CATEGORIFYING JACOBI-TRUDI: A PHENOMENON IN RANK 1

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Abstract. This was an appendix meant for my paper on the Jacobi-Trudi algebra, which is a quasi-hereditary algebra categorifying Jacobi-Trudi. The paper has since grown (both in terms of length and in terms of specialization) to a point that it no longer makes sense for this appendix to be included, but because of the low amount of technology required (you need only know what the degenerate affine Hecke of rank 1 is), I figured this was something good to have on the internet, maybe as a sales pitch for the main paper.

These phenomena were observed around the time of the solar eclipse of 2024.

The point is to realize permutation modules for S_2 as standardization functors applied to appropriate modules. Then the Jacobi-Trudi determinant formula, which should categorify to a statement like "the simples over S_n can be written as a resolution of permutation modules", becomes categorified as a BGG resolution; more precisely, in the rank 1 case, it will be that the triangle

$$\jmath_! \jmath^! \longrightarrow \mathrm{Id} \longrightarrow \imath_* \imath^* \xrightarrow{+1}$$

is exact, or rather that

$$\imath_* \imath^*[-1] \longrightarrow \jmath_! \jmath^! \longrightarrow \mathrm{Id}$$

is exact (the key is that $\imath_* \imath^*$ will be concentrated in degree -1, and this is the analog of Kostant's theorem on the concentration of Lie algebra cohomology). When restricted to S_2 this will say that the sign representation has a two-step resolution by permutation modules,

$$0 \longrightarrow \mathrm{Ind}_{S_2}^{S_2} \mathrm{triv} \longrightarrow \mathrm{Ind}_{S_1 \times S_1}^{S_2} \mathrm{triv} \longrightarrow \mathrm{sgn} \longrightarrow 0;$$

though this level is of course trivial, it is the additional structure which is interesting.

For higher rank we will need the reconstruction-from-stratification machine again in order to replace the open-closed phenomenon. We will also have to utilize the Brundan-Kleshchev bridge to KLR, as well as Soergel calculus.

I have TeXed the Hecke computations here algebraically rather than diagrammatically, but of course one should think of the diagrammatic computations when doing this.

1. The plan

Consider the degenerate affine Hecke algebra $\widehat{\mathcal{H}}_2$. My convention is that $x_2 s - s x_1 = s x_2 - x_1 s = 1$, i.e. right minus left is straights. The motivation is coming from the action of this algebra on $M \otimes V \otimes V$, where V is the standard representation of a Lie algebra $\mathfrak{g}$, which we will fix now to be $\mathfrak{g} = \mathfrak{gl}_2$. To abstract away the dependence on rank perhaps we could just say $\mathfrak{g} = \mathfrak{gl}_\delta$. Let A be the central quotient.

$$A := \widehat{\mathcal{H}}_2 \big/ \langle x_1 + x_2 = \delta + \delta - 1,\ x_1 x_2 = \delta(\delta - 1) \rangle.$$

Note that this would be the same as if we had replaced $\widehat{\mathcal{H}}_2$ by the cyclotomic quotient $\widehat{\mathcal{H}}_2/(x_1 - \delta)(x_1 - (\delta - 1))$; you could think of this as a block of a cyclotomic Hecke if you wanted. For concreteness, we could set $\delta = 2$, but I'll try to remain general.

Certainly A contains as a subalgebra $\mathbb{C}S_2$, and so it has two idempotents,

$$e_{\square\square} = \frac{1}{2}(1 + s), \qquad e_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{1}{2}(1 - s).$$

There is a canonical way to partially order these, under $\square\square > \begin{smallmatrix}\square\\\square\end{smallmatrix}$ (we will think of this theory as lowest-weight). Consider then the algebraic recollement:

$$\mathsf{D\,Mod}\, A \big/ A e_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} A \longrightarrow \mathsf{D\,Mod}\, A \longrightarrow \mathsf{D\,Mod}\, e_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} A e_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}.$$

In standard convention let the left adjoint to the last functor be $\jmath_!$ and the left adjoint to the first functor be $\imath^*$. As usual, $\jmath_! = Ae_{\square} \overset{\mathsf{L}}{\otimes}_{e_{\square}Ae_{\square}} \Box$ and $\imath^* = A\big/ Ae_{\square}A \overset{\mathsf{L}}{\otimes}_A \Box$.

Claim 1.1. The 2-dimensional permutation module of S_2 is (the restriction to S_2 of) $\jmath_!$ applied to the 1-dimensional module over $e_{\square}Ae_{\square} \cong \mathbb{C}$.

This explains why the $\jmath_!\jmath^!$ term in the exact triangle is the 2-dimensional permutation module, which then deserves the name $\Delta_{\square}$. The other remaining term requires the claim

Claim 1.2. $\imath_*\imath^*$ applied to the sign representation is concentrated in cohomological degree -1, and after shifting and restricting to S_2, it corresponds to the trivial permutation module, which deserves name $\Delta_{\square\square}$.

Here 'sign representation' means a simple module $A \circlearrowright L_{\square}$, where S_2 acts like the sign module and the action of the dots is $x_1 = \delta,\ x_2 = \delta - 1$.

Then the above claims, together with the exact triangle, will give a resolution

$$0 \longrightarrow \Delta_{\square\square} \longrightarrow \Delta_{\square} \longrightarrow L_{\square} \longrightarrow 0.$$

Remark. Maybe it's worth saying that we should really be calling this

$$0 \longrightarrow \Delta_{\square/s\circ 0} \longrightarrow \Delta_{\square/0} \longrightarrow L_{\square/0} \longrightarrow 0.$$

This is in reference to the work of Arakawa-Suzuki [AS98] and Orellana-Ram [OR04], where $\Delta_{\lambda/\mu}$ was defined as the space of highest-weight vectors of weight λ in $\Delta_\mu \otimes V^{\otimes n}$, where Δ_μ is the Verma of category $\mathcal{O}$; $L_{\lambda/\mu}$ is the same except for $L_\mu \otimes V^{\otimes n}$. In what follows, I will frequently shorthand $\Delta_{\lambda/s\circ 0}$ to $\Delta_{\lambda/s}$, and similarly for L.

2. Description of the algebras

First we must describe what $e_{\square}Ae_{\square}$ is, as well as the quotient $A/e_{\square\square}$.

Claim 2.1. $e_{\square}\widehat{\mathcal{H}}_2 e_{\square} \cong \mathbb{C}[x_1 + x_2, x_1x_2]$, so that in particular after taking a central quotient we have

$$e_{\square}Ae_{\square} \cong \mathbb{C}.$$

This is true because it is the spherical Hecke algebra. In accordance to ideas present in the literature, this should maybe be called $A^{\square}$ a "Cartan algebra". This claim can be seen to be true by direct computation: it is evident that s is absorbed by the idempotents into a minus sign, and one checks that

$$\begin{aligned}
e_{\square}x_1e_{\square} &= e_{\square}e_{\square}x_1e_{\square} \\
&= \frac{1}{2}e_{\square}(x_1 - sx_1)e_{\square} \\
&= \frac{1}{2}e_{\square}(x_1 - x_2s + 1)e_{\square} \\
&= \frac{1}{2}e_{\square}(x_1 + x_2 + 1)e_{\square} \\
\implies \quad x_1 &= \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2} \quad \text{inside } e_{\square}Ae_{\square} \\
x_1 &= x_2 + 1.
\end{aligned}$$

This holds whether A is cyclotomic or not. Then we can always write x_1 as $\frac{(x_1+x_2)+1}{2}$ and x_2 as $\frac{(x_1+x_2)-1}{2}$, both of which are central elements; multiplication works correctly because of centralness.

The quotient by the ideal generated by $e_{\square}$ can also be described.

Claim 2.2. We also have $e_{\square\square}\widehat{\mathcal{H}}_2 e_{\square\square} \cong \mathbb{C}[x_1 + x_2, x_1 x_2]$, so that after our central quotient we have

$$e_{\square\square} A e_{\square\square} \cong \mathbb{C}.$$

Again, this deserves the name $A^{>\substack{\square\\\square}}$, or $A^{\geq\square\square}$, which in this case coincides with $A^{\square\square}$. This can also be proven directly: as $1 = s$, we have in $A/e_{\substack{\square\\\square}}$ that

$$0 = x_2 - x_2 s = x_2 - s x_1 - 1 = x_2 - x_1 - 1.$$

The following is then easy (remembering that $x_1 + x_2$ is in the center):

Fact 2.3.

$$\begin{aligned} e_{\substack{\square\\\square}} x_2 e_{\substack{\square\\\square}} &= e_{\substack{\square\\\square}} (x_1 - 1) e_{\substack{\square\\\square}}, \\ e_{\substack{\square\\\square}} x_2 e_{\square\square} &= -e_{\substack{\square\\\square}} x_1 e_{\square\square}, \\ e_{\square\square} x_2 e_{\substack{\square\\\square}} &= -e_{\square\square} x_1 e_{\substack{\square\\\square}}, \\ e_{\square\square} x_2 e_{\square\square} &= e_{\square\square} (x_1 + 1) e_{\square\square}. \end{aligned}$$

3. Computation of standardization (Claim 1.1)

Consider the action $e_{\substack{\square\\\square}} A e_{\substack{\square\\\square}} \circlearrowright \mathbb{C}_\delta = \mathbb{C}$ where the action of x_1 is $x_1 = \delta$, and consider $\jmath_! \mathbb{C}_\delta$. Firstly morally $\jmath_!$ should be derived, but in this case $A e_{\substack{\square\\\square}}$ is free over $e_{\substack{\square\\\square}} A e_{\substack{\square\\\square}} = \mathbb{C}$, so the following tensor product is exact

$$\jmath_! \mathbb{C}_\delta = A e_{\substack{\square\\\square}} \otimes_{e_{\substack{\square\\\square}} A e_{\substack{\square\\\square}}} \mathbb{C}_\delta.$$

A more precise version of Claim 1.1 now is that

Claim 3.1. $\jmath_! \mathbb{C}_\delta \cong \operatorname{Ind}_{\widehat{\mathcal{H}}_{1,1}}^{\widehat{\mathcal{H}}_2} \mathbb{C}_{x_1 = \delta,\ x_2 = \delta - 1}$, where the A-module $\jmath_! \mathbb{C}_\delta$ is considered as a module over $\widehat{\mathcal{H}}_2$ in the obvious way. In particular, the restriction to S_2 of both of these modules is the 2-dimensional permutation module.

Indeed, the isomorphism will be

$$\begin{aligned} \jmath_! \mathbb{C}_\delta &\longrightarrow \operatorname{Ind}_{\widehat{\mathcal{H}}_{1,1}}^{\widehat{\mathcal{H}}_2} \mathbb{C}_{\delta,\delta-1} \\ e_{\substack{\square\\\square}} \otimes 1 &\longmapsto v_{\substack{\square\\\square}} \\ e_{\square\square} x_1 e_{\substack{\square\\\square}} \otimes 1 &\longmapsto v_{\square\square}, \end{aligned}$$

where $v_{\substack{\square\\\square}}$ is the vector $1 - s \in \operatorname{Ind}_{\widehat{\mathcal{H}}_{1,1}}^{\widehat{\mathcal{H}}_2} \mathbb{C}_{\delta,\delta-1}$, and $v_{\square\square}$ is the vector $1 + s$. First let us consider the action on the RHS; $v_{\substack{\square\\\square}}$ and $v_{\square\square}$ are of course kept in their own spaces by S_2, and under the action of x_1 $v_{\square\square}$ will be a subspace, while $v_{\substack{\square\\\square}}$ will not:

$$\begin{aligned} x_1 v_{\substack{\square\\\square}} &= x_1(1 - s) \\ &= x_1 - s x_2 + 1 \\ &= \delta - (\delta - 1)s + 1 \\ &= \delta(1 - s) + (1 + s) \\ &= \delta v_{\substack{\square\\\square}} + v_{\square\square}, \end{aligned}$$

$$\begin{aligned} x_1 v_{\square\square} &= x_1(1 + s) \\ &= x_1 + s x_2 - 1 \\ &= \delta + (\delta - 1)s - 1 \\ &= (\delta - 1)(1 + s) \\ &= (\delta - 1) v_{\square\square}. \end{aligned}$$

Since x_1+x_2 is in the center of $\widehat{\mathcal{H}}_2$, it is then easy to argue $x_2 v_{\boxminus} = (2\delta-1)v_{\boxminus} - \delta v_{\boxminus} - v_{\boxbar} = (\delta-1)v_{\boxminus} - v_{\boxbar}$ and similarly $x_2 v_{\boxbar} = \delta v_{\boxbar}$.

On the other side (LHS), one can compute

$$\begin{aligned} x_1(e_{\boxminus}\otimes 1) &= e_{\boxminus}x_1e_{\boxminus}\otimes 1 + e_{\boxbar}x_1e_{\boxminus}\otimes 1 \\ &= e_{\boxminus}\otimes e_{\boxminus}x_1e_{\boxminus} + e_{\boxbar}x_1e_{\boxminus}\otimes 1 \\ &= e_{\boxminus}\otimes \delta + e_{\boxbar}x_1e_{\boxminus}\otimes 1 \\ &= \delta(e_{\boxminus}\otimes 1) + (e_{\boxbar}x_1e_{\boxminus}\otimes 1), \end{aligned}$$

$$\begin{aligned} x_1(e_{\boxbar}x_1e_{\boxminus}\otimes 1) &= \frac{1}{2}(x_1+x_1s)x_1e_{\boxminus}\otimes 1 \\ &= \frac{1}{2}(x_1+sx_2-1)x_1e_{\boxminus}\otimes 1 \\ &= \frac{1}{2}(x_1^2-x_1+sx_1x_2)e_{\boxminus}\otimes 1 \\ &= \frac{1}{2}((2\delta-1)x_1-\delta(\delta-1)-x_1)e_{\boxminus}\otimes 1 + \frac{1}{2}sx_1x_2e_{\boxminus}\otimes 1 \\ x_1x_2 \text{ is in the center} \quad &= (\delta-1)x_1e_{\boxminus}\otimes 1 - \binom{\delta}{2}e_{\boxminus}\otimes 1 - \frac{1}{2}e_{\boxminus}\otimes x_1x_2 \\ &= \delta(\delta-1)(e_{\boxminus}\otimes 1) + (\delta-1)e_{\boxbar}x_1e_{\boxminus}\otimes 1 - \binom{\delta}{2}e_{\boxminus}\otimes 1 - \frac{1}{2}e_{\boxminus}\otimes\delta(\delta-1) \\ &= (\delta-1)(e_{\boxbar}x_1e_{\boxminus}\otimes 1). \end{aligned}$$

It is worth saying that the above computation is still true in the absence of the cyclotomic relation $x_1^2 = (2\delta-1)x_1-\delta(\delta-1)$. Anywho, these two computations complete the isomorphism (x_1+x_2 is in the center so it suffices to do x_1).

Remark. This construction gives a module where the trivial module is a subobject and the sign module is a quotient, but not the other way around. It deserves to be called $\Delta_{\boxminus}$. On the flip side, the $\jmath_*$ construction will give $\jmath_*\mathbb{C}_\delta$ is a module which has the trivial as a quotient and the sign as a sub, but not the other way around; this module would then deserve to be called $\nabla_{\boxminus}$, and it is the dual of $\Delta_{\boxminus}$.

The simple we wish to resolve in the end will be the simple $L_{\boxminus/\emptyset}$ in the notation of Suzuki; this is the sign module on which $x_1=\delta$ and $x_2=\delta-1$. Note then that

$$\jmath_!\jmath^! L_{\boxminus/\emptyset} = \jmath_!\mathbb{C}_\delta = \Delta_{\boxminus} = \Delta_{\boxminus/\emptyset}.$$

This is one term of the exact triangle.

4. Computation of "nilcohomology" (Claim 1.2)

In our case note that $e_{\boxbar}A^{\geq\boxbar}e_{\boxbar} = A^{\geq\boxbar} = A/e_{\boxminus}$. Either way, $\imath^* = \mathbb{C}\overset{\mathsf{L}}{\otimes}_A \square$, where $\mathbb{C}\cong A/e_{\boxminus}$. A more precise version of Claim 1.2 is

Claim 4.1. $\imath_*\imath^*L_{\boxminus/\emptyset}$ is homologically concentrated in degree -1, and $\imath_*\imath^*L_{\boxminus/\emptyset}[-1]\cong\Delta_{\boxminus/s\circ 0} = \Delta_{\boxbar}$, where the RHS is defined to be 1-dimensional trivial module upon which x_1 acts by $\delta-1$ and x_2 acts by δ.

To compute this $\imath^*$ we will need to resolve $\mathbb{C}$, thought of as $e_{\boxbar}(A/e_{\boxminus})e_{\boxbar}$, as a right free module over A. The way to do this diagrammatically is to attach boxes corresponding to idempotents on top of A. Algebraically, we can write a resolution of right A-modules:

$$0\longrightarrow e_{\boxminus}A \xrightarrow{e_{\boxbar}x_1e_{\boxminus}\times} e_{\boxbar}A \longrightarrow e_{\boxbar}\mathbb{C}e_{\boxbar}\longrightarrow 0.$$

Indeed, in the quotient of $e_{\boxbar}A$ by the image, $e_{\boxbar}x_1e_{\boxminus}=0$, so that $e_{\boxbar}x_1 = e_{\boxbar}x_1e_{\boxbar}$.

We can then use this to compute $\mathbb{C}\overset{\mathsf{L}}{\otimes}_A L_{\boxminus/\emptyset}$. Note that $e_{\boxbar}A\otimes_A L_{\boxminus}=0$ since $e_{\boxbar}L_{\boxminus}=0$, so immediately we have the concentration result. We can also tell that the degree -1 part should be 1-dimensional, but the action of $\widehat{\mathcal{H}}_2$ is unclear from this, since this resolution is not of left modules. But if we just remember that $\imath^*$ is supposed to land in modules over $A/e_{\substack{\square\\\square}} \cong e_{\square\square}(A/e_{\substack{\square\\\square}})e_{\square\square}$, then we know that the only 1-dimensional representation possible is the trivial one.

Remark. Another (maybe better) way to see the additional structure is to use the $\bar{\imath}_* \jmath_! \jmath^! \bar{\imath}^*$ business. We've seen that $\jmath^! \bar{\imath}^*$ would be 1-dimensional (in this case $\imath = \bar{\imath}$ since there are only two strata), so after applying $\jmath_!$ we get the Verma we want. Note that this is really the same as the fact that

$$\mathrm{Ext}^1_A(\Delta_{\substack{\square\\\square}/s\circ 0}, L_{\substack{\square\\\square}/0}) = \mathrm{Ext}^1_A(L_{\square\square}, L_{\substack{\square\\\square}}) = \mathbb{C},$$

corresponding to $\nabla_{\substack{\square\\\square}}$ being the nontrivial extension of the two simple modules in the order prescribed by the Ext group.

Hence we have a concentrated $\imath_* \imath^* L_{\substack{\square\\\square}}[-1] = \Delta_{\square\square}$, and hence rotation of the exact triangle gives

Fact 4.2. We have a "BGG resolution" coming from a "open-closed phenomenon" (really a stratification)

$$0 \longrightarrow \Delta_{\substack{\square\\\square}/s\circ 0} \longrightarrow \Delta_{\substack{\square\\\square}/0} \longrightarrow L_{\substack{\square\\\square}/0} \longrightarrow 0.$$

Restricted to S_2, letting $\Sigma_{\substack{\square\\\square}}$ be the sign representation, $E_{\square\square} := \mathrm{Ind}_{S_2}^{S_2} \mathrm{triv}$, and $E_{\substack{\square\\\square}} := \mathrm{Ind}_{S_1 \times S_1}^{S_2} \mathrm{triv}$, we have

$$0 \longrightarrow E_{\square\square} \longrightarrow E_{\substack{\square\\\square}} \longrightarrow \Sigma_{\substack{\square\\\square}} \longrightarrow 0.$$

Such is the motivating phenomenon in rank 1.

References

[AS98] Tomoyuki Arakawa and Takeshi Suzuki. Duality between sln(C) and the degenerate affine Hecke algebra. *Journal of Algebra*, 209(1):288–304, 1998.

[OR04] Rosa Orellana and Arun Ram. Affine braids, Markov traces and the category O. *arXiv preprint math/0401317*, 2004.