8. 04/19 – Degenerate Affine Hecke Algebras and Their Representations (Fan Zhou)

There are no notes for last week's online meeting. This week once again I am dead-TeXing. The material is taken from chapters 3 and 4 of Kleschev's book on linear and projective representations of symmetric groups.

8.1. Basic Structure.

The degenerate affine Hecke algebra, in contrast to the (e.g. rational degeneration of the) double affine Hecke algebra, has a PBW basis with only one 'half'. As a vector space it looks like $\Bbbk S_n \otimes_{\Bbbk} \Bbbk[X_1, \cdots, X_n]$. To be more precise,

Definition 8.1.1. The "degenerate affine Hecke algebra" $\widehat{\mathcal{H}}_n$ is defined as

$$\widehat{\mathcal{H}}_n = \Bbbk[S_n, X_1, \cdots, X_n] / \sim,$$

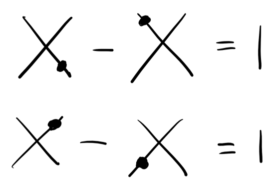
where the relations are that the polynomial ring and the group algebra behave as usual internally, and that when put together they satisfy

$$s_i x_j = x_j s_i \qquad \text{for } j \neq i, i+1$$
$$x_{i+1} s_i - s_i x_i = 1,$$

where the second relation can also be written as

$$s_i x_{i+1} - x_i s_i = 1.$$

Using the usual graphical depictions of crossings as transpositions and dots as X's, the latter relations can be more clearly stated as



Let $\mathcal{P}_n$ denote $\Bbbk[X_1, \cdots, X_n]$. Consider the action $S_n \circlearrowright \mathcal{P}_n$ by

$$w \cdot x_i = x_{w(i)}.$$

Then it is not hard to inductively (on degree) see that

$$s_i f = (s_i \cdot f) s_i + \frac{f - s_i \cdot f}{X_{i+1} - X_i} \tag{$*$}$$

for $f \in \mathcal{P}_n$, so that you can sort of commute polynomials and permutations at a cost. More precisely,

$$wf = (w \cdot f) w + \sum_{u < w} f_u u,$$
$$fw = w(w^{-1} \cdot f) + \sum_{u < w} u f'_u, \tag{$**$}$$

where $\deg f_u, \deg f'_u < \deg f$. So up to lower polynomial-degree terms, polynomials and permutations do (twist-)commute.

It is then intuitively obvious that by using this type of pseudo-commutation you can get a basis of $\widehat{\mathcal{H}}_n$, and this is indeed the case:

Theorem 8.1.2 (PBW)**.** A basis for $\widehat{\mathcal{H}}_n$ over $\Bbbk$ is given by

$$\{x^\alpha w : \alpha \in \mathbb{N}^n, w \in S_n\}.$$

Alternatively you can do

$$\{w x^\alpha : \alpha \in \mathbb{N}^n, w \in S_n\}.$$

This can also be proved using the diamond lemma trick. Note that this gives us a nice injection of $\mathcal{P}_n$ and $\Bbbk S_n$ into $\widehat{\mathcal{H}}_n$. The following is my naive thinking: you would think (if we pretend for instance that $\widehat{\mathcal{H}}_n$ is 'half' of the rational Cherednik) that the analogue of the Cartan would ought to be the semisimple 'middle' part, namely $\Bbbk S_n$, but later we will see that the weight theory is in fact analyzed using the (non-semisimple!) polynomial subalgebra $\mathcal{P}_n$. This is surprising to me, and I wonder what the story looks like (or maybe rather what goes wrong) if you let the simples of $\Bbbk S_n$ serve as 'weight spaces'. Possibly the take-away is that 'mutually commuting' is more important than 'semisimple'.

Another structure result on these algebras is that

Proposition 8.1.3. The center of $\widehat{\mathcal{H}}_n$ is precisely the symmetric polynomials, i.e.
$$Z\widehat{\mathcal{H}}_n = \Bbbk[X_1, \cdots, X_n]^{S_n} =: \mathcal{P}_n^S.$$

Proof. This is an easy check. Clearly symmetric polynomials lie in the center from the starred equations earlier. And if $z = \sum P(X)\sigma$ is central, then by taking the largest (according to Bruhat on permutations) term $P_0(X)\sigma_0$ and choosing i such that $\sigma_0(i) \neq i$, you can compute (again with the starred equations) that $x_i z - z x_i$ is nonzero (due to the basis theorem). Hence picking $\sigma(i) \neq i$ is impossible and $z \in \mathcal{P}_n$. Lastly by thinking of this polynomial as e.g. a polynomial in X_1, X_2 with coefficients in $\Bbbk[X_3, \cdots, X_n]$ and writing $s_1 z - z s_1 = 0$ one can see that z must be symmetric in X_1, X_2, and more generally it must be symmetric in all variables. ■

This is relevant because we will later think of 'central characters' and 'blocks'.

Like in the case of semisimple Lie algebras, there is an antiautomorphism of $\widehat{\mathcal{H}}_n$ given by

$$\begin{aligned} \tau \colon \widehat{\mathcal{H}}_n &\longrightarrow \widehat{\mathcal{H}}_n \\ s_i &\longmapsto s_i \\ X_i &\longmapsto X_i \end{aligned}$$

for appropriate i. This gives duality – for $\widehat{\mathcal{H}}_n \circlearrowright M$, this allows for $\widehat{\mathcal{H}}_n \circlearrowright M^*$. The book denotes this as M^τ, but I will stick to M^*. In fact, it turns out that

$$(\mathrm{Ind}_{m,n}^{m+n} M \boxtimes N)^* \cong \mathrm{Ind}_{n,m}^{n+m} N^* \boxtimes M^*, \tag{$*$}$$

whose meaning, if not already clear, will be made more clear soon. This is a great time to talk about parabolic things.

8.2. **Parabolic Subalgebras.**

Recall the parabolic subalgebras of the symmetric group algebra (cf. Young subgroups). There is a variant for the degenerate affine Hecke.

Definition 8.2.1. For α a composition of n, let $\widehat{\mathcal{H}}_\alpha \subseteq \widehat{\mathcal{H}}_n$ be the "parabolic subalgebra" generated by all the X's and S_α. This has a basis
$$\widehat{\mathcal{H}}_\alpha = \Bbbk\{x^\beta w : \beta \in \mathbb{N}^n, w \in S_\alpha\}$$
and can be written as
$$\widehat{\mathcal{H}}_\alpha = \widehat{\mathcal{H}}_{\alpha_1} \otimes \cdots \otimes \widehat{\mathcal{H}}_{\alpha_r}.$$

These parabolic subalgebras give rise to functors $\mathrm{Ind}_{\widehat{\mathcal{H}}_\alpha}^{\widehat{\mathcal{H}}_n}$ and $\mathrm{Res}_{\widehat{\mathcal{H}}_\alpha}^{\widehat{\mathcal{H}}_n}$, also denoted Ind_α^n and Res_α^n, which elucidates the previous remark about duals of inductions. Note for example that $\widehat{\mathcal{H}}_{(1,\cdots,1)} \cong \mathcal{P}_n$.

Before moving on, let us recall something which perhaps spoils the coming material. Let α, β be compositions of n, and let

$$\begin{aligned} D_\beta &= \text{set of minimal length representatives of } S_n / S_\beta, \\ D_\alpha^{-1} &= \text{set of minimal length representatives of } S_\alpha \backslash S_n, \end{aligned}$$

so that

$$D_{\alpha,\beta} := D_\alpha^{-1} \cap D_\beta = \text{set of minimal length representatives of } S_\alpha \backslash S_n / S_\beta.$$

The following are facts from the theory of symmetric groups.

Fact 8.2.2. (a) For $w \in D_{\alpha,\beta}$, $S_{\alpha\cap w\beta} := S_\alpha \cap wS_\beta w^{-1}$ and $S_{w^{-1}\alpha\cap\beta} := w^{-1}S_\alpha w \cap S_\beta$ are parabolic subgroups, labeled by compositions which are denoted by $\alpha \cap w\beta$ and $w^{-1}\alpha \cap \beta$ respectively.

(b)

$$\begin{aligned} S_{\alpha\cap w\beta} &\longrightarrow S_{w^{-1}\alpha\cap\beta} \\ x &\longmapsto w^{-1}xw \end{aligned}$$

is a length preserving isomorphism.

(c) Elements of $S_\alpha w S_\beta$ can be uniquely written as xwy with $x \in S_\alpha$ and $y \in S_\beta \cap D^{-1}_{w^{-1}\alpha\cap\beta}$, which is the set of minimal length representatives in ${S_{w^{-1}\alpha\cap\beta}}\backslash S_\beta$.

8.3. **Mackey.** As you may have guessed from the previous discussion, in analogy with the symmetric groups and their parabolic subalgebras, there is a Mackey decomposition formula for the degenerate affine Hecke. However, unlike in the symmetric group case where you get a direct sum, here we can only get a filtration on the restriction of the induction of a module, $\operatorname{Res}\operatorname{Ind} M$, with a description of what the quotients look like. This is a first (or maybe second) glimpse into the non-semisimple nature of this representation theory.

We will construct a filtration on $\widehat{\mathcal{H}}_n$. Let $\mathcal{S}_n := \Bbbk S_n$. Let $\prec$ be a total order[28] refining the Bruhat $<$ on $D_{\alpha,\beta}$, and let (here $u \in D_{\alpha,\beta}$)

$$\begin{aligned} \mathcal{B}_{\preceq w} &= \bigoplus_{u \preceq w} \widehat{\mathcal{H}}_\alpha u \mathcal{S}_\beta, \\ \mathcal{B}_{\prec w} &= \bigoplus_{u \prec w} \widehat{\mathcal{H}}_\alpha u \mathcal{S}_\beta, \\ \mathcal{B}_w &= \mathcal{B}_{\preceq w}\big/\mathcal{B}_{\prec w}. \end{aligned}$$

By using the starred relations earlier, we can move all the polynomials to the left. Then, letting w_0 denote the longest element of $D_{\alpha,\beta}$, we get that elements of $\mathcal{B}_{\preceq w_0}$ look like

$$\sum_u P(X)\sigma u \tau \in \bigoplus_{u \in D_{\alpha,\beta}} \widehat{\mathcal{H}}_\alpha u \mathcal{S}_\beta.$$

But as $\bigoplus_{u \in S_\alpha \backslash S_n / S_\beta} \mathcal{S}_\alpha u \mathcal{S}_\beta = \mathcal{S}_n$, we conclude that

$$\bigoplus_{u \in D_{\alpha,\beta}} \widehat{\mathcal{H}}_\alpha u \mathcal{S}_\beta = \widehat{\mathcal{H}}_n.$$

Furthermore, it is easy to check that $\mathcal{B}_{\preceq w}$ is closed under $\widehat{\mathcal{H}}_\beta$ on the right by using the starred relations from earlier to move the new polynomial to the far left. Hence these $\mathcal{B}$'s give a filtration of $\widehat{\mathcal{H}}_n$ as $(\widehat{\mathcal{H}}_\alpha, \widehat{\mathcal{H}}_\beta)$-bimodules. Moreover, part (b) of 8.2.2 tells us that

$$\begin{aligned} \widehat{\mathcal{H}}_{\alpha\cap w\beta} &\xrightarrow{\sim} \widehat{\mathcal{H}}_{w^{-1}\alpha\cap\beta} \\ \sigma &\longmapsto w^{-1}\sigma w \\ X_i &\longmapsto X_{w^{-1}(i)} \end{aligned}$$

is an isomorphism. In particular, if $\widehat{\mathcal{H}}_{w^{-1}\alpha\cap\beta} \circlearrowright N$, then this morphism allows $\widehat{\mathcal{H}}_{\alpha\cap w\beta} \circlearrowright N$ via $\widehat{\mathcal{H}}_{\alpha\cap w\beta} \xrightarrow{\sim} \widehat{\mathcal{H}}_{w^{-1}\alpha\cap\beta} \longrightarrow \operatorname{End} N$. This module is denoted ${}^wN$.

It turns out we can describe the quotients $\mathcal{B}_w$ quite explicitly.

[28] really keep this in mind, that this is total

Lemma 8.3.1. Consider the actions $\widehat{\mathcal{H}}_\alpha \circlearrowright \widehat{\mathcal{H}}_\alpha \circlearrowleft \widehat{\mathcal{H}}_{\alpha\cap w\beta}$ and $\widehat{\mathcal{H}}_{w^{-1}\alpha\cap\beta} \circlearrowright \widehat{\mathcal{H}}_\beta \circlearrowleft \widehat{\mathcal{H}}_\beta$, so that $\widehat{\mathcal{H}}_\alpha \circlearrowright \widehat{\mathcal{H}}_\alpha \circlearrowleft \widehat{\mathcal{H}}_{\alpha\cap w\beta} \circlearrowright {}^w\widehat{\mathcal{H}}_\beta \circlearrowleft \widehat{\mathcal{H}}_\beta$. Then

$$\mathcal{B}_w = \widehat{\mathcal{H}}_\alpha \otimes_{\widehat{\mathcal{H}}_{\alpha\cap w\beta}} {}^w\widehat{\mathcal{H}}_\beta$$

as $(\widehat{\mathcal{H}}_\alpha, \widehat{\mathcal{H}}_\beta)$-bimodules.

Proof. Consider the map

$$\begin{aligned} \widehat{\mathcal{H}}_\alpha \otimes_{\widehat{\mathcal{H}}_{\alpha\cap w\beta}} {}^w\widehat{\mathcal{H}}_\beta &\longrightarrow \mathcal{B}_{\preceq w}/\mathcal{B}_{\prec w} \\ x_1 \otimes x_2 &\longmapsto x_1 w x_2. \end{aligned}$$

Think of $x_i = \sum P_i(X)\sigma_i$; by moving $P_2(X)$ past w and σ_1 using the earlier relations, one can see that this actually lands in the target $\mathcal{B}_w$. By comparing the actions on $x_1 \otimes yx_2$ and $x_1 y \otimes x_2$ for $y = \sum Q(X)\tau$, one can check that they agree up to lower terms in $\mathcal{B}_{\prec w}$. So this gives a legit map.

But by part (c) of 8.2.2, $\{X^\gamma \sigma \otimes \tau\}$ for $\sigma \in S_\alpha$ and $\tau \in S_\beta \cap D^{-1}_{w^{-1}\alpha\cap\beta}$ gives a $\Bbbk$-basis of $\widehat{\mathcal{H}}_\alpha \otimes_{\widehat{\mathcal{H}}_{\alpha\cap w\beta}} {}^w\widehat{\mathcal{H}}_\beta$ (again moving all polynomials to the left). Under the above map, this clearly gets sent to a basis of $\mathcal{B}_w$ (being $\widehat{\mathcal{H}}_\alpha w \mathcal{S}_\beta$). Hence this map is an isomorphism. ∎

Given this, Mackey is easy.

Theorem 8.3.2. For $\widehat{\mathcal{H}}_\beta \circlearrowright M$, the module $\operatorname{Res}^n_\alpha \operatorname{Ind}^n_\beta M$ admits a filtration with subquotients

$$\operatorname{Ind}^\alpha_{\alpha\cap w\beta} {}^w(\operatorname{Res}^\beta_{w^{-1}\alpha\cap\beta} M)$$

over $w \in D_{\alpha,\beta}$. The order of this can be changed up to the refinement $\prec$, and in particular $\operatorname{Ind}^\alpha_{\alpha\cap\beta} \operatorname{Res}^\beta_{\alpha\cap\beta} M$ is a submodule.

Proof. The restriction is equivalent to $\widehat{\mathcal{H}}_n \otimes_{\widehat{\mathcal{H}}_\beta} M$, where $\widehat{\mathcal{H}}_n$ is considered a $(\widehat{\mathcal{H}}_\alpha, \widehat{\mathcal{H}}_\beta)$-bimodule. Hit $\widehat{\mathcal{H}}_n$ with the filtration[29] $\mathcal{B}$; the subquotients in the resulting filtration on $\operatorname{Res}\operatorname{Ind} M$ are $\mathcal{B}_w \otimes_{\widehat{\mathcal{H}}_\beta} M = \widehat{\mathcal{H}}_\alpha \otimes_{\widehat{\mathcal{H}}_{\alpha\cap w\beta}} {}^w\widehat{\mathcal{H}}_\beta \otimes_{\widehat{\mathcal{H}}_\beta} M = \operatorname{Ind}^\alpha_{\alpha\cap w\beta} {}^w(\operatorname{Res}^\beta_{w^{-1}\alpha\cap\beta} M)$. ∎

8.4. **Representations.** Note well all the analogies to how the basic foundations of category $\mathcal{O}$ are laid. Let us preoccupy ourselves with finite-dimensional representations.

8.4.1. *Weight theory.* As we mentioned earlier, weight theory will be studied using the polynomial subalgebra rather than the symmetric subalgebra. Maybe we need $\Bbbk$ to be algebraically closed here. For $a \in \Bbbk$, let $\Bbbk[X_1] \circlearrowright \Bbbk_a$ be the 1-dimensional representation on which X_1 acts by a. Then, as $\mathcal{P}_n \cong \mathcal{P}_1^{\otimes n}$, we exhaust the simples of $\mathcal{P}_n$ by considering

$$\mathcal{P}_n \circlearrowright \Bbbk_{\lambda_1} \boxtimes \cdots \boxtimes \Bbbk_{\lambda_n} =: \Bbbk_\lambda.$$

As $\mathcal{P}_n$ can be considered (the universal enveloping algebra of) a nilpotent Lie algebra (with all brackets zero), we know its representation theory is non-semisimple, so knowing the simples isn't everything; but it almost is, due to Lie's (or Engel's, I forget). Indeed, for $\mathcal{P}_n \circlearrowright M$ and $\lambda \in \Bbbk^n$, let

$$M^\lambda := \bigcap_i \operatorname{Ker}^\infty((X_i - \lambda_i) \circlearrowright M)$$

be the common eventual eigenspace for the mutually commuting X_i. Alternatively you can think of this as the largest $\mathcal{P}_n$-submodule whose Jordan-Holder subquotients are all $\Bbbk_\lambda$. Then, as usual,

$$M = \bigoplus_{\lambda\in\Bbbk^n} M^\lambda$$

as $\mathcal{P}_n$-modules. So think of $\Bbbk^n$ as the weight lattice/space.

If $\widehat{\mathcal{H}}_n \circlearrowright M$, you can then again define the formal character as

$$\chi_M := [\operatorname{Res}^n_{(1,\cdots,1)} M] \in K_0(\mathsf{Mod}\,\mathcal{P}_n),$$

[29] implicitly here we are using that $\widehat{\mathcal{H}}_n$ is a free $\widehat{\mathcal{H}}_\beta$-module

which is really a roundabout way of saying that it is the same information as usual – it keeps track of dimensions of weight spaces. Later it turns out this gives an injection from $K_0(\mathsf{Mod}\,\widehat{\mathcal{H}}_n)$ to $K_0(\mathsf{Mod}\,\mathcal{P}_n)$. The following two little results on characters are easy consequences of Mackey, respectively to $\alpha = \beta = (1, \cdots, 1)$ and $\alpha = (1, \cdots, 1)$ and $\beta = (m, n)$.

Lemma 8.4.1. Let $w^{-1}\lambda = (\lambda_{w^{-1}1}, \cdots, \lambda_{w^{-1}n})$. Then

$$\chi_{\operatorname{Ind}_{\mathcal{P}_n}^{\widehat{\mathcal{H}}_n} \Bbbk_\lambda} = \sum_{w \in S_n} [\Bbbk_{w^{-1}\lambda}].$$

If $\widehat{\mathcal{H}}_m \circlearrowright M$ and $\widehat{\mathcal{H}}_n \circlearrowright N$, then

$$\chi_{\operatorname{Ind}_{m,n}^{m+n} M \boxtimes N} = \sum_{\mu \in \Bbbk^m} \sum_{\nu \in \Bbbk^n} \sum_{\lambda \text{ shuffled from } \mu,\nu} \dim(M^\mu)\dim(N^\nu)[\Bbbk_\lambda].$$

In the above, that λ is shuffled from μ, ν means that $\lambda \in \Bbbk^{m+n}$ and there exists a substring of λ which is a permutation of μ such that the compliment is a permutation of ν.

8.4.2. *Blocks.* Recall that the center of $\widehat{\mathcal{H}}_n$ is the symmetric polynomials. For $\lambda \in \Bbbk^n$, define

$$\begin{aligned} \vartheta_\lambda \colon Z\widehat{\mathcal{H}}_n &\longrightarrow \Bbbk \\ f &\longmapsto f(\lambda_1, \cdots, \lambda_n). \end{aligned}$$

Consider the action of $S_n \circlearrowright \Bbbk^n$ by permutations, in analogy to the (twisted) Weyl action on $\mathfrak{h}^*$, and say $\lambda \sim \mu$ if $w\lambda = \mu$. Then it is clear that

$$\vartheta_\lambda = \vartheta_\mu \iff \lambda \sim \mu.$$

Such things are called "central characters", as usual. Given such a character ϑ and a module over $\widehat{\mathcal{H}}_n$, define

$$M^\vartheta := \{v \in M : (z - \vartheta(z))^\infty v = 0 \ \ \forall\, z \in Z\widehat{\mathcal{H}}_n\} = \bigcap_z \operatorname{Ker}^\infty(z - \vartheta(z)).$$

This is precisely the usual definition in category $\mathcal{O}$. This is a submodule as usual, since $z - \vartheta(z)$ is central. And as $Z\widehat{\mathcal{H}}_n \circlearrowright \Bbbk_\lambda$ via ϑ_λ, we know

$$M^{\vartheta_\lambda} = \bigoplus_{\mu \sim \lambda} M^\mu,$$

so that

$$M = \bigoplus_{\lambda \in \Bbbk^n / S_n} M^{\vartheta_\lambda}.$$

As usual, this breaks up $\mathsf{Mod}\,\widehat{\mathcal{H}}_n$ into a direct sum of categories whose objects are those modules for which $M = M^\vartheta$, and indecomposables lie in precisely one such category. This is block theory.

8.4.3. *Kato.* We can now begin introducing some of the simple modules.

Definition 8.4.2. Let $a \in \Bbbk$. The "Kato module" is defined as

$$L_{a^n} := \operatorname{Ind}_{\mathcal{P}_n}^{\widehat{\mathcal{H}}_n} \Bbbk_{(a,\cdots,a)}.$$

Note that

$$\chi_{L_{a^n}} = n![\Bbbk_{(a,\cdots,a)}].$$

Lemma 8.4.3. Note $L_{a^n} = \widehat{\mathcal{H}}_n \otimes_{\mathcal{P}_n} \Bbbk_{(a,\cdots,a)}$. The claim is that the common a-eigenspace of the X_i is precisely $1 \otimes \Bbbk_{(a,\cdots,a)}$, i.e.

$$\bigcap_{i=1}^{n-1} \operatorname{Ker}((X_i - a) \circlearrowright L_{a^n}) = 1 \otimes_{\mathcal{P}_n} \Bbbk_{(a,\cdots,a)},$$

and the Jordan blocks of the action of X_1 are of size n.

Proof. Let S'_{n-1} be the subgroup of S_n be generated by $s_2, \cdots, s_{n-1}$. The key fact is that any $w \in S_n$ can be written as $us_1 \cdots s_i$ for some $u \in S'_{n-1}$ and $0 \le i < n$. To compute the X_1 a-eigenspace, note that by PBW we can write elements of $\widehat{\mathcal{H}}_n \otimes_{\mathcal{P}_n} \Bbbk_{a^n}$ as sums of $w \otimes v$ for $w \in S_n$. Writing $w = us_1 \cdots s_i$, we can take the expression $(X_1 - a)us_1 \cdots s_i \otimes v$ and keep moving $X_1 - a$ to the right by using the defining relations of $\widehat{\mathcal{H}}_n$ (namely the dots versus crossings; X_1 commutes with $u \in S'_{n-1}$). This gives

$$(X_1 - a)us_1 \cdots s_i \otimes v = -us_1 \cdots s_{i-1} \otimes v + \sum_{j<i-1} \text{blah} \cdot u's_1 \cdots s_j \otimes v.$$

Then if $\sum cw \otimes v$ for constants c is in the a-eigenspace of X_1, then picking out the term where w has the longest string to the right of $u \in S'_{n-1}$ (i.e. largest i in the discussion above), we see that we get a term like $-us_1 \cdots s_{i-1} \otimes v$ which cannot be cancelled (as all other terms have strictly less than $i-1$ things to the right of S'_{n-1}), unless $i = 0$. Hence the a-eigenspace of X_1 looks like $\Bbbk S'_{n-1} \otimes \Bbbk_{a^n}$.

Repeating this argument, the a-eigenspace of X_2 looks like $\Bbbk\langle s_3, \cdots, s_{n-1}\rangle \otimes \Bbbk_{a^n}$, etc., until the a-eigenspace of X_{n-1} looks like $1 \otimes \Bbbk_{a^n}$. Hence the intersection is just $1 \otimes \Bbbk_{a^n}$. Note how the indexing on X must stop at $n-1$. See the book for the last claim. ∎

We can now state and prove the main structure theorem for Kato modules.

Theorem 8.4.4. Let $|\lambda| = n$ be a composition of n. Then L_{a^n} is the only simple in its block, and all JH subquotients of $\operatorname{Res}^n_\lambda L_{a^n}$ are isomorphic to

$$L_{a^{\lambda_1}} \boxtimes \cdots \boxtimes L_{a^{\lambda_r}}.$$

Moreover $\operatorname{Soc} \operatorname{Res}^n_\lambda L_{a^n}$ is irreducible, and $\operatorname{Soc} \operatorname{Res}^n_{n-1} L_{a^n} \cong L_{a^{n-1}}$.

Proof. Consider the restriction of L_{a^n} to $\mathcal{P}_n$; it must contain some simple submodule (maybe itself), which must then be isomorphic to $\Bbbk_{a^n}$ (e.g. by block considerations, since a^n is in its own S_n-orbit): $\operatorname{Res}_{\mathcal{P}_n} L_{a^n} \supseteq N \cong \Bbbk_{a^n}$. Then, since all the X's act on N by a, in particular it is in the common a-eigenspace of $X_1, \cdots, X_{n-1}$, so $N \subseteq 1 \otimes \Bbbk_{a^n}$ as a subspace of $\widehat{\mathcal{H}}_n \otimes \Bbbk_{a^n}$. But they're both 1-dimensional, so they must be the same. So $N = 1 \otimes \Bbbk_{a^n}$ is a subspace of L_{a^n}, but clearly the $\widehat{\mathcal{H}}_n$-action on $1 \otimes \Bbbk_{a^n}$ will generate all of $L_{a^n} = \widehat{\mathcal{H}}_n \otimes \Bbbk_{a^n}$. Hence L_{a^n} is simple.

That the JH subquotients of $\operatorname{Res}^n_\lambda L_{a^n}$ look like as claimed follows from a character computation. What is more interesting is that the socle contains exactly one simple. Note that the submodule $\widehat{\mathcal{H}}_\lambda \otimes_{\mathcal{P}_n} \Bbbk_{a^n} \subseteq \operatorname{Res}^n_\lambda L_{a^n}$ is isomorphic to $L_{a^{\lambda_1}} \boxtimes \cdots \boxtimes L_{a^{\lambda_r}}$, which is simple (being a tensor of simples), so this must be in the socle. On the other hand, by repeating the argument in the previous paragraph, we know any submodule of $\operatorname{Res}_\lambda L_{a^n}$ must have a submodule which is identified with $1 \otimes \Bbbk_{a^n}$, which under the $\widehat{\mathcal{H}}_\lambda$-generation gives a copy of $\widehat{\mathcal{H}}_\lambda \otimes_{\mathcal{P}_n} \Bbbk_{a^n}$; so any simple submodule of $\operatorname{Res}_\lambda L_{a^n}$ must contain $\widehat{\mathcal{H}}_\lambda \otimes_{\mathcal{P}_n} \Bbbk_{a^n}$, so it must be the only simple. ∎