorname{Id}) \otimes_{\Bbbk} (\operatorname{Id}\|f_1\|f_2) + (-1)^{\text{some sign}}(f_0\|f_1\|\operatorname{Id}) \otimes_{\Bbbk} (\operatorname{Id}\|f_2).$$

There is also an algebra structure coming from a "shuffle product", namely that for $f = f_0\|\cdots\|f_{r+1}$ and $g = g_0\|\cdots\|g_{s+1}$,

$$f \star g = (-1)^{|f|s} \sum_{\pi \in S_{(r,s)}} (-1)^{\text{some sign depending on } \pi} (f_0 \boxtimes g_0)\|e_{\pi(1)}\|\cdots\|e_{\pi(r)}\|(f_{r+1} \boxtimes g_{s+1}),$$

where

$$e_i = \begin{cases} f_i \boxtimes \mathrm{Id} & i = 1, \cdots, r \\ \mathrm{Id} \boxtimes g_{i-r} & i = r+1, \cdots, r+s \end{cases}.$$

Together these give $B(\mathcal{C})$ a bialgebra structure.

4.6. **The quadmodule $\mathfrak{X}$.** Consider

$$\mathfrak{X} = \bigoplus_{X_1, X_2, Y_1, Y_2} (Y_1 \boxtimes Y_2)\mathcal{C}(X_1 \boxtimes X_2).$$

We have a $\mathcal{C} \boxtimes \mathcal{C}$-bimodule structure on $\mathfrak{X}$ coming from composition of morphisms. We will consider some more actions: consider the following actions of $\mathcal{C}$ on $\mathfrak{X}$:

$$\begin{aligned} a \cdot_1 f &= (a \boxtimes \mathrm{Id}_f, \\ a \cdot_2 f &= (\mathrm{Id} \boxtimes a) f, \\ f \cdot_1 b &= f(b \boxtimes \mathrm{Id}), \\ f \cdot_2 b &= f(\mathrm{Id} \boxtimes b). \end{aligned}$$

Forget two of these actions (forget the middle two) and define $\mathfrak{X}_{12} = \mathfrak{X}$ with the $\mathcal{C}$-bimodule action

$$afb := a \cdot_1 f \cdot_2 b.$$

Write $\mathfrak{X}_{12}(X', X) = \bigoplus_{Y', Y} (X \boxtimes Y)\mathcal{C}(Y' \boxtimes X')$ as a sub-$\mathcal{C}$-bimodule of $\mathfrak{X}_{12}$.

Remark: the dg Drinfeld center of $\mathcal{C}$ has objects which are pairs $(Z, \tau($ where $Z \in \mathcal{C}$ and $\tau\colon B(\mathcal{C}) \longrightarrow \mathfrak{X}_{12}(Z, Z)$ is a map of $\mathcal{C}$-bimodules *and* dg algebras.

Define two kinds of multiplications:

$$\begin{aligned} \mu_{\nwarrow}(f, g) &= (\mathrm{Id} \boxtimes g) \circ (f \boxtimes \mathrm{Id}), \\ \mu_{\nearrow}(f, g) &= (g \boxtimes \mathrm{Id}) \circ (\mathrm{Id} \boxtimes f). \end{aligned}$$

4.7. **The dg trace.** Consider the $\mathcal{C}$-bimodule $B(\mathcal{C}, \mathfrak{X}_{12}) := B(\mathcal{C}) \otimes_{\mathcal{C}} \mathfrak{X}_{12}$, "the two-sided bar complex with coefficients in $\mathfrak{X}_{12}$", and form its cyclic version

$$C(\mathcal{C}, \mathfrak{X}_{12}) := B(\mathcal{C}, \mathfrak{X}_{12}) / \sim,$$

where the $\sim$ is the same stuff as before.

Definition 4.8. The "derived horizontal trace" of $\mathcal{C}$ is the dg category $\mathsf{Tra}\,\mathcal{C}$ with objects same as those of $\mathcal{C}$, denoted $\mathsf{Tra}\,X$, and morphisms

$$\mathrm{Hom}_{\mathsf{Tra}(\mathcal{C})}(\mathsf{Tra}\,X, \mathsf{Tra}\,X') = C(\mathcal{C}, \mathfrak{X}_{12}(X', X)).$$

Composition is induced by the shuffle product on $B(\mathcal{C}, \mathfrak{X}_{12}(X', X))$.

Here is a diagrammatic for this dg trace category:

pic

The composition is:

pic

ignore the $f \boxtimes g$ at the far right.

Some remarks: If $\mathcal{C} = \mathcal{A}$ is non-dg monoidal, then $\mathsf{Tra}_0(\mathcal{A}) = H^0(\mathsf{Tra}(\mathcal{A}))$. There is also always a functor $\mathsf{Tra}(\mathcal{C}) \longrightarrow \mathsf{Tra}_0(\mathcal{C})$ by killing anything with more than one bar, and sending $\|f \longmapsto f$. Also remark that $\mathrm{End}_{\mathsf{Tra}\,\mathcal{C}}(\mathsf{Tra}\,\mathbf{1}) = C(\mathcal{C})$.

He also says something about a "traciator" ω_{XY}. This traciator is a homotopy equivalence from $\mathsf{Tra}(X \boxtimes Y)$ to $\mathsf{Tra}(Y \boxtimes X)$.

pic

4.8. **Back to Soergel.** Again let $R = \Bbbk[x_1, x_2]$. Define θ_1, θ_2 as in the boardshot below.

pic

Let $\widetilde{\mathsf{TraSBim}}_n$

5. 10/05 – Diagrammatics of Folded Soergel Bimodules of Type $A_1 \times A_1$ (Nicolas Jaramillo)

Nico is from University of Oregon, and this is joint work with Ben Elias.

Unfortunately as soon as the talk began I very quickly realized I stood no chance of live-TeXing this. Thankfully there are slides, and the talk is recorded.