THE COMPOSITION LAW FOR CYCLE INDEX SERIES VIA ANALYTIC SPECIES

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In this expository paper we will consider the idea of a combinatorial species (which is defined as a functor) and its analytic functor (a left Kan extension) and use the theory of symmetric functions to prove a fundamental result on the cycle index series of a composition of species. We will then demonstrate its power by annihilating the Cayley tree theorem.

Contents

1. Combinatorial Species

1.1. **Vanilla Species.** The definition of a combinatorial species will succinctly make rigorous the idea of a "family of combinatorial structures".

Definition. A *(set)-species* is a functor

$$\mathcal{F}\colon \operatorname{Cor}(\operatorname{fnt}\mathbf{Set}) \longrightarrow \operatorname{Cor}(\operatorname{fnt}\mathbf{Set})$$

from the category of finite sets with bijections to itself (here we use the Cor notation from nLab).

The functoriality of this definition succinctly encapsulates the following information: the species $\mathcal{F}$ will

- associate to each set U (the underlying set) a set of structures[1] $\mathcal{F}[U]$:

$$\mathcal{F}\colon U \longmapsto \mathcal{F}[U]$$

  in a way such that

[1]This is the notation used in literature.

- if $\sigma\colon U \to V$ is a bijection (thought of as a renaming) then there is a map (sometimes called the *transport of structures*)

$$\mathcal{F}[\sigma]\colon \mathcal{F}[U] \longrightarrow \mathcal{F}[V]$$

which satisfies

$$\mathcal{F}[\sigma \circ \pi] = \mathcal{F}[\sigma] \circ \mathcal{F}[\pi], \qquad \mathcal{F}[\mathrm{id}_U] = \mathrm{id}_{\mathcal{F}[U]}.$$

That is, a renaming of the underlying set produces an appropriate renaming on the structure set.

Example: An example of a species is the species of sets. This species `Set` will associate to every underlying set U:

$$\texttt{Set}[U] = U.$$

It is clear that this is functorial, with $\texttt{Set}[\sigma] = \sigma$.

Example: Another example is `X`, which has

$$\texttt{X}[U] = \begin{cases} U & |U| = 1 \\ \emptyset & \text{else} \end{cases}.$$

It is also clear this is functorial.

The point of a species is that it allows us to count the number of a certain type of structures. To this end, a common construction given a species $\mathcal{F}$ is to associate the exponential formal series

$$F(x) \coloneqq \sum_{n=0}^{\infty} |\mathcal{F}[n]| \frac{x^n}{n!},$$

where we use the shorthand $\mathcal{F}[n] = \mathcal{F}[[n]]$ the set of structures built on the set $[n]$. It is clear that the number of structures is independent of names of the underlying set (σ a bijection gives rise to $\mathcal{F}[\sigma]$ in the core category of finite sets which must therefore also be a bijection, so $\mathcal{F}[U]$ and $\mathcal{F}[V]$ are equal in size).

The idea is that species formalizes the relationship between relations between combinatorial structures and relations between their formal series. For example, if we wanted to count the number of permutations, we could note that permutations are sets of cycles (composition, so to speak), compose their power series, and read off the coefficients to obtain an answer.

But there is a finer construction, which we will jump to. Given a permutation σ on the underlying set, the corresponding $\mathcal{F}[\sigma]$ permutation on $\mathcal{F}[U]$ will have a cycle type depending only on that of σ. This follows from functoriality, as $\mathcal{F}[\pi\sigma\pi^{-1}] = \mathcal{F}[\pi] \circ \mathcal{F}[\sigma] \circ \mathcal{F}[\pi]^{-1}$, so that $\mathcal{F}[\pi\sigma\pi^{-1}]$ is the same cycle type as $\mathcal{F}[\sigma]$. Hence[2] given a vector[3] of nonnegative integers λ describing a cycle type, it makes sense to speak of $\operatorname{fix} \mathcal{F}[\lambda] = (\mathcal{F}[\lambda])_1$.

To a species $\mathcal{F}$ we then associate

Definition. The *cycle index series* of $\mathcal{F}$ is given by

$$Z_{\mathcal{F}} = \sum_{\lambda} \operatorname{fix} \mathcal{F}[\lambda] \frac{p_\lambda}{\mathbb{Z}_+^\lambda \lambda!},$$

where λ is a vector of nonnegative integers, $p_\lambda = \prod_i p_i^{\lambda_i}$ for $p_i(X) = \sum_{x\in X} x^i$ with X the (infinite) set of variables, and $\mathbb{Z}_+^\lambda \lambda! = \prod_n n^{\lambda_n} \lambda_n!$ is a shorthand.

Note that this is then a symmetric function on the infinite set of variables X by construction. This will be crucial in what follows. At first it may seem that this definition of p_λ is strange, but note

[2]Note in the above we could have taken $V \xrightarrow{\pi} U \xrightarrow{\sigma} U \xrightarrow{\pi^{-1}} V$ to show this does not depend on the label set U either.

[3]Note we do not require λ to be a partition in the sense that $\lambda_1 \geq \lambda_2 \geq \cdots$.

that here we have no weakly decreasing condition on λ and that λ conveys a cycle type; if we rearrange this information into $\lambda^{\vdash} = \{|\text{cycle}|\}_{\text{cycle}\in\sigma}$, this is then a partition, and it is clear that $p_\lambda = p_{\lambda^\vdash}$ where the latter is as defined in $p_{\lambda^\vdash} = \prod_i p_{\lambda_i^\vdash}$. For example, $\lambda = (1,2,1)$ gives a partition $\lambda^\vdash = (3,2,2,1)$. I hope the reader agrees this is a reasonable abuse of notation. Note also the resemblance to the formula for the Schur functions:

$$S_\lambda = \sum_{\mu\vdash|\lambda|_1} \chi_{\Sigma_\lambda}(\mu)\frac{p_\mu}{\mathbb{Z}_+^\mu \mu!}.$$

This expository paper will be dedicated to proving a property of $Z_\mathcal{F}$. It turns out that discussion of $Z_\mathcal{F}$ subsumes discussion of $F(x)$, since one can observe[4]

$$Z_\mathcal{F}(p_1,\cdots,p_\infty)|_{(x,0,\cdots)} = Z_\mathcal{F}(x,0,\cdots) = \sum_{\lambda_1} \operatorname{fix}\mathcal{F}[(\lambda_1,0,\cdots)]\frac{x^{\lambda_1}}{\lambda_1!} = \sum_n |\mathcal{F}[n]|\frac{x^n}{n!},$$

where we observe that $\operatorname{fix}\mathcal{F}[(\lambda_1,0,\cdots)] = |\mathcal{F}[\lambda_1]|$ since all structures are fixed by the permutation with λ_1 1-cycles and nothing else.

1.2. **Operations on Species.** A large part of the allure of species comes from the way in which it handles operations. Some examples of operations are as such:

Definition. Given $\mathcal{F},\mathcal{G}$ species, we define:

- the sum $\mathcal{F}+\mathcal{G}$ as

$$(\mathcal{F}+\mathcal{G})[U] = \mathcal{F}[U]\coprod\mathcal{G}[U],$$

$$(\mathcal{F}+\mathcal{G})[\sigma](s) = \begin{cases}\mathcal{F}[\sigma](s) & s\in\mathcal{F}[U]\\ \mathcal{G}[\sigma](s) & s\in\mathcal{G}[U]\end{cases},$$

  where $\sigma\colon U\longrightarrow V$;
- the product $\mathcal{F}\cdot\mathcal{G}$ as

$$(\mathcal{F}\cdot\mathcal{G})[U] = \coprod_{U_1,U_2\vDash U}\mathcal{F}[U_1]\times\mathcal{G}[U_2],$$

$$(\mathcal{F}\cdot\mathcal{G})[\sigma](s) = \big(\mathcal{F}[\sigma|_{U_1}](f),\mathcal{G}[\sigma|_{U_2}](g)\big),$$

  where U_1 and U_2 partition U (denoted by $\vDash$), $\sigma|_{U_1}$ is the restriction of the bijection to U_1, and f,g are $\mathcal{F},\mathcal{G}$ structures;
- the derivative $\mathcal{F}'$ as

$$(\mathcal{F}')[U] = \mathcal{F}[U+\{*\}],$$

$$(\mathcal{F}')[\sigma](s) = \mathcal{F}[\widehat{\sigma}](s),$$

  where $\widehat{\sigma}$ is the natural extension to $U+\{*\}$ defined by $\widehat{\sigma}|_U = \sigma$ and $\widehat{\sigma}(*) = *$;
- the composition $\mathcal{F}\circ\mathcal{G}$ as

$$(\mathcal{F}\circ\mathcal{G})[U] = \coprod_{\pi\vDash U}\mathcal{F}[\pi]\times\prod_{\pi_i\in\pi}\mathcal{G}[\pi_i],$$

$$(\mathcal{F}\circ\mathcal{G})[\sigma](s) = (\widehat{\sigma}(\pi),\mathcal{F}[\widehat{\sigma}](f),\{\mathcal{G}[\sigma|_{\pi_i}]g_i\}_i),$$

  where $s = (\pi,f,\{g_i\}_i)$ and $\widehat{\sigma}$ is the induced map on the partition π, so that $\widehat{\sigma}(\pi)\vDash V$.

There are many other operations one could consider, such as pointing, Cartesian product, functorial composition, etc., but for basic enumerative purposes the above moves will suffice.

[4]The inclusion of p_∞ below means nothing and is purely stylistic.

The magical thing is that the above correspond to moves on the cycle index series. It is a fact that

Theorem.

$$
\begin{aligned}
Z_{\mathcal{F}+\mathcal{G}} &= Z_{\mathcal{F}} + Z_{\mathcal{G}}, \\
Z_{\mathcal{F}.\mathcal{G}} &= Z_{\mathcal{F}} Z_{\mathcal{G}}, \\
Z_{\mathcal{F}'} &= \frac{\partial}{\partial p_1} Z_{\mathcal{F}}, \\
Z_{\mathcal{F}\circ\mathcal{G}} &= Z_{\mathcal{F}} \circ Z_{\mathcal{G}}.
\end{aligned}
$$

□

The last statement may seem difficult to make sense of; that is the object of this paper. In contrast, the other three relations are not difficult to show, and as the proofs do not have as much to do with symmetric function theory we will skip them. They are included above mostly for completeness' sake.

It turns out proving the composition rule just for the series $F(x) = \sum_{n=0}^{\infty} |\mathcal{F}[n]| \frac{x^n}{n!}$ is not so difficult, but the result for the cycle index series is a little less trivial and also of greater theoretical importance. It is not hard to believe that $Z_{\mathcal{F}}$ contains more information than just $F(x)$, and indeed one example of a greater power lent by the cycle index is its information on what is called the "type generating series". We will not have space to cover that here.

1.3. **Weighted Species.** We can also consider a variation on the vanilla flavor of species. For a commutative ring with unit $R \supseteq \mathbb{Q}$, we can define the category $\mathbf{Set}_R$ as follows:

Definition. Let $\mathbf{Set}_R$ be the category with the following information:

$$
\begin{aligned}
\operatorname{Obj}(\mathbf{Set}_R) &= \Big\{ (A, w) \ \text{ s.t. } \ w \colon A \to R, \ |A|_w := \sum_{a \in A} w(a) \in R \Big\}, \\
\operatorname{Hom}(\mathbf{Set}_R) &= \Big\{ f \colon (A, w) \to (B, v) \ \text{ s.t. } \ v(f(a)) = w(a) \ \ \forall\, a \Big\}.
\end{aligned}
$$

We can then define a weighted species.

Definition. A *weighted species* is a functor

$$
\mathcal{F} \colon \operatorname{Cor}(\operatorname{fnt} \mathbf{Set}) \longrightarrow \mathbf{Set}_R.
$$

It follows from functoriality again that we can speak of $|\operatorname{Fix} \mathcal{F}[\lambda]|_w := \operatorname{fix}_w \mathcal{F}[\lambda]$ for a cycle type λ. This is because $\mathcal{F}[\sigma \circ \sigma^{-1}] = \mathcal{F}[\sigma] \circ \mathcal{F}[\sigma^{-1}]$ implies $\mathcal{F}[\sigma]$ is a bijection of sets, and moreover as a morphism in $\mathbf{Set}_R$ must preserve weights, so $\mathcal{F}[\sigma]$ is a weight-preserving bijection. This way

$$
\begin{aligned}
|\operatorname{Fix} \mathcal{F}[\sigma]|_w &= \sum_{s \in \operatorname{Fix} \mathcal{F}[\sigma]} w(s) \\
&= \sum_{\mathcal{F}[\pi]^{-1}(s) \in \operatorname{Fix} \mathcal{F}[\sigma]} w(\mathcal{F}[\pi]^{-1}(s)) \\
&= \sum_{\mathcal{F}[\pi]^{-1}(s) \in \operatorname{Fix} \mathcal{F}[\sigma]} w(s) \\
&= \sum_{s \in \operatorname{Fix} \mathcal{F}[\pi\sigma\pi^{-1}]} w(s)
\end{aligned}
$$

$$= |\mathrm{Fix}\,\mathcal{F}[\pi\sigma\pi^{-1}]|_w,$$

where we note $\mathcal{F}[\pi]^{-1}(s) \in \mathrm{Fix}\,\mathcal{F}[\sigma] \iff \mathcal{F}[\sigma]\mathcal{F}[\pi]^{-1}(s) = \mathcal{F}[\pi]^{-1}(s) \iff \mathcal{F}[\pi]^{-1}\mathcal{F}[\sigma]\mathcal{F}[\pi](s) = \mathcal{F}[\pi\sigma\pi^{-1}](s) = s$.

One can then construct

Definition. The *cycle index series* for a weighted species $\mathcal{F}$ is given by

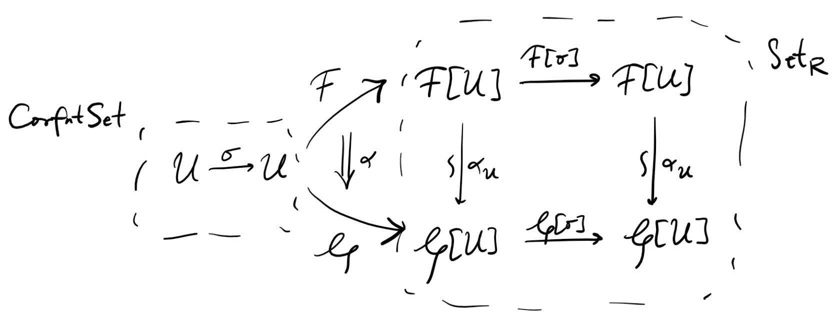
$$Z_{\mathcal{F}} = \sum_{\lambda} \mathrm{fix}_w\,\mathcal{F}[\lambda] \frac{p_\lambda}{\mathbb{Z}_+^\lambda \lambda!}.$$

Remark: For an m-sort weighted species $\mathcal{F}\colon \mathrm{Cor}(\mathrm{fnt}\,\mathbf{Set})^{\times m} \longrightarrow \mathbf{Set}_R$, one can define

$$Z_{\mathcal{F}(X_1,\cdots,X_m)} = \sum_{\lambda_1,\cdots,\lambda_m} \cdots \sum \mathrm{fix}_w\,\mathcal{F}[\lambda_1,\cdots,\lambda_m] \frac{p_{\lambda_1}(X_1)}{\mathbb{Z}_+^{\lambda_1}\lambda_1!} \cdots \frac{p_{\lambda_m}(X_m)}{\mathbb{Z}_+^{\lambda_m}\lambda_m!},$$

where $X_1,\cdots,X_m$ are distinct sets of variables and λ_i are each vectors. We will not pursue this track further.
Remark: Note we can set $w(S) = |S|$ to get back the vanilla species.

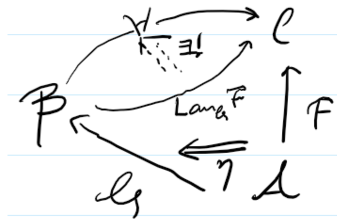
It is fairly obvious by chasing categorical definitions that if $\mathcal{F}$ and $\mathcal{G}$ are naturally isomorphic species, then $Z_{\mathcal{F}} = Z_{\mathcal{G}}$. It suffices to check that $\mathrm{fix}_w\,\mathcal{F}[\sigma] = \mathrm{fix}_w\,\mathcal{G}[\sigma]$, which is true since the diagram is



Since the α_U are isomorphisms, we get a bijection between $\mathrm{Fix}\,\mathcal{F}[\sigma]$ and $\mathrm{Fix}\,\mathcal{G}[\sigma]$; since the isomorphisms α_U are moreover required to be weight preserving (being morphisms in $\mathbf{Set}_R$), we get the sum of weights over $\mathrm{Fix}\,\mathcal{F}[\sigma]$ and over $\mathrm{Fix}\,\mathcal{G}[\sigma]$ are the same. This gives that the cycle index series are also the same.

2. Analytic Functors

2.1. **Abstract Nonsense (can be skipped).** Before we go on to give the concrete way to think of analytic functors, we should maybe say that this analytic functor is a specific instance of a more general, more abstract phenomenon named the "Kan extension". In particular we consider the "left Kan extension of $\mathcal{F}$ along $\mathcal{G}$", which we call $\mathrm{Lan}_{\mathcal{G}}\,\mathcal{F}$. Here the setup is we have $\mathcal{F}\colon \mathcal{A} \longrightarrow \mathcal{C}$, $\mathcal{G}\colon \mathcal{B} \longrightarrow \mathcal{C}$, and then the left Kan extension is a functor $\mathrm{Lan}_{\mathcal{G}}\,\mathcal{F}\colon \mathcal{B} \longrightarrow \mathcal{C}$. The universal property is captured via the following picture (unfortunately very difficult to tex)



When $\mathcal{B}$ is a closed monoidal category and $\mathcal{C}$ is enriched over $\mathcal{B}$ and when copowers[5] exist, we can write this as the coend[6]

$$\operatorname{Lan}_{\mathcal{G}} \mathcal{F}(\square) = \int^{X \in \mathcal{A}} \operatorname{Hom}_{\mathcal{B}}(\mathcal{G}X, \square) \bullet \mathcal{F}X.$$

If $\mathcal{B}$ is enriched in **Set**, then we can moreover write this as

$$\operatorname{Lan}_{\mathcal{G}} \mathcal{F}(\square) = \int^{X \in \mathcal{A}} \coprod_{f \in \operatorname{Hom}_{\mathcal{B}}(\mathcal{G}X, \square)} \mathcal{F}X.$$

We should also briefly explain the coend. Unfortunately this takes a lot of definitions, so we'll try to rush it. a "dinatural transformation" is β such that

$$\beta \colon \mathcal{F} \implies \mathcal{G},$$

where

$$\mathcal{F}, \mathcal{G} \colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathcal{B},$$

consisting of

$$\beta_X \colon \mathcal{F}(X, X) \longrightarrow \mathcal{G}(X, X)$$

such that, for a diagram $X \xrightarrow{f} Y$ in $\mathcal{A}$, the following diagram commutes:

$$\begin{array}{ccccc} & \mathcal{F}(X,X) & \xrightarrow{\beta_X} & \mathcal{G}(X,X) & \\ {}^{\mathcal{F}(f,\mathrm{id})}\nearrow & & & & \searrow^{\mathcal{G}(\mathrm{id},f)} \\ \mathcal{F}(Y,X) & & & & \mathcal{G}(X,Y) \\ {}_{\mathcal{F}(\mathrm{id},f)}\searrow & & & & \nearrow_{\mathcal{G}(f,\mathrm{id})} \\ & \mathcal{F}(Y,Y) & \xrightarrow{\beta_Y} & \mathcal{G}(Y,Y) & \end{array}$$

Then one can define the "coend" of a functor

$$\mathcal{F} \colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathcal{B}$$

to be the initial dinatural transformation from the constant functor to $\mathcal{F}$, namely

$$\int^{\mathcal{A}} \mathcal{F} = B \in \mathcal{B}$$

such that

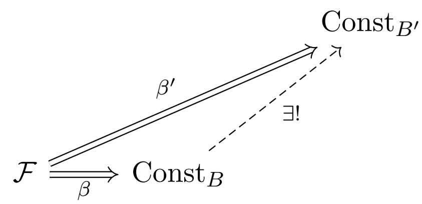
[5]These things are, for $B \in \mathcal{B}$ and $C \in \mathcal{C}$,

$$B \bullet C \in \mathcal{C}$$

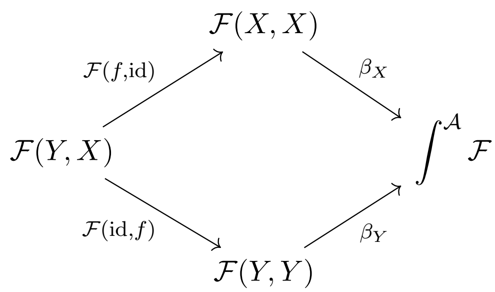
such that

$$\operatorname{Hom}_{\mathcal{C}}(B \bullet C, \square) \overset{\text{nat}}{\cong} \operatorname{Hom}_{\mathcal{B}}(B, \operatorname{Hom}_{\mathcal{C}}(C, \square)).$$

[6]To be explained later.



i.e. the initial object such that the diagram



commutes.

If $\mathcal{B}$ is cocomplete (i.e. small colimits exist) and $\mathcal{A}$ is small (i.e. the objects and morphisms are sets), then the coend can be described as the coequalizer

$$\coprod_{X \xrightarrow{f} Y} \mathcal{F}(Y,X) \underset{\mathcal{F}(\mathrm{id},f)}{\overset{\mathcal{F}(f,\mathrm{id})}{\rightrightarrows}} \coprod_{X \in \mathcal{A}} \mathcal{F}(X,X) \longrightarrow \int^{\mathcal{A}} \mathcal{F}$$

In more concrete terms, this is saying

$$\int^{\mathcal{A}} \mathcal{F} = \coprod_{X \in \mathcal{A}} \mathcal{F}(X,X) \Big/ \mathcal{F}(f,\mathrm{id})s \sim \mathcal{F}(\mathrm{id},f)s.$$

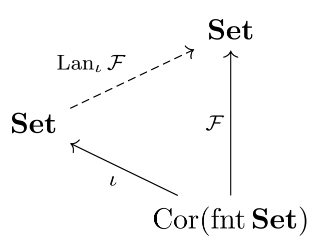
Then, the analytic species associated to a species is simply

$$\widetilde{\mathcal{F}} \colon \mathbf{Set} \longrightarrow \mathbf{Set}$$

defined by

$$\widetilde{\mathcal{F}} \coloneqq \mathrm{Lan}_\iota \mathcal{F}$$

in the diagram



Of course everything described above is rather abstract and maybe doesn't actually give any intuition for what these things actually are, unless one has already drank a lot of categorical juice. So in the next section we discuss how to think about these things concretely. But it is very useful/helpful to read the following below and keep in mind the general form of the coend as this coequalizer, i.e. this coproduct modded out by some relations. In fact this is secretly precisely what the first definition in Section 2.2 is saying.

2.2. **Analytic Functors Associated to Species.** Given a species $\mathcal{F}$, consider the action of the symmetric group on $\mathcal{F}[n] \times \mathrm{Hom}_{\mathrm{Set}}([n], A)$:

$$\begin{aligned} S_n \colon \mathcal{F}[n] \times \mathrm{Hom}_{\mathrm{Set}}([n], A) &\longrightarrow \mathcal{F}[n] \times \mathrm{Hom}_{\mathrm{Set}}([n], A) \\ (s, f) &\longmapsto (\mathcal{F}[\sigma](s), f \circ \sigma^{-1}), \end{aligned}$$

where A is an arbitrary set. We can then define the analytic functor via:

Definition. To each species $\mathcal{F}$ associate the analytic functor

$$\widetilde{\mathcal{F}} \colon \mathbf{Set} \longrightarrow \mathbf{Set}$$

defined via

$$\widetilde{\mathcal{F}}(A) = \coprod_{n=0}^{\infty} \mathcal{F}[n] \times \mathrm{Hom}_{\mathrm{Set}}([n], A) \big/ S_n,$$

where for $g \colon A \longrightarrow B$ in the **Set** category the transport is

$$\begin{aligned} \widetilde{\mathcal{F}}(g) \colon \widetilde{\mathcal{F}}(A) &\longrightarrow \widetilde{\mathcal{F}}(B) \\ \mathrm{orb}_{S_n}(s, f) &\longmapsto \mathrm{orb}_{S_n}(s, g \circ f). \end{aligned}$$

It is readily obvious that a map so defined is indeed a functor[7] and moreover $\widetilde{\mathcal{F}}(g)$ is well-defined[8] (independent of representation). We can generalize this to weighted species, too: the way we give $(s, f) \in \mathcal{F}[n] \times \mathrm{Hom}([n], A)$ a weight is by giving weight

$$w(f) = \prod_{a \in A} w(a)^{|f^{-1}(a)|} = w(f(1)) \cdots w(f(n))$$

to f and giving weight

$$w(s, f) = w(s)w(f)$$

to the pair (s, f). More precisely

Definition. To each weighted species $\mathcal{F}$ associate the weighted analytic functor

$$\widetilde{\mathcal{F}} \colon \mathbf{Set}_R \longrightarrow \mathbf{Set}_R$$

defined via

$$\widetilde{\mathcal{F}}(A) = \coprod_{n=0}^{\infty} \mathcal{F}[n] \times \mathrm{Hom}_{\mathrm{Set}}([n], A) \big/ S_n,$$

where for $g \colon A \longrightarrow B$ in the **Set** category the transport is

$$\begin{aligned} \widetilde{\mathcal{F}}(g) \colon \widetilde{\mathcal{F}}(A) &\longrightarrow \widetilde{\mathcal{F}}(B) \\ \mathrm{orb}_{S_n}(s, f) &\longmapsto \mathrm{orb}_{S_n}(s, g \circ f). \end{aligned}$$

To put a $\mathbf{Set}_R$ structure on this, define

$$w(\mathrm{orb}_{S_n}(s, f)) = w(s)w(f(1)) \cdots w(f(n)) = w(s) \prod_{a \in A} w(a)^{|f^{-1}(a)|}.$$

It is not hard to see that the above weight definition does not depend on choice of representative since $\mathcal{F}[\sigma]$ is weight-preserving and $(f \circ \sigma^{-1})^{-1}(a) = \sigma(f^{-1}(a))$, where σ is a permutation, so that $|\sigma(f^{-1}(a))| = |f^{-1}(a)|$.

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[7] If $g = \mathrm{id}$ then clearly this acts identically, and if $h \colon B \to C$ is another map then clearly $\widetilde{\mathcal{F}}(h \circ g) = \widetilde{\mathcal{F}}(h) \circ \widetilde{\mathcal{F}}(g)$.

[8] For (s, f) and $(\mathcal{F}[\sigma](s), f \circ \sigma^{-1})$ in the same orbit, the former is sent to $(s, g \circ f)$ while the latter is sent to $(\mathcal{F}[\sigma](s), g \circ f \circ \sigma^{-1})$, and it's clear the two are in the same orbit.

One can then define the type series

Definition. Given a weighted analytic functor $\widetilde{\mathcal{F}}$ and a set $A \in \mathbf{Set}_R$, define the *type series* to be
$$Z_{\widetilde{\mathcal{F}}(A)} := |\widetilde{\mathcal{F}}(A)|_w = \sum_{\mathrm{orb} \in \widetilde{\mathcal{F}}(A)} w(\mathrm{orb}).$$

We will next connect this with the cycle index series $Z_{\mathcal{F}}$.

2.3. **Relation to the Cycle Index Series.** Take $R = \mathbb{Q}[[x_1, \cdots, x_\infty]] = \mathbb{Q}[[x_{\mathbb{Z}_+}]]$ the ring of formal power series in infinitely many variables. Consider
$$(\mathbb{Z}_+, w) \in \mathbf{Set}_{\mathbb{Q}[[x_{\mathbb{Z}_+}]]}$$
with w defined by
$$w(n) = x_n.$$
Then, for a weighted species
$$\mathcal{F} \colon \mathrm{Cor}(\mathrm{fnt}\,\mathbf{Set}) \longrightarrow \mathbf{Set}_{\mathbb{Q}[[x_{\mathbb{Z}_+}]]},$$
construct the weighted analytic functor $\widetilde{\mathcal{F}}$. Then an orbit $\mathrm{orb}(s, f) \in \mathcal{F}(\mathbb{Z}_+)$ gets weight
$$w(\mathrm{orb}(s, f)) = w(s) \prod_{n \in \mathbb{Z}_+} w(n)^{|f^{-1}(n)|} = w(s) x_1^{|f^{-1}(1)|} \cdots x_\infty^{|f^{-1}(\infty)|}.$$

We shall show in this section that

Theorem.
$$Z_{\widetilde{\mathcal{F}}(\mathbb{Z}_+)} = Z_{\mathcal{F}}.$$
□

Proof. To compute the type series $\sum_{\mathrm{orb}} w(\mathrm{orb})$ of $\widetilde{\mathcal{F}}(\mathbb{Z}_+)$, we will need the following fact from group theory:

Theorem. For $G \times H$ a product of groups acting on $S \in \mathbf{Set}_R$, we can consider the action of $G \times \mathrm{id}$ on S; then
$$w(\mathrm{orb}_{G \times \mathrm{id}}(s)) = w(s)$$
is well-defined, and if we consider $\mathrm{id} \times H$ acting on $S \big/ G \times \mathrm{id}$, then
$$\mathrm{fix}_w(h) = \frac{1}{|G|} \sum_{g \in G} \mathrm{fix}_w(g, h).$$
□

The left-hand-side makes sense because of the first half of the theorem. Note that we can plug in H the trivial group to recover

Theorem (Burnside)**.**
$$\sum_{\mathrm{orb} \in S/G} w(\mathrm{orb}) = \frac{1}{|G|} \sum_{g \in G} \mathrm{fix}_w(g).$$
□

This theorem is entirely a group-theoretic result, and it doesn't make too much sense to include its proof here, so we will skip it.

We now have the tool we need to compute $Z_{\widetilde{\mathcal{F}}(\mathbb{Z}_+)}$; we will do so by finding the sum of $\mathrm{fix}_w(\sigma)$ for $\sigma \in S_n$. If σ fixes (s, f), we must have $\mathcal{F}[\sigma](s) = s$ and $f \circ \sigma^{-1} = f$, that is, f must be constant on each cycle of σ^{-1} and therefore σ. Note that the two happen independently. Let σ have cycle type λ; then for each cycle we can color it uniformly with some x_i, so that the weight of the cycle is $x_i^{\text{length of cycle}}$. This is done independently for each cycle of σ, so we obtain

$$\begin{aligned}
\mathrm{fix}_w(\sigma) &= \sum_{(s,f)\in\mathrm{Fix}(\sigma)} w(s,f) \\
&= \sum_{(s,f)\in\mathrm{Fix}(\sigma)} w(s)w(f) \\
&= \sum_{s\in\mathrm{Fix}(\mathcal{F}[\sigma])} w(s) \sum_{f:f\circ\sigma^{-1}=f} w(f) \\
&= \mathrm{fix}_w\,\mathcal{F}[\sigma](x_1^1+x_2^1+x_3^1+\cdots)^{\lambda_1}(x_1^2+x_2^2+x_3^2+\cdots)^{\lambda_2}(x_1^3+x_2^3+x_3^3+\cdots)^{\lambda_3}\cdots \\
&= \mathrm{fix}_w\,\mathcal{F}[\sigma]p_1^{\lambda_1}p_2^{\lambda_2}p_3^{\lambda_3}\cdots \\
&= \mathrm{fix}_w\,\mathcal{F}[\sigma]p_\lambda.
\end{aligned}$$

Burnside will then give us

$$\begin{aligned}
\sum_{\mathrm{orb}\in\mathcal{F}[n]\times\mathrm{Hom}([n],\mathbb{Z}_+)/S_n} w(\mathrm{orb}) &= \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{fix}_w(\sigma) \\
&= \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{fix}_w\,\mathcal{F}[\sigma]p_\lambda \\
&= \sum_{\lambda\vDash n}\mathrm{fix}_w\,\mathcal{F}[\lambda]\frac{p_\lambda}{\mathbb{Z}_+^\lambda\lambda!},
\end{aligned}$$

where we again recall $\mathrm{fix}_w\,\mathcal{F}[\sigma]$ depends only on the cycle type of σ and use the fact that there are $\frac{n!}{\mathbb{Z}_+^\lambda\lambda!}$ many permutations of cycle type λ. Here $\lambda \vDash n$ is short for $|\lambda|_1 = \lambda_1 + \lambda_2 + \cdots = n$. We can then see

$$\begin{aligned}
Z_{\widetilde{\mathcal{F}}(\mathbb{Z}_+)} &= \sum_{\mathrm{orb}\in\coprod_n \mathcal{F}[n]\times\mathrm{Hom}([n],\mathbb{Z}_+)/S_n} w(\mathrm{orb}) \\
&= \sum_{n=0}^{\infty}\sum_{\mathrm{orb}\in\mathcal{F}[n]\times\mathrm{Hom}([n],\mathbb{Z}_+)/S_n} w(\mathrm{orb}) \\
&= \sum_{n=0}^{\infty}\sum_{\lambda\vDash n}\mathrm{fix}_w\,\mathcal{F}[\lambda]\frac{p_\lambda}{\mathbb{Z}_+^\lambda\lambda!} \\
&= \sum_{\lambda}\mathrm{fix}_w\,\mathcal{F}[\lambda]\frac{p_\lambda}{\mathbb{Z}_+^\lambda\lambda!} \\
&= Z_{\mathcal{F}},
\end{aligned}$$

as we claimed. ■

We will next use this to prove the result on the cycle index series of a composition of species.

3. The Cycle Index of a Composition

3.1. **Plethysm of Symmetric Functions.** Consider the graded algebra of symmetric functions, $\Lambda(X)$. Recall that $\{p_\lambda : \lambda \vdash n \;\; \exists\, n \in \mathbb{Z}\}$ is a basis of this algebra; in fact, we showed $\{p_\lambda : \lambda \vdash n\}$

is a basis of $\Lambda^n(X)$. Hence we can define plethysm $f \circ g$ as follows:

Definition. Define the *plethysm* operation

$$\circ \colon \Lambda(X) \times \Lambda(X) \longrightarrow \Lambda(X)$$

by requiring

$$p_n \circ p_m = p_m \circ p_n = p_{mn}$$

and bilinearity

$$f \circ (g + h) = f \circ g + f \circ h, \qquad (f + g) \circ h = f \circ h + g \circ h$$

and multiplicativity

$$f \circ (gh) = (f \circ g)(f \circ h), \qquad (fg) \circ h = (f \circ h)(g \circ h).$$

For example, this definition will lend us that

$$\begin{aligned} f \circ p_m &= \left(\sum_\lambda a_\lambda p_\lambda \right) \circ p_m \\ &= \sum_\lambda a_\lambda \left(\prod_i p_{\lambda_i} \right) \circ p_m) \\ &= \sum_\lambda a_\lambda \prod_i (p_{\lambda_i} \circ p_m) \\ &= \sum_\lambda a_\lambda \prod_i p_{\lambda_i m} \\ &= \sum_\lambda a_\lambda \prod_i p_{\lambda_i}(x_1^m, \cdots, x_\infty^m) \\ &= f(x_1^m, \cdots, x_\infty^m) \end{aligned}$$

and similarly for $p_m \circ f$, so that

Fact.

$$f \circ p_n = p_n \circ f = f(x_1^n, \cdots, x_\infty^n).$$

□

And now the statement from earlier that

$$Z_{\mathcal{F} \circ \mathcal{G}} = Z_{\mathcal{F}} \circ Z_{\mathcal{G}}$$

finally makes sense. We now move towards proving this.

3.2. **Proving the Composition Relation.** The desired relation is proved in two steps, the former of which we will follow closely. We will discuss the latter briefly later.

Theorem. For $\mathcal{F}, \mathcal{G}$ weighted species with associated analytic functors $\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}}$,

$$Z_{\widetilde{\mathcal{F}}(\widetilde{\mathcal{G}}(\mathbb{Z}_+))} = Z_{\widetilde{\mathcal{F}}(\mathbb{Z}_+)} \circ Z_{\widetilde{\mathcal{G}}(\mathbb{Z}_+)} = Z_{\mathcal{F}} \circ Z_{\mathcal{G}}.$$

□

Proof. Again we use Burnside to compute

$$Z_{\widetilde{\mathcal{F}}(\widetilde{\mathcal{G}}(\mathbb{Z}_+))} = \sum_{\text{orb} \in \coprod_n \mathcal{F}[n] \times \text{Hom}([n], \widetilde{\mathcal{G}}(\mathbb{Z}_+))/S_n} w(\text{orb}) = \sum_{n=0}^{\infty} \sum_{\text{orb} \in \mathcal{F}[n] \times \text{Hom}([n], \widetilde{\mathcal{G}}(\mathbb{Z}_+))/S_n} w(\text{orb}).$$

To apply Burnside, we will find

$$\text{fix}_w(\sigma),$$

where $\sigma \in S_n$ acts on $\mathcal{F}[n] \times \text{Hom}([n], \widetilde{\mathcal{G}}(\mathbb{Z}_+))$. In order for (s, f) to be fixed, again we must have $\mathcal{F}[\sigma](s) = s$ and $f \circ \sigma^{-1} = f$, which happen independently. Note the latter is true if and only if f is constant on the cycles of σ^{-1}, i.e. of σ.

Since f must be constant per cycle of σ, f will send a cycle c to some $\text{orb}(t, g) \in \widetilde{\mathcal{G}}(\mathbb{Z}_+)$. This contributes weight $w(\text{orb}(t, g))^{\text{length of } c}$ to the weight of f; since this is done independently per cycle, we can see

$$\sum_{f : f \circ \sigma^{-1} = f} w(f) = \left(\sum_{\text{orb} \in \widetilde{\mathcal{G}}(\mathbb{Z}_+)} w(\text{orb})^1 \right)^{\lambda_1} \left(\sum_{\text{orb} \in \widetilde{\mathcal{G}}(\mathbb{Z}_+)} w(\text{orb})^2 \right)^{\lambda_2} \left(\sum_{\text{orb} \in \widetilde{\mathcal{G}}(\mathbb{Z}_+)} w(\text{orb})^3 \right)^{\lambda_3} \cdots .$$

Recall from earlier that

$$w(\text{orb}(t, g)) = w(t)w(g) = w(t) \prod_{\mathbb{Z}_+} x_n^{|f^{-1}(n)|}$$

is a monomial; hence, when we raise $w(\text{orb})^k$ to a power, it is equivalent to substituting x_n^k into $w(\text{orb})$. But this is precisely the action of plethysm with p_k. Hence we see

$$\sum_{\text{orb} \in \widetilde{\mathcal{G}}(\mathbb{Z}_+)} w(\text{orb})^k = \left(\sum_{\text{orb} \in \widetilde{\mathcal{G}}(\mathbb{Z}_+)} w(\text{orb}) \right) \circ p_k = p_k \circ \left(\sum_{\text{orb} \in \widetilde{\mathcal{G}}(\mathbb{Z}_+)} w(\text{orb}) \right) = p_k \circ Z_{\widetilde{\mathcal{G}}(\mathbb{Z}_+)},$$

so that

$$\begin{aligned}
\sum_{f : f \circ \sigma^{-1} = f} w(f) &= (p_1 \circ Z_{\widetilde{\mathcal{G}}(\mathbb{Z}_+)})^{\lambda_1} (p_2 \circ Z_{\widetilde{\mathcal{G}}(\mathbb{Z}_+)})^{\lambda_2} (p_3 \circ Z_{\widetilde{\mathcal{G}}(\mathbb{Z}_+)})^{\lambda_3} \cdots \\
&= (p_1^{\lambda_1} p_2^{\lambda_2} p_3^{\lambda_3} \cdots) \circ Z_{\widetilde{\mathcal{G}}(\mathbb{Z}_+)} \\
&= p_\lambda \circ Z_{\widetilde{\mathcal{G}}(\mathbb{Z}_+)}.
\end{aligned}$$

We conclude

$$\begin{aligned}
\text{fix}_w(\sigma) &= \sum_{(s,f) \in \text{Fix}(\sigma)} w(s, f) \\
&= \sum_{s \in \text{Fix}\, \mathcal{F}[\sigma]} w(s) \sum_{f : f \circ \sigma^{-1} = f} w(f) \\
&= \text{fix}_w\, \mathcal{F}[\sigma] p_\lambda \circ Z_{\widetilde{\mathcal{G}}(\mathbb{Z}_+)},
\end{aligned}$$

so that Burnside tells us

$$\sum_{\text{orb}\in\mathcal{F}[n]\times\mathrm{Hom}([n],\widetilde{\mathcal{G}}(\mathbb{Z}_+))/S_n} w(\text{orb}) = \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{fix}_w\,\mathcal{F}[\sigma](p_\lambda\circ Z_{\widetilde{\mathcal{G}}(\mathbb{Z}_+)})$$

$$= \sum_{\lambda\vDash n}\mathrm{fix}_w\,\mathcal{F}[\lambda]\frac{1}{\mathbb{Z}_+^\lambda\lambda!}(p_\lambda\circ Z_{\widetilde{\mathcal{G}}(\mathbb{Z}_+)}),$$

$$Z_{\widetilde{\mathcal{F}}(\widetilde{\mathcal{G}}(\mathbb{Z}_+))} = \sum_{n=0}^{\infty}\sum_{\text{orb}\in\mathcal{F}[n]\times\mathrm{Hom}([n],\widetilde{\mathcal{G}}(\mathbb{Z}_+))/S_n} w(\text{orb})$$

$$= \sum_{n=0}^{\infty}\sum_{\lambda\vDash n}\mathrm{fix}_w\,\mathcal{F}[\lambda]\frac{1}{\mathbb{Z}_+^\lambda\lambda!}(p_\lambda\circ Z_{\widetilde{\mathcal{G}}(\mathbb{Z}_+)})$$

$$= \left(\sum_{\lambda}\mathrm{fix}_w\,\mathcal{F}[\lambda]\frac{p_\lambda}{\mathbb{Z}_+^\lambda\lambda!}\right)\circ Z_{\widetilde{\mathcal{G}}(\mathbb{Z}_+)}$$

$$= Z_{\widetilde{\mathcal{F}}(\mathbb{Z}_+)}\circ Z_{\widetilde{\mathcal{G}}(\mathbb{Z}_+)},$$

as to be shown. The last equality in the claim follows from our earlier conclusion that $Z_{\widetilde{\mathcal{F}}(\mathbb{Z}_+)} = Z_{\mathcal{F}}$. ■

Before we can conclude that $Z_{\mathcal{F}\circ\mathcal{G}} = Z_{\mathcal{F}}\circ Z_{\mathcal{G}}$, we still need one last detail:

Theorem. For weighted species $\mathcal{F}, \mathcal{G}, \mathcal{F}\circ\mathcal{G}$, the analytic functors $\widetilde{\mathcal{F}}\circ\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{F}\circ\mathcal{G}}$ are naturally isomorphic.
□

The proof of this theorem does not relate as much to symmetric polynomials, so we skip it. In fact, even though this is a categorical statement, since our definition of the $\mathcal{F}\circ\mathcal{G}$ functor is somewhat "dirty" (i.e. not categorical), one should expect the proof to also be dirty (not categorical). This is indeed the case, and the proof is mostly just defining a map and checking details.

Once we have this, we can directly see

Theorem (Main Theorem: Composition Law)**.**

$$Z_{\mathcal{F}\circ\mathcal{G}} = Z_{\mathcal{F}}\circ Z_{\mathcal{G}}.$$

□

since

$$Z_{\mathcal{F}\circ\mathcal{G}} = Z_{\widetilde{\mathcal{F}\circ\mathcal{G}}(\mathbb{Z}_+)} = Z_{\widetilde{\mathcal{F}}(\widetilde{\mathcal{G}}(\mathbb{Z}_+))} = Z_{\mathcal{F}}\circ Z_{\mathcal{G}},$$

as desired.

4. Application

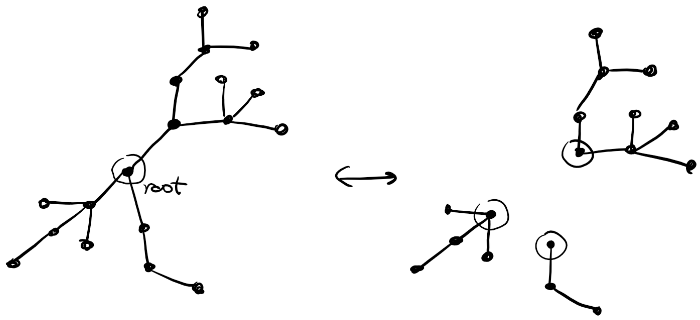
"Now witness the firepower of this fully armed and operational battle station."

The result on the composition of cycle index series in particular implies that $(F\circ G)(x) = F(G(x))$ by plugging in $(x, 0, \cdots)$.

Note that the species of rooted trees can be written as

$$\texttt{rtTree} = \texttt{X}\cdot(\texttt{Set}\circ\texttt{rtTree})$$

since every rooted tree can be broken down into smaller rooted trees by taking away its root. This is schematically represented below.



Since the generating series for `Set` is $\sum_n \frac{x^n}{n!} = e^x$ (for each U, $|\texttt{Set}[U]| = |\{U\}| = 1$) and since the generating series for the `X` is x, we get the series relation

$$T(x) = xe^{T(x)}.$$

We can then apply a theorem from analysis. Recall Lagrange inversion says that if $y = xf(y), f(0) = 1$, then

$$[x^n]\{y(x)\} = \frac{1}{n}[y^{n-1}]\{f(y)^n\}.$$

Applying this to $f(y) = e^y$, $y = T(x)$, we get the coefficient in front of x^n in $T(x)$, $[x^n]\{T(x)\}$, is given by $\frac{1}{n}[y^{n-1}]\{e^{ny}\} = \frac{1}{n}\frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{n!}$, so that

$$T(x) = \sum_{n=0}^{\infty} n^{n-1}\frac{x^n}{n!},$$

which gives n^{n-1} many rooted trees on n vertices, as desired.

5. References

Bergeron, Francois; Labelle, Gilbert; Leroux, Pierre. "Combinatorial Species and Tree-Like Structures". 1998.

Travis, Leopold. "Graphical Enumeration: A Species-Theoretic Approach". 1999. https://arxiv.org/pdf/math/9811127.pdf