CATEGORIFICATIONS OF HERMITE POLYNOMIALS

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1. Introduction

The following inner product of one-variable polynomials $f(x)$ and $g(x)$:

$$(f,g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)g(x)e^{-\frac{x^2}{2}}dx \tag{1}$$

has nonnegative integer coefficients when evaluated on monomials:

$$(x^n, x^m) = \begin{cases} (n+m-1)!! & n+m \quad \text{even} \\ 0 & n+m \quad \text{odd} \end{cases}$$

where $(2k-1)!! = 1 \cdot 3 \cdot \cdots \cdot (2k-1)$. Hence, (x^n, x^m) equals the number of perfect pairings of an $(n+m)$-element set. Pictorially, (x^n, x^m) is the number of homotopy classes of planar diagrams, referred to as (n,m)-diagrams, connecting n points on the left and m points on the right in pairs, see Figure 1.

Let us try to categorify the inner product (1) on the ring of polynomials $\mathbb{Z}[x]$. Monomials x^n should become objects P_n of an additive category and the inner product (x^n, x^m) the dimension of

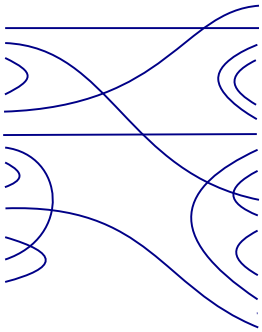


Figure 1. An example of an (n,m)-diagram.

the vector space $\mathrm{Hom}(P_n, P_m)$. We choose a field $\Bbbk$ and define $\mathrm{Hom}(P_n, P_m)$ as the $\Bbbk$-vector space with the basis of (n, m)-diagrams. Next, we need associative compositions

$$\mathrm{Hom}(P_n, P_m) \otimes \mathrm{Hom}(P_m, P_k) \longrightarrow \mathrm{Hom}(P_n, P_k).$$

Some compositions of this kind are already well-known. For instance, if V is a real vector space with a Euclidean inner product and $\dim(V) > n + m$, then the vector space $\mathrm{Hom}_{SO(V)}(V^{\otimes n}, V^{\otimes m})$ of representation's homomorphisms has a natural basis given by (n, m)-diagrams. When $n = m$, this vector space specializes to the Brauer centralizer algebra. When the diagrams are composed, all homotopies are allowed and each closed loop evaluates to $\dim(V)$.

A generalization of this construction was given by Birman, Wenzl [BW89] and Murakami [Mur87], in which intersections become braidings and the value of a closed loop is the quantum dimension of the representation V of the quantum group $U_q(so(r))$.

The BMW algebra relates the 2-variable Kauffman polynomial to representation theory. A similar construction extends the Jones polynomial (equal to the one-variable Kauffman bracket) to tangles via the Temperley-Lieb algebra, see [KL94], and references therein. The Temperley-Lieb (TL) algebra, Brauer centralizer algebra, as well as the BMW algebra have many idempotents. If we adopt the composition rules of TL, Brauer or BMW, the objects P_n will have many idempotent endomorphisms and will decompose into a direct sum of smaller objects if the ambient category is abelian. Either way, they will fall apart into pieces in the Karoubian envelope of the original additive category. Existence of many direct summands of P_n will make the Grothendieck group much larger than the desired $\mathbb{Z}[x]$ with the basis of elements $x^n = [P_n]$.

To avoid this and keep the size of the Grothendieck group to the minimum, we look for a degeneration of the Brauer algebra and its distributive version (the sum over all n and m) in which the only idempotent of $\mathrm{Hom}(P_n, P_n)$ is the identity map 1_n, since this will ensure indecomposability of P_n.

First off, the value of the closed loop must be zero, otherwise we get a nontrivial idempotent in $\mathrm{Hom}(P_2, P_2)$ proportional to the diagram U in the Figure 2. Specifically, if d denotes the value of the loop, $U^2 = dU$ and $\frac{1}{d}U$ is an idempotent if $d \neq 0$.

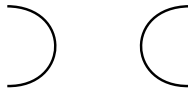


FIGURE 2. A diagram $U \in \mathrm{Hom}(P_2, P_2)$.

Next, the permutation endomorphism of $V^{\otimes 2}$ generates a subalgebra of the Brauer algebra isomorphic to the group algebra of the symmetric group S_n. This group algebra has many idempotents. The standard way to get rid of them is by degenerating the group algebra to the nilCoxeter algebra, with the same generators s_i as the symmetric group and relations

$$\begin{aligned} s_i^2 &= 0, \\ s_i s_j &= s_j s_i \quad \text{if } |i-j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}. \end{aligned}$$

The nilCoxeter algebra is local, with the unique idempotent 1, and has already appeared in the categorification theory [KL09, Kho01]. Pictorially, we do not allow arbitrary homotopies of arcs anymore, instead imposing the relations in Figure 3.

Thus, we are already allowing only isotopies and triple point homotopies. Furthermore, the relations we introduced so far force us into setting the curl to zero, see the leftmost diagram in Figure 4. There are still some idempotents left, the simplest one given by the middle diagram in Figure 4.

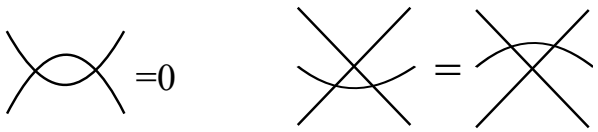


FIGURE 3. Relations in the nilCoxeter algebra.

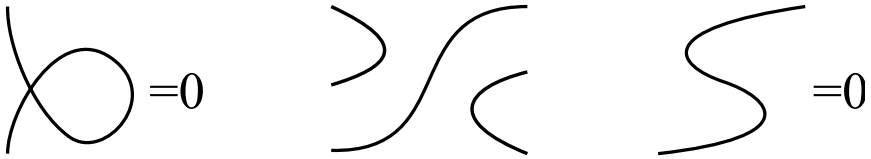


FIGURE 4. The value of a diagram containing a curl is zero, on the left, a diagram representing an idempotent element in the middle, and a diagram which evaluates to zero because it has a strand with 2 critical points with respect to the projection to the x-axis on the right.

In order to avoid unwanted idempotents we disallow isotopies that change the number of critical points in the projection of arcs on the x-axis, and require that the value of a diagram is zero if it contains an arc whose projection to the x-axis has more than one critical point, on the right of Figure 4.

To summarize, we switched from homotopies to controlled isotopies while keeping track of the projection to the x-axis, so that the basis ${}_nB_m$ of $\mathrm{Hom}(P_n, P_m)$ is given by minimal diagrams of pairings, up to allowed isotopies. Each perfect pairing of (n, m) points admits a unique isotopy class of planar diagrams with the minimal number of intersection points and critical points of the x-axis projection.

The resulting structure can be encapsulated by an algebra

$$A = \bigoplus_{n,m\geq 0} \mathrm{Hom}(P_n, P_m).$$

This is a non-unital algebra, with the family $\{1_n\}_{n\geq 0}$ of mutually-orthogonal idempotents serving as a substitute for the unit element. In this setting, each object P_n can be viewed as an indecomposable left projective A-module, and its symbol $[P_n]$ is an element of the Grothendieck group $K_0(A)$ of a category $A-$pmod. The Grothendieck group of category $A-$pmod of left finitely-generated projective A-modules is free abelian, with basis $\{[P_n]\}_{n\geq 0}$. $A-$pmod has a monoidal structure, and there is a natural isomorphism of rings

$$K_0(A) \cong \mathbb{Z}[x]$$

taking $[P_n]$ to x^n.

We will construct so-called big standard modules $\widetilde{M}_n$, each admitting finite projective resolution, and show that in the Grothendieck group $[\widetilde{M}_n]$ corresponds to the n-th Hermite polynomial H_n. Module $\widetilde{M}_n$ is built out of $n!$ copies of small standard modules M_n by iterated extensions. This structure gives an example of the generalized BGG reciprocity (with two types of standard modules: big and small) in the context of non-unital rings.

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2. A categorification of the Hermite polynomials

Hermite polynomials. There are a few equivalent ways of defining Hermite polynomials:

(1) **Rodrigues's representation:**

$$H_n(x) = (-1)^n e^{x^2/2} \frac{\partial^n}{\partial x^n} e^{-x^2/2}. \tag{2}$$

(2) **Differential operator representation:**

$$H_n(x) = e^{-D^2/2}(x^n) = \sum_{i=0}^{\infty} (-1)^i \frac{D^{2i}(x^n)}{2^i i!}, \tag{3}$$

where D denotes differentiation with respect to x.

(3) **Orthogonality:** $H_n(x)$ is the unique monic degree n polynomial orthogonal to x^m for all $0 \leq m < n$ with respect to the inner product

$$(f, g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)g(x)e^{-\frac{x^2}{2}} dx. \tag{4}$$

(4) **Combinatorial interpretation:**

$$H_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k u_{n,k} x^{n-2k}, \tag{5}$$

where

$$u_{n,k} = \binom{n}{n-2k} (2k-1)!! = \frac{n!}{2^k k!(n-2k)!}$$

Coefficient $u_{n,k}$ have a diagrammatic interpretation as the number of partial matchings of n points into $n-2k$ points connected by through arcs and k pairs connected by returns.

Equivalence of the second and the fourth definition can be explained by interpreting the action of the differential operator $\frac{D^2}{2}$ on the monomial x^n in a pictorial way. The coefficients $u_{n,k}$ of the Hermite polynomial expansion can be presented as in the Figure 5. On the other hand, merging two slarcs so that there is an arc connecting the corresponding points can be viewed as an action of D^2 on two x's in x^n. More precisely, say that connecting k pairs of points reflects applying the product rule to finding $2k$-th derivative of x^n. Unmatched $n-2k$ points correspond to multiples of x, and k pairs correspond to $2k$-th derivative. The formula is obtained in the following way: $n(n-1)\dots(n-(k-1))$ corresponds to choosing endpoints of arcs but as the order does not matter, we need to divide by 2^k. In addition, k arcs are disjoint so they can be arranged in $k!$ ways that we do not distinguish which gives the formula $u_{n,k} = \frac{n!}{2^k k!(n-2k)!}$.

$H_n(x)$ contains only powers of x of the same parity as n. For small values of n the Hermite polynomials are:

$$\begin{aligned}
H_0(x) &= 1, \\
H_1(x) &= x, \\
H_2(x) &= x^2 - 1, \\
H_3(x) &= x^3 - 3x, \\
H_4(x) &= x^4 - 6x^2 + 3, \\
H_5(x) &= x^5 - 10x^3 + 15x, \\
H_6(x) &= x^6 - 15x^4 + 45x^2 - 15.
\end{aligned}$$

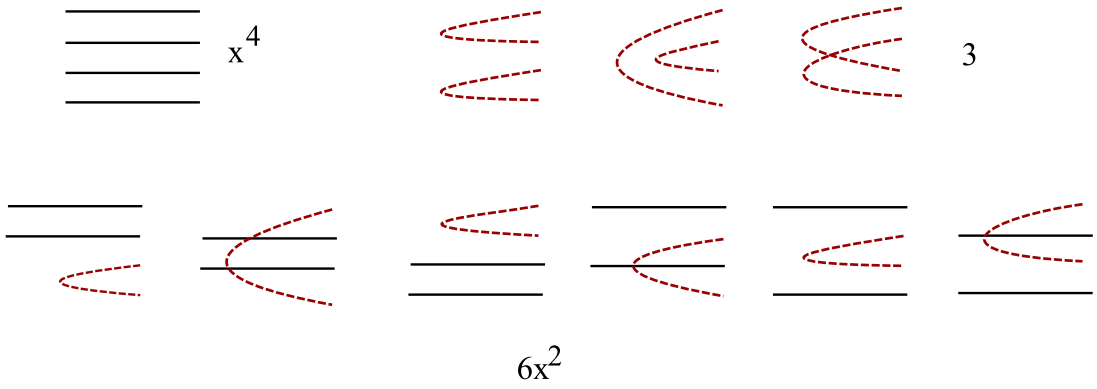


FIGURE 5. Equivalence of the Rodrigues's and the fourth definitions: through arcs correspond to deriving $x\ \frac{\partial^i}{\partial x^i} e^{\frac{-x^2}{2}}$ and closing an arc to $e^{\frac{-x^2}{2}}$.

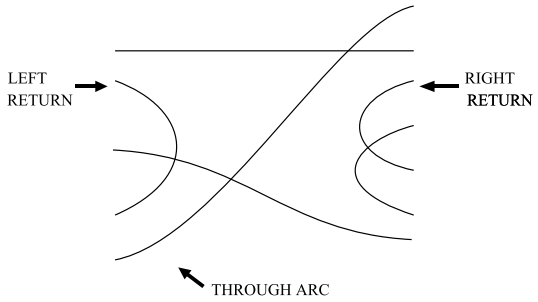


FIGURE 6. A plane diagram for the categorification of Hermite polynomials

Interestingly, coefficients of x^n written in the basis of Hermite polynomials are the absolute values of the corresponding coefficients of the Hermite polynomials:

$$x^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} u_{n,k} H_{n-2k}(x).$$

Using this representation it is easy to see that Hermite polynomials are orthogonal with respect to the inner product (1):

$$(f,g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)g(x)e^{-\frac{x^2}{2}}\,dx$$

namely

$$(H_n, H_m) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} H_n(x)H_m(x)e^{-\frac{x^2}{2}}\,dx = \begin{cases} n!, & n = m; \\ 0, & n \neq m. \end{cases} \tag{6}$$

Categorification of Hermite polynomials.

Multiplication in A and the basis B. For $n, m \geq 0$ denote by ${}_nB_m$ the set of isotopy classes of plane diagrams that connect n points on the line $x = 0$ and m points on the line $x = 1$ in pairs by $\frac{n+m}{2}$ arcs. The positions of points are fixed once and for all. Arcs that connect a point on $x = 0$ line with a point on $x = 1$ line are called *through* arcs. Arcs that connect two points on the same vertical line are called *returns*. They are further separated into *left returns* and *right returns* depending on whether the points lie on $x = 0$ or $x = 1$ line (see Figure 6).

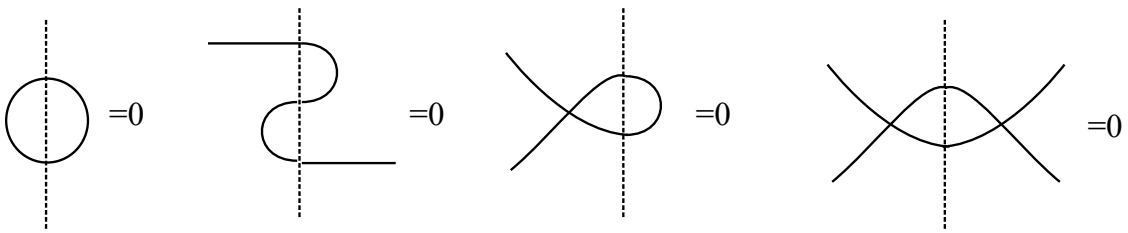


FIGURE 7. Examples of products equal to zero.

Plane diagrams and their isotopy classes are subject to the following conditions. Through arcs are not allowed to have critical points relative to the vertical projection onto the x-axis. Each return arc has one critical point when projected onto the x-axis. Each arc is simple, i.e. without self-intersections. Each pair of arcs has at most one intersection. Only isotopies that preserve these conditions are allowed (and triple intersections of three distinct arcs are allowed during isotopies).

Each isotopy class in ${}_nB_m$ determines a perfect matching of the set of $n+m$ endpoints. In fact, elements of ${}_nB_m$ are in a bijection with perfect matchings of the endpoints. The set ${}_nB_m$ is empty if $n+m$ is odd and has cardinality

$$(n+m-1)!! = (n+m-1)(n+m-3)\cdot\cdots\cdot 3\cdot 1$$

if $n+m$ is even.

Let ${}_nA_m$ be the $\Bbbk$-vector space with the basis ${}_nB_m$ and define

$$A = \bigoplus_{n,m\geq 0} {}_nA_m.$$

We equip A with the following $\Bbbk$-linear multiplication. The product xy of $x \in {}_nB_m$ and $y \in {}_kB_r$ is the concatenation of diagrams x and y if $m = k$ and the concatenation belongs to ${}_nB_r$. Otherwise, the product is zero, Figure 7.

The multiplication is associative and the unique diagram in ${}_nB_n$ without returns and intersections, denoted 1_n, is an idempotent in A:

$$1_n x = x, x \in {}_nA_m,$$
$$y1_n = y, y \in {}_mA_n.$$

The ring A has no unit element, and the collection $\{1_n\}_{n\geq 0}$ of mutually orthogonal idempotents makes the ring a *locally unital algebra*.

We say that a left A-module M is *unital* if

$$M = \bigoplus_{n\geq 0} 1_n M$$

as a $\Bbbk$-vector space. From now on we work with unital left A-modules, unless otherwise specified. Let $A-$mod denote the category of left unital A-modules.

Projective, standard and simple modules. Let $P_n := A1_n$. As a $\Bbbk$-vector space, P_n has a basis given by diagrams in B_n. P_n is an indecomposable projective left A-module.

A (left) A-module M is called *locally finite-dimensional*, or *lfd*, for short, if $\dim 1_n M < \infty$ for all $n \geq 0$. Denote the category of left *lfd* A-modules by $A-$lfd. This category is abelian.

Let L_n be the simple A-module with $1_n L_n \cong \Bbbk$ and $1_m L_n = 0$ for $m \neq n$. This module is isomorphic to the unique simple quotient of P_n, and any simple A-module is isomorphic to L_n, for a unique n.

There is a functorial in M isomorphism $\mathrm{Hom}_A(P_n, M) \cong 1_n M$.

Theorem 2.1. Any projective left lfd A-module P is isomorphic to a finite direct sum of P_n's,

$$P \cong \bigoplus_{n\geq 0} P_n^{a_n}.$$

The multiplicities $a_n \in \mathbb{Z}_+$ are invariants of P.

In particular, any indecomposable projective A-module is isomorphic to P_n, for some n. It might be desirable to have an abelian category that contains projective modules P_n and their finite direct sums.

If a module is projective in $A-$lfd then it is projective in $A-mod$. Note that the category $A-$lfd does not have enough projectives: $\bigoplus_{n\geq 0} L_n \in A-$lfd does not have an lfd projective cover, since $\bigoplus_{n\geq 0} P_n$ is not lfd.

We denote by $A-$mod the category of lfd projective modules. Any object of $A-$plfd is isomorphic to a finite direct sum of P_n's. The Grothendieck group $K_0(A-\text{plfd})$ is free abelian with the basis $\{[P_n]\}_{n\geq 0}$.

Let $\mathcal{C}(A-\text{plfd})$ be the category of bounded complexes of objects in $\mathcal{P}$ modulo chain homotopies. This category is triangulated and is a full subcategory of the bounded derived category $D^b(A-\text{plfd})$, as well as $D^b(A-\text{lfd})$.

For any triangulated category $\mathcal{C}$ its Grothendieck group $K_0(\mathcal{C})$ has symbols $[X]$ as generators, where X runs over all objects of $\mathcal{C}$ and defining relations $[Y] = [X] + [Z]$ over all distinguished triangles $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$, and $[T(X)] = -[X]$, over all X, where T is the translation functor.

There is a canonical isomorphism

$$K_0(\mathcal{C}(A-\text{plfd})) \cong K_0(A-\text{plfd})$$

which takes the symbol of a bounded complex $X = (\dots \longrightarrow X^i \longrightarrow X^{i+1} \longrightarrow \dots)$ to the alternating sum of symbols of its terms:

$$[X] \longmapsto \sum_{i\in\mathbb{Z}} (-1)^i [X^i].$$

For the reasons that will soon become clear, we identify $K_0(A-\text{plfd})$ with the abelian group $\mathbb{Z}[x]$ of polynomials in an indeterminant x with integer coefficients by taking $[P_n]$ to x^n.

In addition to $A-$plfd and $\mathcal{C}(A-\text{plfd})$, we will use abelian category $A-$lfd to make computations in the Grothendieck group. The Grothendieck group $G_0(A-\text{lfd})$ has symbols $[X]$ of lfd A-modules as generators and defining relations $[Y] = [X] + [Z]$ over all short exact sequences

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

For each $n \geq 0$ there is a homomorphism

$$\psi_n \ : \ G_0(A-\text{lfd}) \longrightarrow \mathbb{Z}$$

taking $[X]$ to $\dim(1_n X)$. Let ψ be the product homomorphism

$$\psi \ : \ G_0(A-\text{lfd}) \longrightarrow \prod_{n\geq 0} \mathbb{Z}.$$

This homomorphism is surjective, since for any infinite sequence $(a_0, a_1, \dots) \in \prod_{n\geq 0} \mathbb{Z}_+$ of nonnegative integers the module $L = \bigoplus_{n\geq 0} L_n^{a_n}$ is lfd and $\psi([L]) = (a_0, a_1, \dots)$. Thus, $G_0(A-\text{lfd})$ surjects onto the countable direct product of $\mathbb{Z}$'s, thus it is sufficiently large. We don't know the size of the kernel of ψ.

For an lfd M we can informally think of $\psi_n([M]) = \dim(1_n M)$ as the multiplicity of the simple module L_n in M.

For each $p, r \in \mathbb{Z}_+$ the ring A has a 2-sided ideal $J_{p,r}$ spanned by diagrams with at least p left and at least r right returns. The *big standard* module $\widetilde{M}_n$, $n \geq 0$ is defined as the quotient $P_n/J_{0,1}P_n$. $\widetilde{M}_n$ has a basis of diagrams in P_n with no right returns. The left action of A on $\widetilde{M}_n$ is as follows. For a diagram $b \in {}_mB_n$ with no right returns and $a \in {}_kB_m$ the product $ab \in \widetilde{M}_n$ is the concatenation if it has no right returns and belongs to ${}_kB_n$, otherwise the product is zero.

Notice that ${}_1{}_m\widetilde{M}_n = 0$ if $m < n$ and

$$\dim({}_{1_{n+2k}}\widetilde{M}_n) = \binom{n+2k}{n}(2k-1)!! = \frac{(n+2k)!}{2^k k!}.$$

This formula gives the "multiplicity" of the simple module L_{n+2k} in the big standard module $\widetilde{M}_n$.

The small standard module M_n is the quotient of $\widetilde{M}_n$ by the span of diagrams with at least one intersection between the n through arcs of the diagram. Basis of M_n consist of diagrams in $\widetilde{M}_n$ with no intersections between through arcs.

Each diagram d in $\widetilde{M}_n$ defines a permutation $p(d)$ of the n through strands of d. The diagrams without returns give a basis of the algebra $\mathrm{Hom}_A(\widetilde{M}_n, \widetilde{M}_n)$ and correspond to the elements of the symmetric group.

Notice that $\mathrm{Hom}_A(\widetilde{M}_n, \widetilde{M}_n)$ is isomorphic to the nilCoxeter algebra NC_n, with the basis t_w labeled by elements w of the symmetric group and multiplication

$$t_w t_s = \begin{cases} t_{ws}, & \text{if } l(ws) = l(w) + l(s); \\ 0, & \text{otherwise.} \end{cases},$$

with $l(w)$ the length of the permutation.

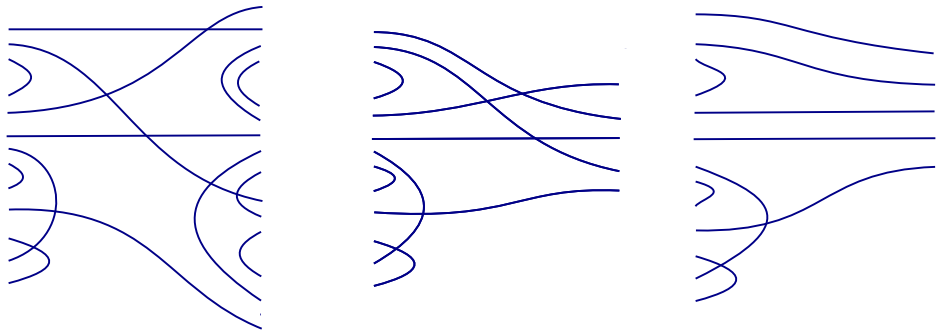


FIGURE 8. Basis elements in projective, big standard and standard/Verma modules

We can filter $\widetilde{M}_n$ by the submodules $\widetilde{M}_n^{(m)}$ spanned by diagrams d with $l(p(d)) \geq m$ for $m \in \{0, 1, \ldots, \frac{n(n-1)}{2}\}$,

$$\widetilde{M}_n = \widetilde{M}_n^{(0)} \supset \widetilde{M}_n^{(1)} \supset \cdots \supset \widetilde{M}_n^{(\frac{n(n-1)}{2})} = 0.$$

Subsequent quotients $\widetilde{M}_n^{(i)}/\widetilde{M}_n^{(i+1)}$ are isomorphic to the direct sum of j copies of M_n, where j is the number of permutations of length i.

We see that the big standard module $\widetilde{M}_n$ has a filtration of length $n!$ with each subsequent quotient isomorphic to the small standard module M_n (and any filtration with this property has length $n!$). We say that the multiplicity of M_n in $\widetilde{M}_n$ is $n!$ and write $[\widetilde{M}_n : M_n] = n!$. In the Grothendieck group of the category of lfd modules we have $[\widetilde{M}_n] = n![M_n]$.

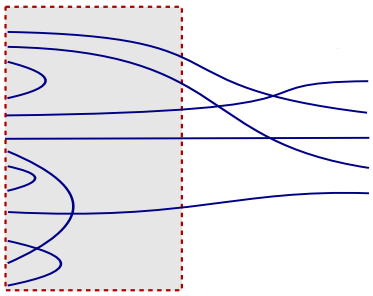


FIGURE 9. Multiplicity of small Verma modules in big standard/Verma modules.

Consider finite decreasing filtration

$$P_n = P_n(n) \supset P_n(n-2) \supset P_n(n-4) \supset \dots,$$

where $P_n(n-2k)$ is spanned by the diagrams in B_n of width at most $n-2k$ (equivalently, with at least k right returns). Left multiplication by a basis vector cannot increase the thickness, hence $P_n(n-2k)$ is a submodule of P_n. The quotient $P_n(n-2k)/P_n(n-2k-2)$ has a basis consisting of diagrams in B_n whose thickness is exactly $n-2k$, or equivalently, which have exactly k right returns. Any such diagram factors uniquely into the product ab where a has only left and b only right returns and through strands of b do not intersect, see Figure 10.

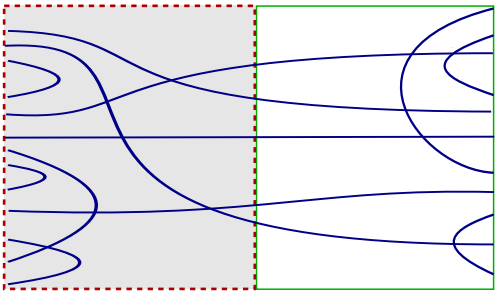


FIGURE 10. Factoring a diagram of thickness $n-2k$ into a product of diagrams with right and left returns only.

Such a diagram $b \in {}_{n-2k}B_n$ has exactly k right returns and no left returns. These diagrams correspond to partial matchings of n points $\{1, 2, \dots, n\}$ into k pairs I and $n-2k$ leftover points $\underline{n} \setminus I$, where $\underline{n} = \{1, 2, \dots, n\}$. The number of such matchings is

$$u_{n,k} = \binom{n}{n-2k} (2k-1)!! = \frac{n!}{2^k k!(n-2k)!},$$

a coefficient of the Hermite polynomial. Viewed as an element of $P_n(n-2k)/P_n(n-2k-2)$, b yields a copy of the standard module $\widetilde{M}_{n-2k}$ in this quotient. In fact, the quotient is isomorphic to the direct sum of these standard modules over all b as above. Thus,

$$P_n(n-2k)/P_n(n-2k-2) \cong \widetilde{M}_{\lfloor n-2k \rfloor}^{u_{n,k}}.$$

In the Grothendieck group, we obtain the following equality

$$\begin{aligned}[P_n] &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} [P_n(n-2k)/P_n(n-2k-2)] \\ &= \sum_{k=0}^{\frac{n}{2}} u_{n,k}[\widetilde{M}_{n-2k}]\end{aligned}$$

By looking at the dimension of $1_m P_n$ one can check that any filtration of P_n by $\widetilde{M}_{n-2k}$ has a finite length and that the multiplicity in P_n of $\widetilde{M}_{n-2k}$ is equal to the number of diagrams $b \in {}_{n-2k}B_n$ described above.

Moreover, denote by $b_{n-2k,n}$ the number of diagrams in ${}_{n-2k}B_n$ without left returns.

Let us now specialize to standard modules M_m. We have

$$[M_m : L_n] = dim(1_n M_m) = \binom{n}{m} (n-m-1)!!.$$

Recall that $[P_n : \widetilde{M}_m] = \binom{n}{m} (n-m-1)!!$ and hence

$$[P_n : \widetilde{M}_m] = [M_m : L_n] \tag{7}$$

Thus, our diagrammatically defined algebra possesses the genaralized Bernstein–Gelfand–Gelfand (BGG) reciprocity. Indecomposable projective modules P_n have filtration by standard modules M_m, with $m \leq n$ and $[P_n : M_n] = 1$. In the equality (7) the multiplicity in the RHS is understood in the generalized sense.

Define the Cartan matrix $C(A)$ by

$$C(A)_{i,j} := \dim \mathrm{Hom}(P_i, P_j) \tag{8}$$

and by $m(A)$ the multiplicity matrix $m(A)_{i,j} := [P_i : \widetilde{M}_j]$. Then we have the following equality:

$$C(A) = m(A)D(A)m(A)^t, \tag{9}$$

where $D(A)_{i,j} := [\widetilde{M}_i : M_j]$. Indeed,

$$\begin{aligned}C(A)_{i,j} &= \dim \mathrm{Hom}(P_i, P_j) = [P_i : L_j] \\ &= \sum_{k,l} [P_i : \widetilde{M}_k][\widetilde{M}_k : M_l][M_l : L_j] = \sum_{k,l} m(A)_{i,k} D(A)_{k,l} m(A)_{j,l} \\ &= \sum_{k,l} m(A)_{i,k} D(A)_{k,l} m(A)^t_{l,j} = (m(A)D(A)m(A)^t)_{i,j}.\end{aligned}$$

Projective resolution of $\widetilde{M}_n$. Notice that $P_0 = \widetilde{M}_0$ and $P_1 = \widetilde{M}_1$. Moreover we can think of $\widetilde{M}_2$ as a quotient of P_2 by all diagrams which have right returns which is isomorphic to $P_0 = \widetilde{M}_0$ (actually just a reflection with respect to vertical mirror/axes). Hence, $\widetilde{M}_2 = P_2/P_0$ i.e. we have the following resolution of $\widetilde{M}_2$: $0 \to P_0 \to P_2 \to \widetilde{M}_2 \to 0$. Using induction on odd and even n and the fact that every projective module has a filtration by standard modules we show that every standard module $\widetilde{M}_n$ has a finite resolution by projective modules P_{n-2k}, $k \in (1, \lfloor n/2 \rfloor)$.

Now let us explicitly construct this resolution. The resolution will be of the form:

$$P_0 \xrightarrow{\partial} P_2 \xrightarrow{\partial} \ldots \xrightarrow{\partial} P_{n-2k}^{u_{n,k}} \xrightarrow{\partial} \ldots P_{n-2}^{u(n,1)} \xrightarrow{\partial} P_n \xrightarrow{\partial} \widetilde{M}_n.$$

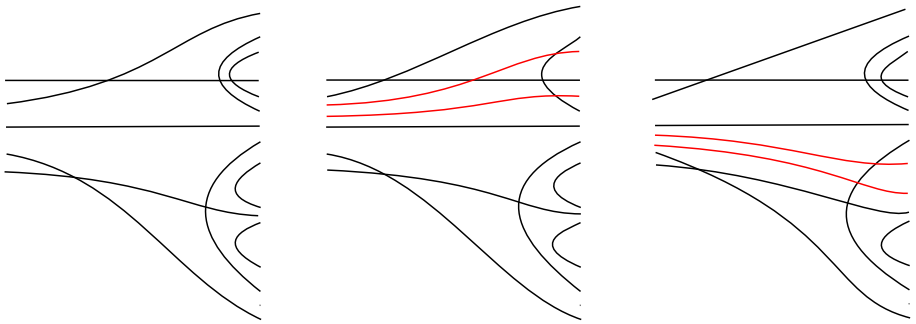


FIGURE 11. Resolution of Verma modules M_n

Each copy of P_{n-2k} corresponds to the partial matchings of n points $\{1, 2, \ldots, n\}$ into k pairs I and $n-2k$ leftover points $\underline{n}/I$ where $\underline{n} = \{1, 2, \ldots, n\}$. $u_{n,k}$ denotes the number of such matchings, and it is equal to:

$$u_{n,k} = \binom{n}{n-2k} (2k-1)!! = \frac{n!}{2^k k!(n-2k)!}$$

The correspondence between projective modules P_{n-2k} and partial matchings is realized through diagrams in ${}_{n-2k}B_{n-2(k-1)}$ without left returns and intersections on $\underline{n}/I$ through arcs assigned to the unmatched points. Hence, we can label different copies of P_{n-2k} by P_I where I is a set of k ordered pairs (i, j) denoting matchings i.e. right returns.

The differential $\partial_{I,J} : P_I \to P_J$, $I = J \sqcup (i, j)$ where (i, j) is one of the innermost returns in I, is given by an alternating sum of maps determined by composing with the minimal diagram corresponding to P_I with its innermost right return "pulled out"; otherwise it is zero. The differential is defined as the following alternating sum:

$$\partial_{P_I}(D) = \sum_{(i',j')\text{innermost returns inside } (i,j)} (-1)^{t_{(i',j')}} \partial_{I,J} \tag{10}$$

where $t_{i',j'}$ is the number of first endpoints of right returns before point i' in the diagram corresponding to I.

Homological dimension of A, computation of various Ext *groups.* If we choose to work in the category of finite complexes of modules in $A-$plfd everything functions. Unfortunately we can not extend it to the category of lfd A-modules since the ring A is not Nöetherian. As a (counter) example we use the following infinitely generated submodule M^∞ of the projective module P_0 generated by the diagrams b_i^h (see Figure 12 which have no through arcs and no returns on the right, just the following left returns: $n+\widehat{1, n}+2$, $n+\widehat{3, n}+4$ inside nested arcs $j, \widehat{2n+5}-j$, $j \in (1, n)$.

Functors on A-mod and operators they categorify.

Functor categorifying $e^{-\frac{D^2}{2}}$. The operator $e^{-\frac{D^2}{2}}$ acting on polynomials (where D is differentiation) restricts to an endomorphism of the abelian group $\mathbb{Z}[x]$ that takes x^n to $H_n(x)$. Since in our framework these correspond to modules P_n and $\widetilde{M}_n$, to categorify this operator we look for an additive functor E that takes P_n to $\widetilde{M}_n$. Recall that $\widetilde{M}_n \cong P_n/J_{0,1}P_n$, where $J_{0,1}$ is a two-sided ideal in A spanned by diagrams with at least one right return. It is immediate to see that for morphism $\varphi : P_n \to P_m$, $\varphi(J_{0,1}P_n) \subset J_{0,1}\varphi(P_n)$, so that φ descends to $\widetilde{\varphi} : \widetilde{M}_n \to \widetilde{M}_m$. Imposing

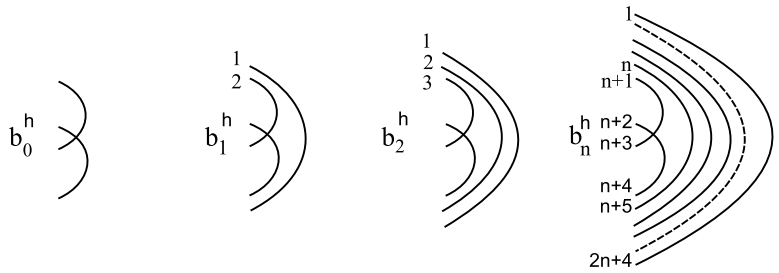


FIGURE 12. Generators of the submodule Q of P_0 which can not be finitely generated.

additivity we obtain a functor $E : A-\text{pmod} \to A-\text{plfd}$. Since $\widetilde{M}_n$ has a finite projective resolution, we can extend E to an endofunctor in $C(A-\text{pmod})$, also denoted by E. To define this endofunctor on a bounded complex $P = (\dots \longrightarrow P^i \longrightarrow P^{i+1} \longrightarrow \dots)$ of lfd projective modules, choose a finite projective resolution R^i of each quotient module $E(P^i)$, fit them together into a bicomplex and collapse the bigrading to a single grading. The resulting complex is $E(P)$. Summarizing, we have

Proposition 2.2. The exact endofunctor E on the triangulated category $C(A-\text{pmod})$ descends to the operator $e^{-\frac{D^2}{2}}$ on the abelian group $\mathbb{Z}[x]$ under the canonical isomorphism between the latter and the Grothendieck group of $C(A-\text{pmod})$.

More generally, we can define functor E on the category of all A-modules $E : A-\text{plfd} \to A-\text{plfd}$ as $E(M) = M/J_{0,1}M$. The functor E is right exact and extends to an endofunctor $E : D^-(A-\text{plfd}) \to D^-(A-\text{plfd})$.

We now specialize to our A and the inclusion $\iota : A \xhookrightarrow{\subseteq} A$ induced by adding a straight through line at the top of every diagram i.e. diagram $d \in {}_mB_n$ goes to $\iota(d) \in {}_{m+1}B_{n+1}$. In particular, the system of idempotents $\{1_n\}_{n\geq 0}$ goes to $\{1_{n+1}\}_{n\geq 0}$ missing 1_0. This inclusion ι induces both induction and restriction functors, with $\text{Res}(N) \cong N/1_0N$ with A acting on the left via ι.

Proposition 2.3. $\text{Res}\,\widetilde{M}_n \cong \widetilde{M}_{n+1} \oplus n\widetilde{M}_{n-1}$ for $n > 0$, and $\text{Res}\,\widetilde{M}_0 \cong \widetilde{M}_1$

Proof. Let $\widetilde{M}_n^L$ and $\widetilde{M}_n^\emptyset$ denote spans of diagrams in $\widetilde{M}_n$ with top left point part of a left return or a through line, respectively. Then $\widetilde{M}_n \cong \widetilde{M}_n^L \oplus \widetilde{M}_n^\emptyset$ as left A-modules. Furthermore, $\widetilde{M}_n^\emptyset \cong \bigoplus_{i=1}^n \widetilde{M}_n^\emptyset(i)$ where $\widetilde{M}_n^\emptyset(i)$ denotes the submodule of $\widetilde{M}_n^\emptyset$ spanned by diagrams containing an arc connecting a top left point and an i-th point on the right. It is easy to see that, as A-modules $\widetilde{M}_n^L \cong \widetilde{M}_{n+1}$ and $\widetilde{M}_n^\emptyset(i) \cong \widetilde{M}_{n-1}$.

■

As we already know, big standard modules $\widetilde{M}_n$ on the level of Grothendieck group correspond to Hermite polynomials H_n.

Proposition 2.4. Induction and restriction functors on the category $C\mathcal{P}$ categorify the multiplication by x acting on the ring of polynomials $\mathbb{Z}[x]$. In other words, $(xf, g) = (f, xg)$ for any $f, g \in \mathbb{Z}[x]$ is categorified to $\text{Hom}_{C\mathcal{P}}(IndP, Q) \cong \text{Hom}_{C\mathcal{P}}(P, \text{Res}\,Q)$ where P, Q are bounded complexes of projective modules in $\mathcal{P}$.

Proof. First let us see how restriction and induction functors act on $\mathcal{P}$ starting with $P_n = P1_n$, It is easy to see that the induction functor takes P_n into P_{n+1}. However, $\text{Res}\,P$ is not a projective module but it has a finite filtration by the big standard modules $\widetilde{M}_n$ (since P has finite filtration and Res is an exact functor). ■

Categorification of kernels $K_n(x,y)$. Consider a Hilbert space $\mathcal{H}$ a completion of polynomials with respect to

$$(f,g)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)g(x)e^{-\frac{x^2}{2}}dx.$$

Hermite polynomials $\{H_i(x)\}_{i=0}^{\infty}$ form an orthogonal basis (see equation 6). For every $i\geq 0$ let $h_i(x)=H_i(x)/i!$ and define an operator Ψ on $\mathcal{H}$ by $\Psi(f)=\sum_{i=0}^{\infty}(f,h_i(x))H_i(x)$. Operator Ψ acts like identity on $\mathcal{H}$ and can be approximated by the following partial sums $\Psi_n(f)=\sum_{i=0}^{n}(f,h_i(x))H_i(x)$. The goal is finding a functor which categorifies Ψ_n. Consider A-bimodules $A(\leq\omega)$ spanned by diagrams of width at most k. This is a submodule of A. Every $A(\leq\omega)$ defines a functor Φ_ω of a category of A-modules given by $\Phi_\omega(M)=A(\leq k)\otimes_A M\subset M$. Hence, functor Φ_ω acts like identity on projective modules P_m for $m\leq\omega$ while for $m>\omega$ $\phi(P_m)$ is spanned by all diagrams of width less than $\omega+1$. Action of Φ on big standard modules resembles action of Ψ on Hermite polynomials:

$$\Phi_\omega(\widetilde{M}_m)=\begin{cases}\widetilde{M}_m, & m\leq\omega;\\ 0, & m>\omega.\end{cases}$$

Proposition 2.5. Functor Φ_ω and its derived functor $L\Phi_\omega$ on the category $C\mathcal{P}$ are categorifying the operator Ψ.

Proof sketch: We want to have $L^i\Phi_\omega(\widetilde{M}_m)=0$ for every $i>0$. Notice that $L^i\Phi_\omega(\widetilde{M}_m)=0$ for $m=0,1$ and $i>0$ since right exact functors are zero on projective modules and $\widetilde{M}_0=P_0$ and $\widetilde{M}_1=P_1$. By induction on m we obtain $L^i\Phi_\omega(\widetilde{M}_m)=0$.

3. The Hermite algebra is triangular-based

In this section we briefly recall Brundan's theory of (graded) triangular bases from [Bru23] and check that the Hermite algebra defined above fits the bill.

Recalling (graded) triangular bases. Let A be any locally graded-finite-dimensional locally unital algebra. Let I index the local units (orthogonal and homogeneous) of A, $\Phi\subseteq I$ label the "distinguished idempotents", Θ be a (lower-finite)[1] poset of "weights", and $\varpi\colon\Phi\longrightarrow\Theta$ be a map (with finite fibers) from distinguished idempotents to the weight poset. Diagrammatically this is:

$$\begin{array}{ccc} I & & \\ \uparrow & & \\ \Phi & \xrightarrow{\varpi}\!\!\!\!\rightarrow & \Theta \end{array}$$

We will let symbols like $i,j\in I$ and $\alpha,\beta\in\Phi$ and $\theta,\phi,\psi\in\Theta$. In the interest of appealing to weight notation later, we will denote the local unit labelled by i as 1^i.

Let A have a "(graded) triangular basis" in the sense of Brundan, which is to say that

Definition 3.1. A is "graded triangular based" if there are (homogeneous) sets $\mathrm{X}(i,\alpha)\subseteq 1^iA1^\alpha$, $\mathrm{H}(\alpha,\beta)\subseteq 1^\alpha A1^\beta$, $\mathrm{Y}(\beta,j)\subseteq 1^\beta A1^j$ such that

(1) products of these things in these sets give a basis for A, i.e.

$$\left\{xhy:(x,h,y)\in\bigcup_{i,j,\alpha,\beta}\mathrm{X}(i,\alpha)\times\mathrm{H}(\alpha,\beta)\times\mathrm{Y}(\beta,j)\right\}$$

[1]Here "lower-finite" means for any θ the set $\{\phi:\phi\leq\theta\}$ is a finite set.

forms a basis of A;

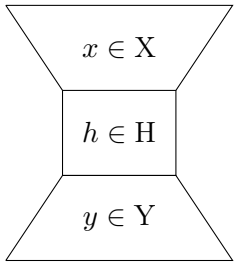
(2) $\mathrm{X}(\alpha,\alpha) = \mathrm{Y}(\alpha,\alpha) = \{1^\alpha\}$;

(3) for $\alpha \neq \beta$,

$$\begin{aligned} \mathrm{X}(\alpha,\beta) \neq \emptyset &\implies \varpi(\alpha) > \varpi(\beta), \\ \mathrm{H}(\alpha,\beta) \neq \emptyset &\implies \varpi(\alpha) = \varpi(\beta), \\ \mathrm{Y}(\alpha,\beta) \neq \emptyset &\implies \varpi(\alpha) < \varpi(\beta); \end{aligned}$$

(4) for each $i \in I - \Phi$, there are only finitely many $\alpha \in \Phi$ such that $\mathrm{X}(i,\alpha) \cup \mathrm{Y}(\alpha, i) \neq \emptyset$.

One colloquial description for this is that of the "shirt-and-belt-and-pants", or alternatively the "sandwich":



$= xhy.$

Note that for instance we have depicted the diagram for y to be wider on the bottom than the top, because $y \in \mathrm{Y}$ should go from a 'larger' idempotent to a 'smaller' one.

Let

$$e^\theta \coloneqq \sum_{\alpha \in \varpi^{-1}\theta} 1^\alpha.$$

Let $A^{\geq\theta}$ be defined as

$$A^{\geq\theta} \coloneqq A \Big/ \langle e^\phi \colon \phi \not\geq \theta \rangle,$$

let e.g. $A^{\leq\theta}, A^{>\theta}$ be defined similarly, and let

$$A^\theta \coloneqq e^\theta A^{\geq\theta} e^\theta.$$

This plays the role of Cartan in the sense that modules over it are induced to form standard objects. Note that

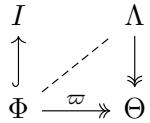
$$A^{>\theta} = A^{\geq\theta} \Big/ A^{\geq\theta} e^\theta A^{\geq\theta};$$

hence, by the general theory of Cline-Parshall-Scott [CPS88], one expects a recollement $\mathsf{D}\,\mathsf{Mod}\, A^{>\theta} \longrightarrow \mathsf{D}\,\mathsf{Mod}\, A^{\geq\theta} \longrightarrow \mathsf{D}\,\mathsf{Mod}\, A^\theta$. For M a module over A, we write

$$M^\theta \coloneqq e^\theta M$$

in analogy with weight space notation. We often write simply $\alpha \in \theta$ as shorthand for $\alpha \in \varpi^{-1}(\theta)$, so that $e^\theta = \sum_{\alpha\in\theta} 1^\alpha$.

Let Λ_θ label the simple modules $L_\lambda(\theta)$ of A^θ as well as their projective covers $P_\lambda(\theta)$ and injective hulls $Q_\lambda(\theta)$, and let $\Lambda = \bigsqcup_\theta \Lambda_\theta$ be the disjoint union of these label sets. Then one can show that Λ labels all simples of A. Frequently it is the case (for example if the semisimplification of the Cartan is $A^\theta/\mathfrak{j} \cong \prod_{\alpha\in\varpi^{-1}(\theta)} \Bbbk$) that Λ can be identified with Φ; in this case we will instead use the symbols $\lambda, \mu \in \Lambda$ for the distinguished idempotents. Here is a picture of the situation:



There is moreover often a "split triangular decomposition", which is to say that

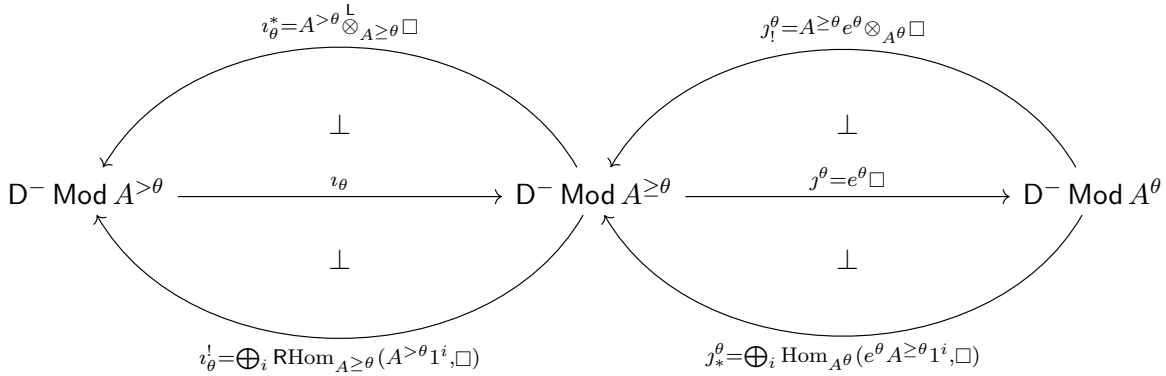
Definition 3.2. A has a "split triangular decomposition" if there are A^-, A^0, A^+ locally unital algebras such that

(1) $A^\flat = A^0A^-$ and $A^\sharp = A^+A^0$ are subalgebras;
(2) the multiplication map $A^+ \otimes_{\mathbb{K}} A^0 \otimes_{\mathbb{K}} A^- \longrightarrow A$ is a linear isomorphism;
(3) $1^iA^\flat 1^i = 1^iA^\sharp 1^i = 1^iA^01^i$ for each i and

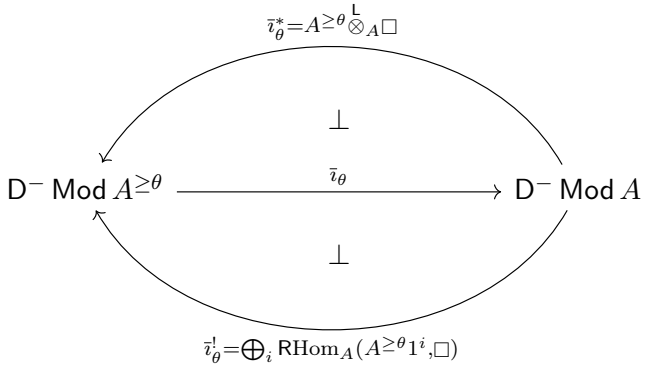
$$1^iA^01^j \neq 0 \implies \varpi(i) = \varpi(j),$$
$$1^jA^-1^i, 1^iA^+1^j \neq \Bbbk 1^i \implies \varpi(i) > \varpi(j).$$

Roughly this is saying that the sets X, H, Y generate actual subalgebras of A. We will later see that Hermite falls into this set-up.

Recollement. The following recollement diagrams are considered in [Bru23]:



and



This is an artifact of the Cline-Parshall-Scott [CPS88] theory of algebraic recollements. Brundan proved in [Bru23] that for triangular-based algebras the functors $\jmath_!^\theta$ and $\jmath_*^\theta$ are exact. Hence we may define the "(co)standard modules" and "proper (co)standard modules" Δ and $\overline{\Delta}$ (resp. ∇ and $\overline{\nabla}$) by:

$$\Delta_\lambda = \jmath_!^\theta P_\lambda(\theta), \qquad \overline{\Delta}_\lambda = \jmath_!^\theta L_\lambda(\theta),$$
$$\nabla_\lambda = \jmath_*^\theta Q_\lambda(\theta), \qquad \overline{\nabla}_\lambda = \jmath_*^\theta L_\lambda(\theta).$$

Quite often colloquially these modules are referred to as big or small (co)Vermas respectively. This is because these modules enjoy many homological properties in close analogy with highest weight categories such as category $\mathcal{O}$, and for these many properties we refer the reader to [Bru23] for a detailed account.

Also define the module

$$\Delta(\theta) := A^{\geq\theta}e^\theta = \bigoplus_{\lambda\in\varpi^{-1}\theta} \Delta_\lambda^{\overline{l_\lambda(\theta)}},$$

where $l_\lambda(\theta) = \dim L_\lambda(\theta)$; the benefit of this module is that it has a right action $A^{\geq\theta}e^\theta \circlearrowleft e^\theta A^{\geq\theta}e^\theta$, so that we have

$$\Delta(\theta) \circlearrowleft A^\theta.$$

Note that when $\Lambda = \Theta$ and $l_\lambda(\theta) = 1$, the notions of $\Delta(\theta)$ and Δ_λ coincide.

Hermite fits the bill. The earlier basis ${}_nB_m$ for the algebra A gives us a good idea for how to define the triangular basis of A. Indeed, in the previous language, we can let

X $\coloneqq$ set of diagrams with only left returns, no right returns, and no crossings of through arcs,

H $\coloneqq$ set of diagrams with only through arcs and crossings thereof,
no left returns and no right returns

Y $\coloneqq$ set of diagrams with only right returns, no left returns, and no crossings of through arcs

Note well that for example in X, the left returns are allowed to possibly intersect each other, but at most once. Also not well that in X and Y, diagrams with no left/right returns are also allowed. We may also sometimes refer to left returns as "cups", through arcs as "propagating strings", and right returns as "caps", coming from the vertical stacking convention in diagrammatic algebra[2]. Letting the weight poset be $\Theta = \Phi = I = \mathbb{N}$ and the set of orthogonal idempotents e^θ be simply the diagrams with θ straight propagating strings, it is straightforward to see that the above gives a triangular basis of A:

Lemma 3.3. The above setup makes the Hermite algebra triangular-based.

Proof. This is essentially due to the discussion in 2. ■

The general theory then gives us a great deal of structure for free. We can form as prescribed the Cartan algebras (here $\theta \in \mathbb{N}$)

$$A^\theta = \mathrm{NC}_\theta,$$

whose unique simple module is trivial module $\Bbbk e^\theta$ and whose unique projective indecomposable is the regular representation. Since then we have that $\Lambda = \Theta = \mathbb{N}$, moving forward we no longer distinguish between the two sets. Also note that since the unique simple is 1-dimensional, we have that the notions of $\Delta(\theta)$ and Δ_λ coincide for the Hermite algebra.

We can then form, again as prescribed above, the standard modules (here $n \in \mathbb{N}$)

$$\Delta_n = \jmath_!^n \,\mathrm{NC}_n$$

and proper standard modules

$$\overline{\Delta}_n = \jmath_!^n \mathbb{C}.$$

Note well that these modules are exactly the big and small Vermas defined in the previous sections:

$$\Delta_n = \widetilde{M}_n, \qquad \overline{\Delta}_n = M_n.$$

Properties such as BGG reciprocity then immediately follow from [Bru23].

4. Nil-Koszulity of Hermite and some consequences

In this section we prove that Hermite is "nil-Koszul", which roughly says that half of this algebra is Koszul.

[2] In such a way that xy refers to x stacked on top of y.

The Hermite nil-algebra. We previously gave a triangular basis for A. In fact, it turns out that this triangular basis is actually a split triangular basis.

It is not hard to see that

Lemma 4.1. Each of the sets $\mathrm{X}, \mathrm{H}, \mathrm{Y}$ generate a subalgebra of A, named A^+, A°, A^- respectively, which make A a split triangular-based algebra. Each of A^+, A^- can then be referred to as a "nil-algebra".

Proof. It is easy to see that the space generated by diagrams in H, which are nil-Coxeter diagrams, is closed under multiplication. Hence H generates a subalgebra $A^\circ \subset A$. Similarly it is straightforward to see that Y is closed under multiplication, for multiplying caps cannot introduce cups.

It is also straightforward to see from the definition of a triangular basis that $A^+ \otimes A^\circ \otimes A^- \longrightarrow A$ is an isomorphism. In particular note that A is free as a right module over A^-. ■

In what follows, when we speak of the "Hermite nil-algebra", we refer to this A^-. This is to be thought of as the analogue of $\mathfrak{n}^+$ in Lie theory[3]. It turns out that we can grade this nil-algebra, and under this grading it is quadratic. Indeed, let

$$A_i^- \coloneqq \Bbbk\text{-span of all Y-diagrams with exactly } i \text{ caps;}$$

then it is clear that

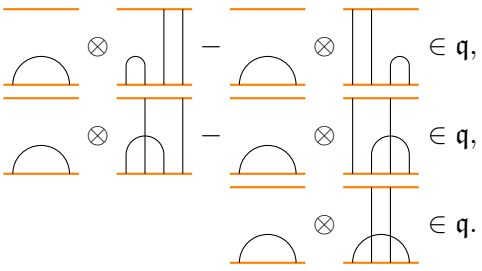
$$A^- = \bigoplus_{i=0}^{\infty} A_i^-.$$

This gives a notion of degree on A^-, which we simply denote as $\deg(x)$. We sometimes call this the "cap-grading" or "cap-degree".

Recall that a graded algebra R is said to be "one-generated" if there is a surjection $\bigotimes^\bullet R_1 \twoheadrightarrow R$, and if the kernel $\mathfrak{q}$ of this map is moreover generated by a subspace of $R_1 \otimes R_1$ then R is said to be "quadratic".

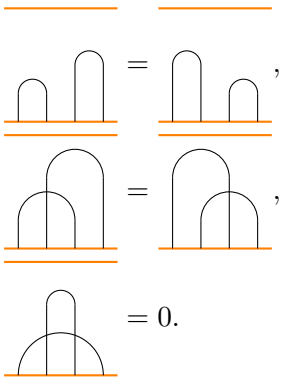
Lemma 4.2. Under the cap-grading, A^- is quadratic.

Proof. This is not difficult to see. In fact, the kernel $\mathfrak{q}$ is generated by the following elements (here we draw diagrams vertically, and the orange lines denote the boundaries of our diagrams):



[3]The signs are flipped here because category $\mathcal{O}$ is typically defined to be *highest* weight, whereas diagram algebras such as ours are typically *lowest* weight.

In other words, these elements of the kernel are telling us that A^- is defined by the relations



■

is Koszul. For more on Koszul theory, we refer the reader to the text [PP05].

The amazing thing is that A^- turns out to be not just quadratic – it is Koszul[4]. First let us briefly recall what this means.

Given a quadratic algebra R, we can define its "quadratic dual", denoted $R^!$, as

$$R^! := \bigotimes^\bullet R_1^* \Big/ \mathfrak{q}^\perp,$$

where $\mathfrak{q}^\perp \subset \bigotimes^\bullet R_1^*$ is the orthogonal complement to $\mathfrak{q}$ under the pairing $(\phi\otimes\psi)(v\otimes w) = \phi(w)\psi(v)$. Note well that this is defined using the *left* dual $V^* := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$.

A graded (locally unital) algebra R with $R_0 = \mathbb{K}$ is called "Koszul" if the following equivalent conditions hold:

- $\mathrm{Ext}^i_{R\,j} = 0$ for $i \neq j$;
- R is one-generated and the algebra $\mathrm{Ext}^\bullet_R(\mathbb{K}, \mathbb{K})$ is generated by $\mathrm{Ext}^1_R(\mathbb{K}, \mathbb{K})$;
- R is quadratic and $\mathrm{Ext}^\bullet_R(\mathbb{K}, \mathbb{K}) \cong R^!$;
- the algebra $\mathrm{Ext}^\bullet_R(\mathbb{K}, \mathbb{K})$, equipped with the cap grading, is one-generated.

Let us define a notion of nil-Koszulity.

Definition 4.3. A triangular-based algebra is said to be "nil-Koszul" if it has a nil-algebra which is Koszul.

The punchline is that the Hermite algebra satisfies this.

Theorem 4.4. The Hermite algebra A is nil-Koszul. In other words, the Hermite nil-algebra A^- is Koszul. (By symmetry, A^+ is also Koszul.)

One way to see this theorem is by observing that A^- is isomorphic to the associated graded $\mathrm{gr}_0 \mathrm{N}\mathcal{B}^-$ of the dot filtration on the nil-Brauer nil-algebra from [Zho24]; there, it is proven that the associated graded of the nil-Brauer nil-algebra is Koszul, and hence in this paper we have A^- is Koszul. However we understand that outsourcing this proof is likely unsatisfactory for the reader, so we reproduce in the next subsection a proof.

Our philosophy is that most diagrammatic algebras appearing in nature are nil-Koszul in some way. The point of proving this nil-Koszulity is that we will later use it to construct BGG resolutions by using some further machinery. But before delving into that, we already have an immediate consequence just from basic Koszul theory, see subsection 4.

[4]This sentence was written by the third author in 2023, but since then this type of sentence structure has become a smoking gun for AI-produced slop. While regrettable, the third author refuses to change what he had already written on account of AI's behaviors.

Proof of nil-Koszulity. The proof is slightly technical so we put it in a separate header.

Our strategy for proving such a thing is 'divide-and-conquer' – we will aim to prove a equivalent formulation of Koszulity, namely that of "distributive lattices", and in order to prove this equivalent formulation we will divide-and-conquer first by looking at idempotents and second by looking at "nesting patterns".

Let us lay some relevant background, first defining what a distributive lattice is. Note that they are defined differently in [PP05]; we take here as definition an equivalent condition, see Proposition 1.7.1 of [PP05].

Definition 4.5 ([PP05] Chapter 1 Proposition 7.1)**.** For V a vector space, a collection of its subspaces $U_1, \cdots, U_n$ is said to be "distributive" (or "generate a distributive lattice") if there exists a basis of V such that each U_i is the span of a subset of this basis. Such a basis is called a "distributing basis".

The significance of this definition is that it can be used to characterize Koszulity of a quadratic algebra.

Proposition 4.6 ([PP05] Chapter 2 Theorem 4.1)**.** A quadratic algebra R with quadratic kernel $\mathfrak{q}$ is Koszul if and only if for every $n \geq 0$ the collection of subspaces

$$U_i = R_1^{\otimes i-1} \otimes \mathfrak{q} \otimes R_1^{\otimes n-i-1} \subset R_1^{\otimes n}$$

for $1 \leq i \leq n-1$ is distributive. More precisely, the following are equivalent:

- $\mathrm{Ext}^i_{R\,j}(\mathbb{K}, \mathbb{K}) = 0$ for all $i < j \leq n$;
- the Koszul complex is acyclic in positive internal degrees $\leq n$;
- the collection $\{U_1, \cdots, U_{n-1}\}$ in $R_1^{\otimes n}$ is distributive.

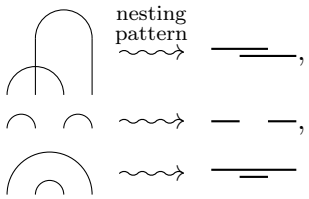
Hence we seek a distributing basis of $A_1^{-,\otimes n}$ for each $n \geq 0$, distributing the collection of spaces $A_1^{-,\otimes i-1} \otimes \mathfrak{q} \otimes A_1^{-,\otimes n-i-1}$ ranging over i. This is a massive space, so we will simplify the job by using divide-and-conquer first via idempotents.

As diagrams with different ending/beginning idempotents are linearly independent, it suffices to give a distributing basis for $e^{m-2n}A_1^{-,\otimes n}e^m$ for each m. Moreover, if we have a basis for $m = 2n$, we can add propagating strings in appropriate places to all the basis diagrams to obtain a basis for any desired m; hence it suffices to give a distributing basis for $e^0 A_1^{-,\otimes n} e^{2n}$. Most importantly, note that the spaces $A_1^{-,\otimes i-1} \otimes \mathfrak{q} \otimes A_1^{-,\otimes n-i-1}$ respect these reductions, in the sense that each relation in $\mathfrak{q}$ is homogeneous with respect to the beginning/ending idempotents.

Things are already looking much more tractable, but we can make things even easier by further performing divide-and-conquer via "nesting patterns of shadows".

Definition 4.7. The "nesting pattern" of a diagram is given by looking at its "shadow". More precisely, color each cap in the diagram, and project the diagram to the horizontal axis; the result is a 1-dimensional topological subspace of the horizontal axis, consisting of segments of different colors, possibly overlapping, with their intersection inheriting both colors. This, considered up to isotopy within the 1-dimensional parent space, is the nesting pattern of this diagram.

Here's a quick diagram depicting this:

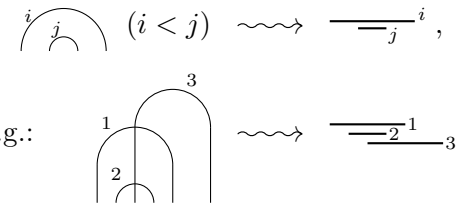


(NP)

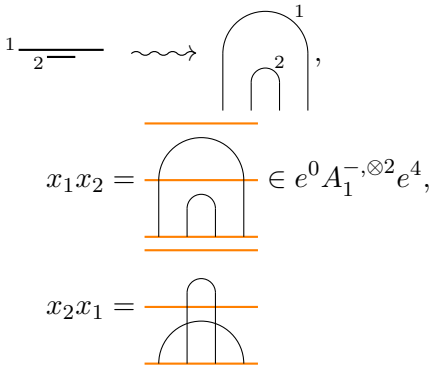
Here to make the picture clearer we have staggered the shadows of the caps in drawing the nesting pattern; it does not matter which one goes on top; all that matters is where they overlap, if they do. It is clear that diagrams with different nesting patterns are linearly independent. Also importantly, all relations in $\mathfrak{q}$ are homogeneous with respect to nesting patterns.

Hence it suffices to provide a distributing basis of the subspace of $e^0 A_1^{-,\otimes n} e^{2n}$ consisting of a fixed nesting pattern P; denote this subspace $e^0 A_1^{-,\otimes n} e^{2n}(\mathrm{P})$.

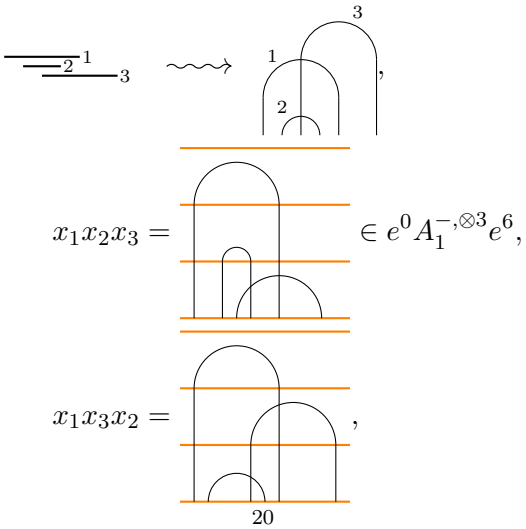
So let us fix a nesting pattern P. Also fix a numbering of each segment, in such a way that a segment labeled j nested inside a segment labeled i has $i < j$:

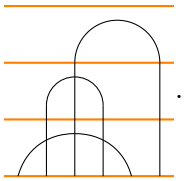


For any two segments, diagram NP exhausts the possibilities of their nesting patterns. Let each segment i have a formal algebraic symbol x_i attached to it. We think of x_i as 'the cap for segment i'. Given a monomial in the x_i's, we read it as right to left corresponding to bottom to top in terms of A^- diagrams. For example, if one has a pattern P of a segment labeled 2 sitting inside a segment labeled 1, then x_1x_2 (cf. x_2x_1) corresponds to the following element of $e^0 A_1^{-,\otimes 2} e^4$:



Note that x_2x_1 is then an element in $\mathfrak{q}$. Similarly, if the pattern was two segments neither of which sits entirely inside the other, then $x_1x_2 - x_2x_1$ is an element in $\mathfrak{q}$. Here's a slightly more complicated example:



$x_3x_2x_1 =$  .

More generally, given any two segments i, j in P, corresponding to two caps x_i, x_j, either

$$x_ix_j - x_jx_i \in \mathfrak{q} \quad \text{or} \quad x_jx_i \in \mathfrak{q}, \ \ i < j;$$

the latter case corresponds to if segment j lies inside segment i, and the former is otherwise. Since the relation of 'lying inside of' is transitive, namely $k \subset j, \ j \subset i \implies k \subset i$, we have

$$x_kx_j \in \mathfrak{q}, \ x_jx_i \in \mathfrak{q} \implies x_kx_i \in \mathfrak{q}.$$

Lastly, we only want each cap to appear once, so we must impose for each i

$$x_i^2 \in \mathfrak{q}.$$

We want a distributing basis for $e^0A_1^{-,\otimes n}e^{2n}(\mathrm{P})$. There will be n segments in P, corresponding to caps $x_1, \cdots, x_n$. Consider the algebra $R = R(\mathrm{P})$ over $\mathbb{C}$ with generators $x_1, \cdots, x_n$ subject to the quadratic relations

(1) $x_i^2 = 0$ for each i,
(2) either $x_ix_j = x_jx_i$ or $x_jx_i = 0$ for every $i < j$, depending on P; this will automatically be in such a way that $x_kx_j = 0$ and $x_jx_i = 0 \implies x_kx_i = 0$;

clearly, if this were n-Koszul[5], then one has a basis of $R_1^{\otimes n}$ distributing the collection $\{R_1^{\otimes i-1} \otimes \mathfrak{q} \otimes R_1^{\otimes n-i-1}\}_i$, and this same basis will be a distributing basis for $e^0A_1^{-,\otimes n}e^{2n}(\mathrm{P})$ under the obvious isomorphism. So we have reduced what we want to a (semi?)commutative algebra question. Thankfully it turns out such algebras $R(\mathrm{P})$ are not only n-Koszul but Koszul.

At this point we may safely cite a theorem from [Zho24, Proposition 5.13]:

Proposition 4.8. Consider an algebra R which is generated by $x_1, \cdots, x_n$ subject to the relations

(1) $x_i^2 = 0$ for each i;
(2) for each $i < j$ either:
   (a) $x_ix_j = c_{ij}x_jx_i$ for $c_{ij} \in \Bbbk^\times$, or
   (b) $x_jx_i = 0$,
   such that $x_kx_j = 0, \ x_jx_i = 0 \implies x_kx_i = 0$.

Then R is Koszul.

This fact can be proven by using classical tools from [PP05]; we refer the reader to [Zho24] for a proof. Then we can see the algebras $R(\mathrm{P})$ considered above are exactly examples of algebras in Proposition 4.8, where we set $c_{ij} = +1$.

So we have proved $R(\mathrm{P})$ is Koszul, which gives a distributing basis for $R(\mathrm{P})_1^{\otimes n}$, which gives a distributing basis for $e^0A_1^{-,\otimes n}e^{2n}(\mathrm{P})$, which shows A^- is Koszul via divide-and-conquer. This concludes the proof that the Hermite algebra is nil-Koszul.

[5]We will not use the definition of n-Koszul in this paper, but for completeness let us recall it is defined as follows (see [PP05, Chapter 2, Theorem 4.1]): a quadratic algebra is n-Koszul if $\mathrm{Ext}^i_{R\,j}(\mathbb{K}, \mathbb{K}) = 0$ for all $i < j \leq n$, or equivalently the collection $\{R_1^{\otimes i-1} \otimes \mathfrak{q} \otimes R_1^{\otimes n-i-1}\}_{1 \leq i \leq n-1}$ is distributive in $R_1^{\otimes n}$.

Projectively resolving the big Verma. Using just the nil-Koszulity of A and some basic Koszul theory, we can already projectively resolve standard modules. Recall that given a Koszul algebra R, we can consider the "Koszul complex" constructed as follows: consider the complex

$$K^\bullet := R \otimes R^{!,\vee,\bullet},$$

where $V^\vee$ denotes the *right* linear dual $\mathrm{Hom}_{-\mathbb{K}}(V, \mathbb{K})$ and the convention is that $R^{!,\vee,\bullet}$ indicates that the complex is infinite to the left and ends at $\bullet = 0$, and the differential is given by the following (in the first incarnation): consider

$$\mathrm{mult}\colon R^!_n \otimes R^*_1 \longrightarrow R^!_{n+1} \quad \rightsquigarrow \quad \mathrm{mult}^\vee\colon R^{!,\vee}_{n+1} \longrightarrow R_1 \otimes R^{!,\vee}_n,$$

so that

$$R \otimes R^{!,\vee}_{n+1} \longrightarrow R \otimes R_1 \otimes R^{!,\vee}_n \longrightarrow R \otimes R^{!,\vee}_n,$$

where the second map is multiplying the first two components.

Then it is well-known from Koszul theory that the fact that R is Koszul is corresponding to the fact that

$$K^\bullet \simeq \mathbb{K}.$$

Then let us consider the Koszul complex for A^-. We have a resolution

$$\cdots \longrightarrow A^- \otimes_{\mathbb{K}} A^{-,!,\vee,1} \longrightarrow A^- \otimes_{\mathbb{K}} A^{-,!,\vee,0} \longrightarrow \mathbb{K} \longrightarrow 0,$$

and multiplying on the right by e^n gives us

$$\cdots \longrightarrow A^- \otimes_{\mathbb{K}} A^{-,!,\vee,1} e^n \longrightarrow A^- \otimes_{\mathbb{K}} A^{-,!,\vee,0} e^n \longrightarrow \Bbbk e^n \longrightarrow 0;$$

since A is free over A^-, the functor $A \otimes_{A^-} \square$ is exact, and we can apply this functor to this resolution to obtain

$$\cdots \longrightarrow A \otimes_{\mathbb{K}} A^{-,!,\vee,1} e^n \longrightarrow A \otimes_{\mathbb{K}} A^{-,!,\vee,0} e^n \longrightarrow A \otimes_{A^-} \Bbbk e^n \longrightarrow 0.$$

Using the fact that

$$A \otimes_{A^-} \Bbbk e^n = A^{\geq n} \otimes_{\mathrm{NC}_n} \Bbbk = \Delta_n,$$

that

$$A^{-,!,\vee,k} e^n = e^{n-2k} A^{-,!,\vee,k} e^n,$$

and that

$$A \otimes_{\mathbb{K}} e^{n-2k} A^{-,!,\vee,k} e^n \cong P_{n-2k} \otimes_{\mathbb{K}} e^{n-2k} A^{-,!,\vee,k} e^n$$

as A-modules (with the space $e^{n-2k} A^{-,!,\vee,k} e^n$ as a multiplicity space), we can then obtain a finite resolution

$$\cdots \longrightarrow P_{n-4}^{\oplus \dim e^{n-4} A^{-,!,\vee} e^n} \longrightarrow P_{n-2}^{\oplus \dim e^{n-2} A^{-,!,\vee} e^n} \longrightarrow P_n \longrightarrow \Delta_n \longrightarrow 0,$$

The multiplicities here, being dimensions of the Koszul dual of A^-, can be computed easily as

$$\begin{aligned}
\dim e^{n-2k} A^{-,!,\vee} e^n &= \text{number of ways to choose } k \text{ pairs of points from } n \text{ points} \\
&= \binom{n}{n-2k} (2k-1)!! \\
&= \frac{n!}{2^k k! (n-2k)!} \\
&= u_{n,k}
\end{aligned}$$

agreeing with the multiplicities calculated in an earlier section.

Note well that the maps in the resolution constructed this way are baked into the Koszul theory of A^- – they are defined by the multiplications on A^- and $A^{-,!}$. We would like to remark that a similar strategy will work for any nil-Koszul diagrammatic algebra; the problem is computing the dimensions of the Koszul dual may not be so easy in other cases.

5. Computing Ext groups via nil-cohomology

Before giving the main consequence of the nil-Koszulity of A, we must compute some Ext groups for later use. In particular, we are interested in computing the Ext groups[6]

$$\mathrm{Ext}^\bullet_A(A^{\geq n}e^n, L_m) = \mathrm{Ext}^\bullet_A(\Delta_n, L_m).$$

Note that this is the analogue of the Kazhdan-Lusztig problem. However, unlike the classical category $\mathcal{O}$ case, these Ext groups have richer structure – since we have an action $A^{\geq n}e^n \circlearrowright A^n \cong \mathrm{NC}_n$, these Ext groups are actually left NC_n-modules:

$$\mathrm{NC}_n \circlearrowright \mathrm{Ext}^\bullet_A(\Delta_n, L_m).$$

Moreover, note that the Hermite algebra A has an anti-involution given by turning diagrams upside-down; this gives an anti-involution on the subquotients $A^n = \mathrm{NC}_n$, also given by turning diagrams upside-down. Using this anti-involution τ one can define a duality (which one might call 'contragredient duality') on left NC_n-modules in the usual way, by having $a \in \mathrm{NC}_n$ send $f \in \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$ to $(a \cdot f)(v) = f(\tau(a)v)$; see [Bru23] for more details. So we also have

$$\mathrm{NC}_n \circlearrowright \mathrm{Ext}^\bullet_A(\Delta_n, L_m)^*.$$

It turns out we will not only want the dimensions of these Ext groups but also the NC_n-structure of their duals. To solve this Kazhdan-Lusztig analogue, let us first recall the notion of "nil-cohomology".

Nil-cohomology. Recall that $\mathrm{Ext}^\bullet(\Delta_\lambda, \square) = H^\bullet(\mathfrak{n}^+ : \square)^\lambda$ in category $\mathcal{O}$, where the key was to write this as $\mathsf{RHom}_{\mathfrak{n}^+}(\mathbb{C}_0, \square)$, which admits an action of $\mathfrak{h}$. This in turn can be computed with the Koszul resolution of $\mathbb{C}_0$, which importantly is a finite resolution (because it is built from wedge powers of $\mathfrak{n}^+$, which is finite-dimensional). This is the story we will try to emulate.

For the remainder of this subsection let A be an arbitrary triangular-based algebra. If the reader prefers, they can pretend that all instances of θ are replaced with $n \in \mathbb{N}$ to recover the Hermite case. Let $\mathbb{K} = \bigoplus \Bbbk e^\theta$, and let the "nilalgebra" be defined as

$$A^- \coloneqq \mathbb{K}\mathrm{Y} = \mathbb{K} \oplus \bigoplus_{\psi \not\geq \theta} \bigoplus e^\psi \mathbb{C}\mathrm{Y}_+ e^\theta,$$

where the subscript in Y_+ indicates that the diagram e^θ is not included (i.e. we are looking at diagrams with positive cap-degree). This is the analogue of $U\mathfrak{n}^+$. Of course one must check on a case-by-case basis that, so defined, this A^- is actually a subalgebra of A, i.e. that it is closed under multiplication. This is typically not difficult, and indeed we have done it for the Hermite algebra earlier.

This A^- then has an ideal

$$I \coloneqq A^-_+ \coloneqq \bigoplus_{\psi \not\geq \theta} \bigoplus e^\psi \mathbb{C}\mathrm{Y}_+ e^\theta,$$

which is analogous to the positive degree elements of $U\mathfrak{n}^+$. Note that $A^-/I = \mathbb{K}$, and that there is an action

$$A^- \circlearrowright \Bbbk e^\theta$$

where all of I acts by zero and $\mathbb{K}$ acts in the obvious way by multiplication. We can consider the obvious modification of the bar resolution in order to resolve $\Bbbk e^\theta$ with free A^--modules (here all tensors are over $\mathbb{K}$):

$$\cdots \longrightarrow A^- \otimes I^{\otimes k} e^\theta \longrightarrow \cdots \longrightarrow A^- \otimes I \otimes I e^\theta \longrightarrow A^- \otimes I e^\theta \longrightarrow A^- e^\theta \longrightarrow \Bbbk e^\theta,$$

[6]The fact that $A^{\geq n}e^n = \Delta_n$ here is a phenomenon specific to Hermite.

where the maps are (for brevity we write $(a_1, \cdots, a_k)$ for $a_1 \otimes \cdots \otimes a_k$)

$$\partial_k \colon A^- \otimes_{\mathbb{K}} I^{\otimes k} e^\theta \longrightarrow A^- \otimes_{\mathbb{K}} I^{\otimes k-1} e^\theta$$

$$x \otimes (a_1, \cdots, a_k e^\theta) \longmapsto xa_1 \otimes (a_2, \cdots, a_k e^\theta) + \sum_{i=1}^{k-1} (-1)^i x \otimes (a_1, \cdots, a_i a_{i+1}, \cdots, a_k e^\theta).$$

This can be shown to be exact by using the obvious modification of the usual homotopy: send $x \otimes a_1 \otimes \cdots \otimes a_k$ to $1 \otimes \pi_+(x) \otimes a_1 \otimes \cdots \otimes a_k$, where $\pi_+ \colon A^- \longrightarrow I$ is projection away from $A_0^- = \mathbb{K}$.

Now, note that (the emphasis being on the last equality)

$$\Delta(\theta) \coloneqq A^{\geq \theta} e^\theta = A \otimes_{A^-} \Bbbk e^\theta,$$

where we consider $A^- \circlearrowright \Bbbk e^\theta$ by having I kill all vectors. Hence

$$\mathrm{Hom}_A(\Delta(\theta), M) = \mathrm{Hom}_A(A \otimes_{A^-} \Bbbk e^\theta, M) = \mathrm{Hom}_{A^-}(\Bbbk e^\theta, M).$$

In fact, since A is free as a right A^--module, the derived version is true as well:

$$\mathsf{RHom}_A(\Delta(\theta), M) = \mathsf{RHom}_A(A \overset{\mathsf{L}}{\otimes}_{A^-} \Bbbk e^\theta, M) = \mathsf{RHom}_{A^-}(\Bbbk e^\theta, M).$$

Note that a map in $\mathrm{Hom}_{A^-}(\Bbbk e^\theta, M)$ is determined by where it sends e^θ, provided that the image is killed by I (since e^θ is killed by I), so we have

$$\mathrm{Hom}_A(\Delta(\theta), M) = \mathrm{Hom}_{A^-}(\Bbbk e^\theta, M) = (M^{A^-})^\theta,$$

where

$$M^{A^-} \coloneqq \{v \in M : I \cdot v = 0\}$$

is the submodule of vectors killed by I. Then

$$\mathsf{RHom}_A(\Delta(\theta), M) = \mathsf{RHom}_{A^-}(\Bbbk e^\theta, M) = \mathsf{R}(M^{A^-})^\theta \tag{VH}$$

is the derived functor of this. This is all in analogy to the observation from Lie theory of $\mathrm{Hom}_{\mathcal{O}}(\Delta_\lambda, M) = \mathrm{Hom}_{\mathfrak{n}^+}(\mathbb{C}_\lambda, M) = (M^{\mathfrak{n}^+})^\lambda$.

Hence we may define (the θ-weight space of) the "nil-cohomology" as

$$H^\bullet(A^- : M)^\theta \coloneqq \mathsf{RHom}_A(A^{\geq \theta} e^\theta, M) \overset{\mathsf{Vec}}{\cong} \mathsf{RHom}_{A^-}(\Bbbk e^\theta, M)$$

which can be computed as the cohomology of the complex

$$0 \longrightarrow \mathrm{Hom}_{A^-}(A^- e^\theta, M) \longrightarrow \mathrm{Hom}_{A^-}(A^- \otimes I e^\theta, M) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{Hom}_{A^-}(A^- \otimes I^{\otimes k} e^\theta, M) \longrightarrow \mathrm{Hom}_{A^-}(A^- \otimes I^{\otimes k+1} e^\theta, M) \longrightarrow \cdots$$

This complex can be rewritten using tensor-Hom as

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{K}}(\Bbbk e^\theta, M) \longrightarrow \mathrm{Hom}_{\mathbb{K}}(I e^\theta, M) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{Hom}_{\mathbb{K}}(I^{\otimes k} e^\theta, M) \longrightarrow \mathrm{Hom}_{\mathbb{K}}(I^{\otimes k+1} e^\theta, M) \longrightarrow \cdots$$

where each differential $\mathrm{d}^k(f) = f(\partial_{k+1} \square)$ can be simply found to be

$$\mathrm{d}^k \colon \mathrm{Hom}_{\mathbb{K}}(I^{\otimes k} e^\theta, M) \longrightarrow \mathrm{Hom}_{\mathbb{K}}(I^{\otimes k+1} e^\theta, M)$$

$$f \longmapsto \left((a_1, \cdots, a_{k+1} e^\theta) \mapsto a_1 f(a_2, \cdots, a_{k+1} e^\theta) + \sum_{i=1}^{k} (-1)^i f(a_1, \cdots, a_i a_{i+1}, \cdots, a_{k+1} e^\theta) \right).$$

This story can be made to even more closely resemble the Lie algebra one by taking a more 'uniform' approach – consider the action $A^- \circlearrowright \mathbb{K}$ instead of the action on $\Bbbk e^\theta$; then $\mathsf{RHom}_{A^-}(\mathbb{K}, M)$ has a left action of $\mathbb{K}$ given by the right action $\mathbb{K} \circlearrowleft \mathbb{K}$, and we can take the weight spaces with respect to this action to get each

$$H^\bullet(A^- : M)^\theta = e^\theta \operatorname{Ext}^\bullet_{A^-}(\mathbb{K}, M) = \operatorname{Ext}^\bullet_A(\Delta(\theta), M).$$

This $\mathsf{RHom}_{A^-}(\mathbb{K}, M)$ then deserves the name "nil-cohomology":

Definition 5.1. The (negative) "nil-cohomology" of a module $A \circlearrowright M$ is defined as

$$H^\bullet(A^- : M) \coloneqq \operatorname{Ext}^\bullet_{A^-}(\mathbb{K}, M) \simeq \mathsf{R}(M^{A^-}).$$

This is computed with the complex

$$\cdots \longrightarrow A^- \otimes I^{\otimes k} \longrightarrow \cdots \longrightarrow A^- \otimes I \otimes I \longrightarrow A^- \otimes I \longrightarrow A^- \longrightarrow \mathbb{K},$$

which 'weight-specializes' to the complex $A^- \otimes I^{\otimes \bullet} e^\theta$ considered above. More explicitly,

$$H^\bullet(A^- : M) \simeq \operatorname{Hom}_{\mathbb{K}}(I^{\otimes \bullet}, M),$$

where the differentials are identical to d^k computed above except erasing each appearance of e^θ. Worth saying is that, also in analogy with Lie theory, $\mathsf{RHom}^\bullet_{A^-}(\mathbb{K}, M)$ is the right derived functor of the invariant functor $M \mapsto M^{A^-}$ which picks out the set of vectors killed by I. So again one has a cohomology theory obtained from requiring a 'nil-algebra' to act by zero.

Remark 5.2. A warning though: this incarnation of the nilcohomology, using the bar complex above, loses the left A^θ-action on $H^\bullet(A^- : M)^\theta$ due to the lack of an obvious right A^θ-action on $A^- \otimes I^{\otimes k}$. We will fix this later in this section.

It is perhaps worth pointing out, as a special case of the constructions above, that for the module $A^- \circlearrowright \mathbb{K}$, the nilcohomology $H^\bullet(A^- : \mathbb{K}) = \mathsf{RHom}^\bullet_{A^-}(\mathbb{K}, \mathbb{K})$ can be computed as

$$\begin{aligned} \operatorname{Ext}^\bullet_{A^-}(\mathbb{K}, \mathbb{K}) &= H^\bullet\big(\operatorname{Hom}_{A^-}(A^- \otimes I^{\otimes \bullet}, \mathbb{K})\big) \\ &= H^\bullet\big(\operatorname{Hom}_{\mathbb{K}}(I^{\otimes \bullet}, \mathbb{K})\big) \\ &= H^{-\bullet}(I^{\otimes \bullet}). \end{aligned}$$

Also worth noting is that, in the case that A^- is Koszul, we know that

$$\operatorname{Ext}^\bullet_{A^-}(\mathbb{K}, \mathbb{K}) \cong A^{-,!,\mathrm{op}}.$$

Using the right actions of $\mathbb{K} \circlearrowleft \mathbb{K}$ for both entries to obtain left and right $\mathbb{K}$-actions on the Ext group, we then have

$$e^\phi A^{-,!} e^\theta = e^\theta \operatorname{Ext}^\bullet_{A^-}(\mathbb{K}, \mathbb{K}) e^\phi = \operatorname{Ext}^\bullet_{A^-}(\Bbbk e^\theta, \Bbbk e^\phi).$$

Computing. Now let us return to letting A be the Hermite algebra. Recall from Section 2 that

$$L_m \cong \Bbbk e^m.$$

Then the Koszul result guarantees that $\mathsf{RHom}_A(\Delta_n, L_m) = \mathsf{RHom}_{A^-}(\Bbbk e^n, \Bbbk e^m)$ is concentrated in homological degree $\frac{n-m}{2}$, so we can pick a resolution (namely the truncated bar resolution from before) and take alternating sums to compute the dimension. However, we also want the left NC_n-structure on these Ext groups. Since NC_n is a graded algebra by letting

$$\deg \diagup\!\!\!\!\diagdown = -2,$$

and since the Hermite algebra can also be graded by extending this (letting caps and cups be of degree 0), we can instead compute the graded dimension $\dim_q \operatorname{Ext}_{A^-}^{\frac{n-m}{2}}(\Bbbk e^n, \Bbbk e^m)^*$ – as the simples

of NC_n are concentrated in degree 0, this will be sufficient to tell us a 'Loewy filtration' on $\mathrm{NC}_n \circlearrowright \mathrm{Ext}_{A^-}^{\frac{n-m}{2}}(\Bbbk e^n, \Bbbk e^m)^*$.

Remark 5.3. Here for the graded dimension we are adopting the conventions of [Bru23], which means that the graded dimension of a vector space $V = \bigoplus V_k$ is

$$\dim_q V = \sum_k \dim(V_k) q^{-k};$$

note well the minus sign. The third author seems to recall that the two choices here both have upsides and downsides and there is no obvious choice.

Remark 5.4. A brief warning: we are here computing the quantum dimension of the dual $\mathrm{Ext}_{A^-}^{\frac{n-m}{2}}(\Bbbk e^n, \Bbbk e^m)^*$, which means that the powers on q are flipped from the quantum dimension of $\mathrm{Ext}_{A^-}^{\frac{n-m}{2}}(\Bbbk e^n, \Bbbk e^m)$.

Note also that n, m must then be of the same parity.

The case $m = 0$. This is the easiest case, being a corollary of the computation in [Zho24]. In this case, since n must be of the same parity as 0, to simplify expressions let us compute $\dim_q \mathrm{Ext}_{A^-}^n(\Bbbk e^{2n}, \Bbbk e^0)^*$.

Proposition 5.5. The quantum dimension (where crossings have degree -2) can be computed as

$$\dim_q \mathrm{Ext}_{A^-}^n(\Bbbk e^{2n}, \Bbbk e^0)^* = (1+q^2+q^4)(1+q^2+q^4+q^6+q^8)\cdots(1+q^2+\cdots+q^{4n-4}).$$

Note that at $q = 1$ this is $(2n-1)!!$.

Proof. The proof of this is simply by computing the quantum dimensions of $e^0 I^{\otimes i} e^{2n}$ and taking the alternating sum; it boils down to a combinatorial identity of binomial coefficients.

We wish to compute the dimension of $\mathrm{Ext}_{A^-}^\bullet(\Bbbk e^{2n}, \Bbbk e^0)^* \simeq \mathsf{RHom}_{A^-}^\bullet(A^- \otimes_{\mathbb{K}} I^{\otimes \bullet} e^{2n}, \Bbbk e^0)^* = \mathsf{RHom}_{\mathbb{K}}^\bullet(I^{\otimes \bullet} e^{2n}, \Bbbk e^0)^* = e^0 I^{\otimes \bullet} e^{2n}$. This is the complex (here the heart symbol indicates homological degree 0)

$$0 \longrightarrow e^0 I^{\otimes n} e^{2n} \longrightarrow \cdots \longrightarrow e^0 I^{\otimes 2} e^{2n} \longrightarrow e^0 I e^{2n} \longrightarrow \overset{\heartsuit}{e^0 \mathbb{K} e^{2n}} \longrightarrow 0.$$

Since we know that this Ext group is concentrated in one degree (namely n), we know

$$\sum_{k=0}^{k} (-1)^k \dim_q e^0 I^{\otimes k} e^{2n} = (-1)^n \dim_q \mathrm{Ext}_{A^-}^n(\Bbbk e^{2n}, \Bbbk e^0)^*.$$

In order to compute the LHS, let us define

$$Y_{i,j} := \text{ be the generating function for dotless Y diagrams from } j \text{ to } i;$$

in other words, the coefficient in front of q^k in $Y_{i,j}$ is the number of cap diagrams from j strings to i strings of degree $-k$, i.e. with exactly $k/2$ crossings (note k must be even). Then it is easy to see

$$\dim_q e^0 I^{\otimes k} e^{2n} = \sum_{1 \le i_1 < \cdots < i_{k-1} \le n-1} Y_{0,2i_1} \cdots Y_{2i_{k-1},2n}.$$

So the claim is then saying that

$$\sum_{k=0}^{n} (-1)^{n-k} \sum_{1 \le i_1 < \cdots < i_{k-1} \le n-1} Y_{0,2i_1} \cdots Y_{2i_{k-1},2n} = (1+q^2+q^4)(1+q^2+q^4+q^6+q^8)\cdots(1+q^2+\cdots+q^{4n-4}).$$

This is morally a PIE (principle of inclusion-exclusion) argument. On the RHS is the generating function for number of ways to go from $2n$ strings to 0 strings in n steps, where each step has exactly one cap whose left-most vertex is at the far left (or alternatively whose right-most vertex is at the far right). On the LHS one starts at $k = n$ with the product of all the ways to go from

$2n$ strings to 0 strings in n steps except with no requirement on where the endpoints of the cap are, and each subsequent term consists of ways of combining multiple steps into one. To make this identity easier to show, we can apply a reduction using the idea of nesting patterns from earlier: it is clear that for each nesting pattern, there is exactly one cap diagram counted by the RHS, namely the one for which the caps are arranged like a staircase from left to right. Then it suffices to show this for a fixed nesting pattern, as the terms on the LHS can also be organized according to nesting patterns. WLOG it suffices to show this for the nesting pattern of n disjoint segments, where the desired identity is

$$1 = \sum_{k=1}^{n} \sum_{1 \le i_1 < \cdots < i_{k-1} \le n-1} (-1)^{n-k} \binom{i_1}{0} \cdots \binom{n}{i_{k-1}}.$$

This is now a purely elementary combinatorial identity which can be proved with some elbow grease – we refer to [Zho24] for the proof of this combinatorial identity. ∎

The upshot of having the quantum dimension is that the Loewy filtration of $\operatorname{Ext}^n_{A^-}(\Bbbk e^{2n}, \Bbbk e^0)^*$ is given exactly by the quantum dimension. This is because the simples of NC_{2n} are only 1-dimensional and the Jacobson radical $\mathfrak{j}_{2n} \subset \mathrm{NC}_{2n}$ consists of precisely all crossings, so we can use the usual filtration $\mathfrak{j}_{2n}^k \otimes_{\mathrm{NC}_{2n}} \operatorname{Ext}^n_{A^-}(\Bbbk e^{2n}, \Bbbk e^0)^*$ to get associated graded pieces

$$\operatorname{gr}^k \operatorname{Ext}^n_{A^-}(\Bbbk e^{2n}, \Bbbk e^0)^* = \left(\mathfrak{j}_{2n}^k \big/ \mathfrak{j}_{2n}^{k+1}\right) \otimes_{\mathrm{NC}_{2n}} \operatorname{Ext}^n_{A^-}(\Bbbk e^{2n}, \Bbbk e^0)^*,$$

which is semisimple. Hence

Proposition 5.6. The module $\mathrm{NC}_{2n} \circlearrowright \operatorname{Ext}^n_{A^-}(\Bbbk e^{2n}, \Bbbk e^0)^*$ has a (decreasing) filtration with associated graded pieces

$$\operatorname{gr}^k \operatorname{Ext}^n_{A^-}(\Bbbk e^{2n}, \Bbbk e^0)^* \cong \Bbbk^{\oplus [q^{2k}]\left\{(1+q^2+q^4)(1+q^2+q^4+q^6+q^8)\cdots(1+q^2+\cdots+q^{4n-4})\right\}},$$

where $\Bbbk$ is the (trivial) simple of NC_{2n} and $[q^{2k}]\{(1+q^2+q^4)(1+q^2+q^4+q^6+q^8)\cdots(1+q^2+\cdots+q^{4n-4})\}$ denotes the coefficient in front of q^{2k} in the expansion of $(1+q^2+q^4)(1+q^2+q^4+q^6+q^8)\cdots(1+q^2+\cdots+q^{4n-4})$.

The case $m \neq 0$. In this case it is still not hard to see that the classical dimension of the Ext groups is

$$\dim \operatorname{Ext}^{\frac{n-m}{2}}_{A^-}(\Bbbk e^n, \Bbbk e^m) = \binom{n}{m}(n-m-1)!!,$$

simply by doing the same computation as above (using the classical dimensions of $e^0 I^{\otimes k} e^n$ instead), but without the quantum dimension I cannot say anything about a Loewy filtration.

6. BGG resolutions

In accordance with the philosophy of [Zho24], in this section we give the main consequence of the nil-Koszulity of the Hermite algebra, namely that we can construct BGG resolutions. To do so we will require some categorical machinery, which will be powered by the Ext groups we computed earlier. Let us first recall how this machine works. There is quite a bit of setup which is required to state the full functorial statement; to just see the classical spectral sequence 'shadow' of this, the reader can skip to Theorem 6.5.

Reconstruction from stratification. We use highest weight notation in this section on general machinery, and will return to lowest weight notation when we return to the setting of Hermite or other specific diagram algebras. Let $\mathsf{St}\infty\mathsf{Cat}$ denote the ∞-category of stable ∞-categories, and let Λ be a poset with a unique final object (our convention on arrows is that final means maximal).

Definition 6.1. A "filtration" on $\mathcal{C} \in \mathsf{St}\infty\mathsf{Cat}$ is a functor

$$\mathcal{F}\colon \Lambda \longrightarrow \mathsf{St}\infty\mathsf{Cat}$$

such that all arrows go to fully faithful embeddings and the final object goes to $\mathcal{C}$. For $\lambda \in \Lambda$, we denote $\mathcal{C}^{\leq\lambda} = \mathcal{F}(\lambda)$ and $\mathcal{C}^{<\lambda}$ as the smallest stable subcategory containing all $\mathcal{C}^{\leq\mu}$ for all $\mu < \lambda$. Let $\mathcal{C}^{=\lambda} = \mathcal{C}^{\leq\lambda}/\mathcal{C}^{<\lambda}$ be the Verdier quotient.

Our convention is to write $\mu \to \lambda$ for $\mu < \lambda$. Let us name

$$\begin{aligned}
\bar{\imath}_\lambda &\colon \mathcal{C}^{\leq\lambda} \longrightarrow \mathcal{C},\\
\imath_\mu^{<\lambda} &\colon \mathcal{C}^{\leq\mu} \longrightarrow \mathcal{C}^{<\lambda},\\
\imath_\mu^{\lambda} &\colon \mathcal{C}^{\leq\mu} \longrightarrow \mathcal{C}^{\leq\lambda},\\
\imath_\lambda &\colon \mathcal{C}^{<\lambda} \longrightarrow \mathcal{C}^{\leq\lambda},\\
\jmath^\lambda &\colon \mathcal{C}^{\leq\lambda} \longrightarrow \mathcal{C}^{=\lambda};
\end{aligned}$$

for our purposes let us require both the inclusions of each $\mathcal{C}^{\leq\lambda}$ as well as the Verdier quotient functors to have both adjoints. This is for example the case with recollement situations. Let the adjoints, like in the recollement case, be named so that $\bar{\imath}_\lambda^*$ and $\jmath^\lambda_!$ are left adjoints.

In another direction, one can filter an object of an ∞-category by a poset Λ:

Definition 6.2. A "Λ-filtered object of $\mathcal{C}$" is a functor

$$\mathcal{X} \in \mathsf{Fun}(\Lambda, \mathcal{C}).$$

Letting Λ^0 be the 0-skeleton of Λ, the "associated graded" of this filtered object is

$$\begin{aligned}
\mathrm{gr}\colon \mathsf{Fun}(\Lambda,\mathcal{C}) &\longrightarrow \mathsf{Fun}(\Lambda^0,\mathcal{C})\\
\mathcal{X} &\longmapsto \mathrm{gr}\,\mathcal{X}
\end{aligned}$$

defined by

$$(\mathrm{gr}\,\mathcal{X})(\lambda) = \mathrm{gr}^\lambda \mathcal{X} = \mathrm{Fib}\left(\mathcal{X}(\lambda) \longrightarrow \lim_{\mu\leftarrow\lambda} \mathcal{X}(\mu)\right),$$

assuming such limits exist.

An object $X \in \mathcal{C}$ is said to have a Λ-filtration if after adding an initial/minimal element $\lambda_{-\infty}$ to Λ there is a functor $\mathcal{X}\colon \Lambda \cup \{\lambda_{-\infty}\} \longrightarrow \mathcal{C}$ such that $\mathcal{X}(\lambda_{-\infty}) = X$.

In the previous discussion we alluded to a general fact that said that in appropriate stratification/recollement situations, the identity functor on the big category admits a filtration. More precisely, the statement is

Theorem 6.3 ([AMGR22], folklore, [Zho24])**.** Let $\mathcal{C}$ admit a filtration by Λ such that the inclusions $\imath_\lambda$ and quotients $\jmath^\lambda$ have both adjoints. Then the identity functor $\mathrm{Id}_\mathcal{C} \in \mathsf{End}\,\mathcal{C}$ admits a Λ^{op}-filtration with terms

$$\mathcal{I}\mathrm{d}(\lambda) = \bar{\imath}_\lambda \bar{\imath}_\lambda^*,$$

and the associated graded can be computed as

$$\mathrm{gr}^\lambda \mathcal{I}\mathrm{d} = \bar{\imath}_\lambda \jmath^\lambda_! \jmath^\lambda \bar{\imath}_\lambda^*,$$

provided that Λ is locally-finite and "eventually disjoint-totally-ordered" (which means that for any λ, there exists $\mu \leq \lambda$ such that $\{\nu : \nu \leq \mu\}$ is totally ordered).

If there is a "dimension/length function" $\ell\colon \Lambda \longrightarrow \mathbb{Z}^{\mathrm{op}}$ and $\mathcal{C}$ moreover has a t-structure, then there is a spectral sequence

$$E_1^{p,q} = \bigoplus_{\lambda\in\ell^{-1}(-p)} \pi^{p+q}(\mathrm{gr}^\lambda \mathcal{I}\mathrm{d}) \quad\Longrightarrow\quad E_\infty^{p,q} = \mathrm{gr}^{-p}\,\pi^{p+q}(\mathrm{Id}_\mathcal{C}),$$

where $\pi^\bullet$ are the homotopy groups associated to the t-structure.

Remark 6.4. Also note that if Λ is down-finite (namely that for any λ the set $\{\mu : \mu \leq \lambda\}$ is finite), then Λ automatically satisfies the conditions of the theorem. This is for instance the case with (a block of) category $\mathcal{O}$.

Let us say a few words about the maps in the spectral sequence in Theorem 6.3 above. The maps on the first page E_1 are coming from the functoriality of the (co)cone; indeed, one has a $\mathbb{Z}$-filtration

$$\cdots \longrightarrow \bigoplus_{\ell(\lambda)=i} \mathcal{I}\mathrm{d}(\lambda) \longrightarrow \bigoplus_{\ell(\lambda)=i+1} \mathcal{I}\mathrm{d}(\lambda) \longrightarrow \cdots,$$

where the morphisms are as constructed in the proof of [Zho24, Theorem 3.4], so that applying the functoriality of the fiber to

$$\bigoplus_{\ell(\lambda)=i} \mathcal{I}\mathrm{d}(\lambda) \longrightarrow \bigoplus_{\ell(\lambda)=i+1} \mathcal{I}\mathrm{d}(\lambda) \longrightarrow \bigoplus_{\ell(\lambda)=i+2} \mathcal{I}\mathrm{d}(\lambda)$$

we obtain the exact triangle

$$\bigoplus_{\ell(\lambda)=i} \mathrm{gr}^\lambda \mathcal{I}\mathrm{d} \longrightarrow \mathrm{Fib}\left(\bigoplus_{\ell(\lambda)=i} \mathcal{I}\mathrm{d}(\lambda) \to \bigoplus_{\ell(\lambda)=i+2} \mathcal{I}\mathrm{d}(\lambda)\right) \longrightarrow \bigoplus_{\ell(\lambda)=i+1} \mathrm{gr}^\lambda \mathcal{I}\mathrm{d} \xrightarrow{+1};$$

rotating this gives a map

$$\bigoplus_{\ell(\lambda)=i+1} \mathrm{gr}^\lambda \mathcal{I}\mathrm{d} \longrightarrow \bigoplus_{\ell(\lambda)=i} \mathrm{gr}^\lambda \mathcal{I}\mathrm{d}[1],$$

which under the action of taking π gives the differential maps of E_1.

Application to triangular-based algebras. Now let us return to the setting of the triangular based algebra A, which has lowest weight notation. Note that the recollements described in Section 3 give rise to a Θ^{op}-filtration of the stable ∞-category $\mathsf{D}^-\,\mathsf{Mod}\,A$ where

$$(\mathsf{D}^-\,\mathsf{Mod}\,A)^{\leq\theta} = \mathsf{D}^-\,\mathsf{Mod}\,A^{\geq\theta}.$$

It is easy to see that

$$(\mathsf{D}^-\,\mathsf{Mod}\,A)^{<\theta} = \mathsf{D}^-\,\mathsf{Mod}\,A^{>\theta} = \mathsf{D}^-\,\mathsf{Mod}(A^{\geq\theta}\big/A^{\geq\theta}e^\theta A^{\geq\theta}),$$

so that the Verdier quotient is $\mathsf{D}^-\,\mathsf{Mod}\,A^\theta$. Then, by Theorem 6.3,

Theorem 6.5. Let A be a triangular-based algebra with weight poset Θ locally-finite and eventually disjoint-totally-ordered. There exists a Θ-filtration on the identity functor $\mathrm{Id}_{\mathsf{D}^-\,\mathsf{Mod}\,A}$ whose terms are $\mathcal{I}\mathrm{d}(\theta) = \bar{\imath}_\theta \bar{\imath}_\theta^*$ and whose associated graded is

$$\mathrm{gr}^\theta \mathcal{I}\mathrm{d} = \bar{\imath}_\theta \jmath_!^\theta \jmath^\theta \bar{\imath}_\theta^* = \Delta(\theta) \otimes_{A^\theta} \mathrm{RHom}_A\Big(\Delta(\theta), \square^\dagger\Big)^*,$$

where we recall the shorthand $\Delta(\theta) \coloneqq \bigoplus_{\lambda\in\theta} \Delta_\lambda^{\overline{l_\lambda(\theta)}} = A^{\geq\theta}e^\theta$ and $\overline{l_\lambda(\theta)}$ is the q-conjugate of the quantum dimension $\dim L_\lambda(\theta)$. Moreover, provided there is a length function[7] $\ell\colon \Theta \longrightarrow \mathbb{Z}$, there is a spectral sequence (functorial in the input $\square$)

$$E_1^{p,q} = \bigoplus_{\ell(\theta)=-p} \Delta(\theta) \otimes_{A^\theta} \mathrm{Ext}_A^{-(p+q)}(\Delta(\theta), \square^\dagger)^* \quad \Longrightarrow \quad E_\infty^{p,q} = \mathrm{gr}^{-p} H^{p+q}(\square).$$

[7] Satisfying that $\ell(\Theta)$ has a minimal element.

Execution. Now let us power up this machine with the Ext group computations we did earlier. At this point there is no more work to be done.

Theorem 6.6. For the trivial simple $L_0 = \Bbbk e^0$ of the Hermite algebra, we have a BGG resolution

$$\cdots \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow L_0 \longrightarrow 0,$$

where the terms have (graded) character

$$\chi(C_n) = (1+q^2+q^4)(1+q^2+q^4+q^6+q^8)\cdots(1+q^2+\cdots+q^{4n-4})\chi(\overline{\Delta}_{2n})$$

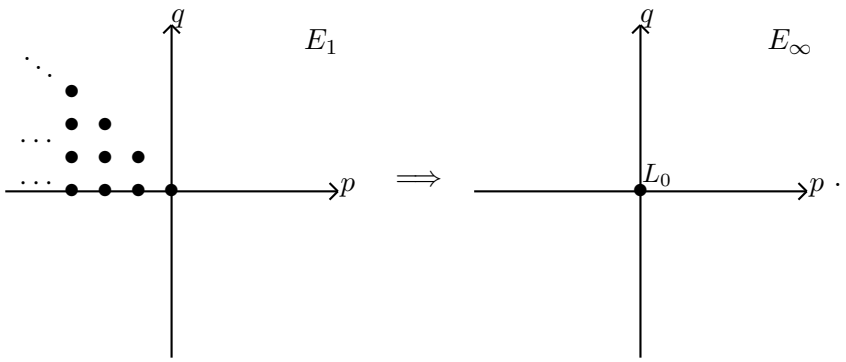
and admit decreasing filtrations with associated graded pieces

$$\operatorname{gr}^k C_n = \overline{\Delta}_{2n}^{\oplus [q^{2k}]\left\{(1+q^2+q^4)(1+q^2+q^4+q^6+q^8)\cdots(1+q^2+\cdots+q^{4n-4})\right\}},$$

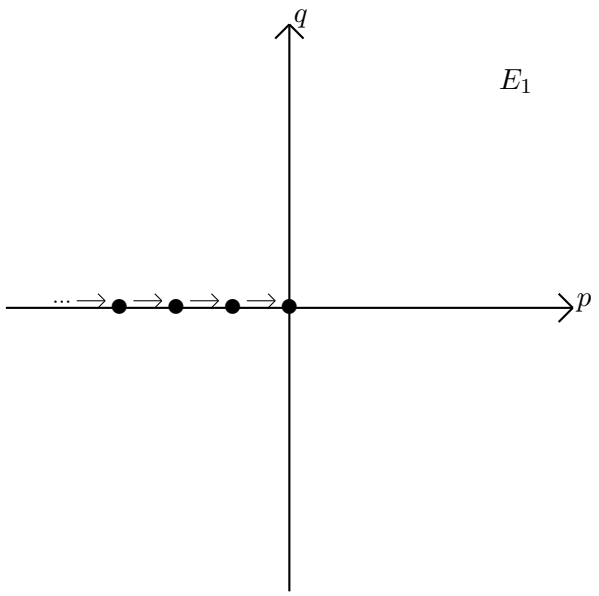
where again $[q^{2k}]\{(1+q^2+q^4)(1+q^2+q^4+q^6+q^8)\cdots(1+q^2+\cdots+q^{4n-4})\}$ denotes the coefficient in front of q^{2k} in the expansion of $(1+q^2+q^4)(1+q^2+q^4+q^6+q^8)\cdots(1+q^2+\cdots+q^{4n-4})$.

Proof. Let us endow our weight poset $\Theta = \mathbb{N}$ with a length function ℓ sending $\ell(\theta) = \lfloor \theta/2 \rfloor$. It is easy to see this satisfies the conditions.

By plugging in $\square = L_0$ to Theorem 6.5, we see that the shape of the spectral sequence on the first page is roughly (our convention is that $\deg \mathrm{d}_r = (r, 1-r)$, so here the differentials go to the right by one step)



But we moreover know that the Ext groups $\operatorname{Ext}_A^{-(p+q)}(\Delta_{2n}, L_0)^*$ are appropriately concentrated, namely in $-p-q = n$, where $n = \ell(2n) = -p$ forces $q = 0$. Hence the spectral sequence for us secretly looks like



The spectral sequence must then immediately collapse, $E_2 = E_\infty$, so that the first page *is* the BGG resolution.

We can compute precisely what the terms of E_1 are. Unwinding Theorem 6.5, we see that

$$C_n = E_1^{-n,0} = A^{\geq 2n} e^{2n} \otimes_{\mathrm{NC}_{2n}} \mathrm{Ext}_A^n(\Delta_{2n}, L_0)^* = \jmath_!^{2n} \mathrm{Ext}_A^n(\Delta_{2n}, L_0)^*,$$

where the term corresponding to Δ_{2n+1} does not appear because $2n+1$ is not the same parity as 0. Then by our earlier work we have already a Lowey filtration on $\mathrm{Ext}_A^n(\Delta_{2n}, L_0)^*$, so under the exact functor $\jmath_!^{2n}$ we obtain exactly the advertised resolution with the advertised structure. ■

Remark 6.7. We would like to remark that the terms in this complex are neither proper standard modules nor standard modules; rather, they are something in between. It would likely have been quite difficult to guess this resolution and in particular the nontrivial extension structure of the terms without this sophisticated machinery.

The case $m \neq 0$. Let us endow Θ with a length function ℓ sending $\ell(\theta) = \lfloor \theta/2 \rfloor$. By unwinding Theorem 6.5, one then finds that the concentration result above gives that for each L_λ, $\lambda \in \mathbb{N}$, there must be spectral sequence converging to (the associated graded of a filtration of) L_λ whose first page is concentrated on the ray $q = \lfloor \lambda/2 \rfloor$, $p \leq -\lfloor \lambda/2 \rfloor$; then this must converge on the second page, which is necessarily L_λ in position $(p, q) = (-\lfloor \lambda/2 \rfloor, \lfloor \theta/2 \rfloor)$, corresponding to a trivial filtration of L_λ where $\mathrm{gr}^{\lfloor \theta/2 \rfloor} = L_\lambda$. In other words:

Theorem 6.8. For other simples $L_m = \Bbbk e^m$ $(m \neq 0)$ of the Hermite algebra, we have a BGG resolution

$$\cdots \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow L_m \longrightarrow 0,$$

where the terms have ungraded character

$$\chi(C_n) = \binom{m+2n}{m} (2n-1)!! \chi(\overline{\Delta}_{m+2n}).$$

We cannot say anything about the Loewy filtration on these terms without the graded dimension computation for Ext groups.

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