Koszul duality and categorifying Jacobi-Trudi

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Link

These slides can be found on my webpage:

math.columbia.edu/~fanzhou/files/beamer-KIAS2026.pdf

Part I: Koszul

Slogan

Slogan

Koszulity of half of A ("nil-Koszulity" of A) is intimately connected to BGG resolutions.

It seems like many (most?) algebras appearing in categorification are nil-Koszul.

Slogan

In good cases, Koszul duality with respect to the nil-algebra *is* the BGG resolution.

This is the case for Soergel, which we discuss here, and Temperley-Lieb.

Today's example

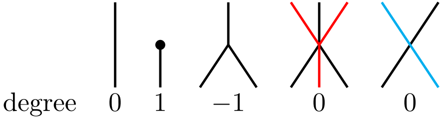
What to think of:

- If you are a representation theorist:
  - (central block of) category $\mathcal{O}$ for $\mathfrak{sl}_n$ $(n = 2, 3)$.
- If you are a geometer:
  - constructible sheaves on the flag variety SL_n / B w.r.t. the Bruhat stratification $(n = 2, 3)$.
- If you do diagrammatic algebra (the central perspective today):
  - modules over "(cyclotomic) Soergel calculus" (endomorphism algebra of some Soergel/Bott-Samelson (bi)modules).

Soergel calculus – generators

Soergel calculus $\mathcal{S}$ is a (graded) algebra spanned by diagrams. Multiplication is stacking diagrams vertically.

Diagrams are generated by:



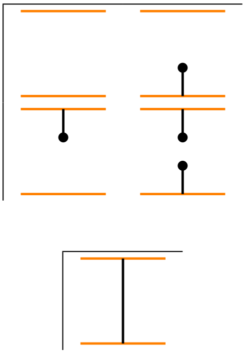
as well as their upside-down flips.

Each string is colored by a node of the Dynkin diagram. Red and black are adjacent, blue and black are distant.

We will consider only diagrams whose top/bottom boundaries are subwords of some redex for $w_0 \in S_n$. (More generally subwords of some redex for $w_\lambda \in S_{\ell(\lambda)}$.)

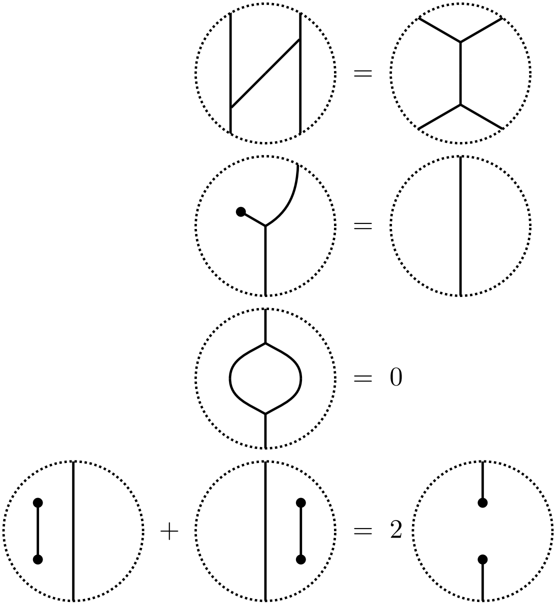
Example: $\mathfrak{sl}_2$

$\mathcal{S}$ for $\mathfrak{sl}_2$ is a 5-dimensional algebra equivalent to $\mathcal{O}^0(\mathfrak{sl}_2)$:



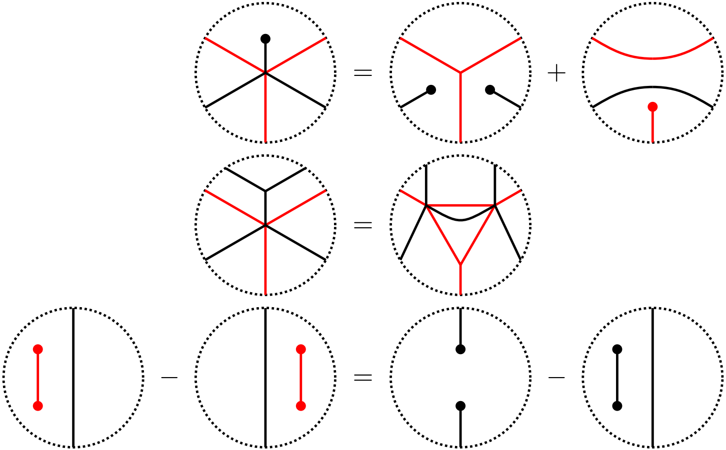
Soergel calculus – relations

1-color:



Soergel calculus – relations

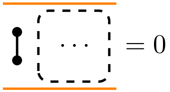
2-color (adjacent):



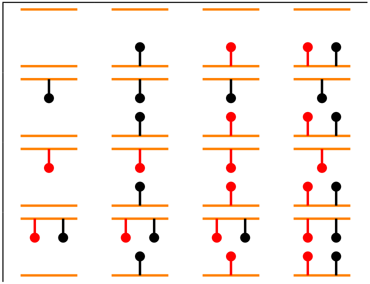
Distant colors pull apart. (There is also a tetrahedral relation.)

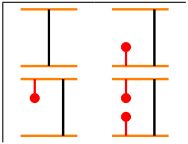
Soergel calculus – cyclotomic relation

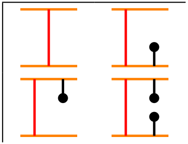
The "cyclotomic condition" for Soergel calculus is setting barbells at the far left to zero:

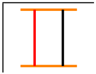


Example: $\mathcal{O}^0(\mathfrak{sl}_3)$ mod simples labeled by $s_1s_2, s_1s_2s_1$









Koszulity

A graded algebra is "quadratic" if it is generated in degree 1 and relations are generated in degree 2:

$$0 \longrightarrow \mathfrak{q} \longrightarrow \bigotimes{}^{\bullet} A_1 \longrightarrow A \longrightarrow 0.$$

Definition

A quadratic graded algebra A ($A_0 = \mathbb{K}$) is "Koszul" if $\mathrm{Ext}_A(\mathbb{K}, \mathbb{K})$ is nonzero only when the homological degree agrees with the Koszul degree.

Koszul dual algebra

If A is Koszul, then

$$\mathrm{Ext}_A(\mathbb{K},\mathbb{K}) \cong A^!,$$

where

$$A^! := \bigotimes\nolimits^{\bullet} A_1^* \big/ \mathfrak{q}^{\perp}$$

is the "quadratic dual" or "Koszul dual" algebra. (Here $\mathfrak{q}^{\perp} \subset \bigotimes^{\bullet} A_1^*$ is the orthogonal complement to $\mathfrak{q}$ under the pairing $(\phi \otimes \psi)(v \otimes w) = \phi(w)\psi(v)$.)

Multiplication is inherited from the tensor algebra.

Warning: need to be careful about left vs right.

Koszul dual coalgebra

There is also a dual *coalgebra*,

$$A^{¡} := A^{!,\vee},$$

with

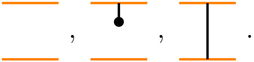
$$A^{¡}_n = \bigcap_i A_1^{\otimes i} \otimes \mathfrak{q} \otimes A_1^{\otimes n-i-2}.$$

Comultiplication is

$$x_1 \otimes \cdots \otimes x_n \longmapsto \sum_i (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n).$$

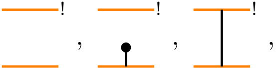
Soergel ($\mathfrak{sl}_2$) example

Consider the subalgebra $\mathcal{S}^+$ for $\mathfrak{sl}_2$ spanned by

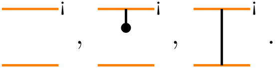


This is Koszul by letting deg() $= 1$. This is not so surprising because $\mathcal{S}^+$ is "morally" $\mathbb{C}[x]/x^2$.

The Koszul dual algebra $\mathcal{S}^{+,!}$ is spanned by

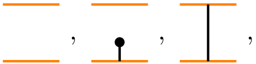


while the Koszul dual coalgebra $\mathcal{S}^{+,¡}$ is spanned by



Note: the upside-down flipping is happening because we need to be careful about left vs right.

Note we could just as easily have considered $\mathcal{S}^-$ spanned by

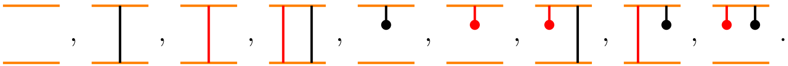


which is also Koszul.

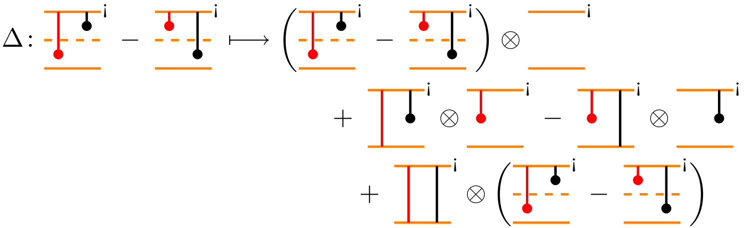
This is simply the "upside-down flipping" anti-involution applied to $\mathcal{S}^+$, i.e. $\mathcal{S}^- = \tau(\mathcal{S}^+)$.

Soergel ($\mathfrak{sl}_3$) example

For the example with 4 cells, consider the subalgebra $\mathcal{S}^+$ spanned by

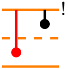
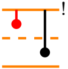


The comultiplication on the Koszul dual coalgebra $\mathcal{S}^{+,¡}$ sends e.g.



This is secretly saying something about BGG differentials.

Another way to think of this is the relation in $\mathcal{S}^{-,!}$

 $=\ -$  $\quad \in \mathcal{S}^{-,!}$

which is because

 $=$  $\quad \in \mathcal{S}^{-}.$

(This is morally the Sym-$\bigwedge$ duality.)

Nil-Koszulity

Definition

Let A be a "highest weight" algebra (e.g. quasi-hereditary or triangular-based [Bru23]). Suppose A has a "nil-algebra" $A^- \subset A$ such that

$$A \overset{\mathsf{L}}{\otimes}_{A^-} \Bbbk e^w = \Delta(w) := A^{\geq w} e^w.$$

Then

$$\mathsf{R}\mathrm{Hom}_A(\Delta(w), \Box) = \mathsf{R}\mathrm{Hom}_{A^-}(\Bbbk e^w, \Box).$$

If A^- is Koszul, then A is said to be "nil-Koszul."

Then

Theorem (Z.)

$\mathcal{S}$ is nil-Koszul, with $\mathcal{S}^- := \tau(\mathcal{S}^+)$ as the nil-algebra with the Soergel grading.

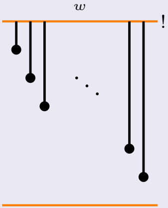
Homological information

Then we have the analogue of Kostant:

Proposition

$$\begin{aligned}\mathsf{RHom}_{\mathcal{S}}(\Delta_w, L_{\mathrm{id}}) &= \mathsf{RHom}_{\mathcal{S}}(\mathcal{S} \overset{\mathsf{L}}{\otimes}_{\mathcal{S}^-} \Bbbk e^w, L_{\mathrm{id}}) \\ &= \mathsf{RHom}_{\mathcal{S}^-}(\Bbbk e^w, \Bbbk e^{\mathrm{id}}) \\ &= e^w \mathcal{S}^{-,!} e^{\mathrm{id}} \\ &\cong q^{\ell(w)} \Bbbk[-\ell(w)],\end{aligned}$$

spanned by the diagram



.

Reconstruction

This type of homological information already allows us to "reconstruct" the trivial module via Vermas. There is some (infinity-)categorical machinery allowing one to reconstruct the identity functor from a stratification using this type of homological information. A concrete shadow is a (functorial) spectral sequence of Vermas.

In this talk we will forgo this in favor of Koszul duality.

Koszul duality

One perspective on Koszul duality, for example in Positselski's work, is that Koszul duality is a functor (equivalence) between derived dg-modules over an algebra and coderived dg-comodules over its Koszul dual coalgebra:

$$\mathsf{D}\,\mathsf{Mod}^{\mathrm{dg}}\,A \;\underset{A\otimes^{\tau}_{\mathbb{K}}\square}{\overset{A^{¡}\otimes^{\tau}_{\mathbb{K}}\square}{\rightleftarrows}}\; \mathsf{coD}\,\mathsf{coMod}^{\mathrm{dg}}\,A^{¡} \qquad (\sim)$$

Here $\otimes^{\tau}$ is a tensor product "twisted" by a "twisting cochain" $\tau\colon A^{¡} \longrightarrow A$ defined by killing everyone except $A^{¡}_1 \cong A_1$.

The twisting

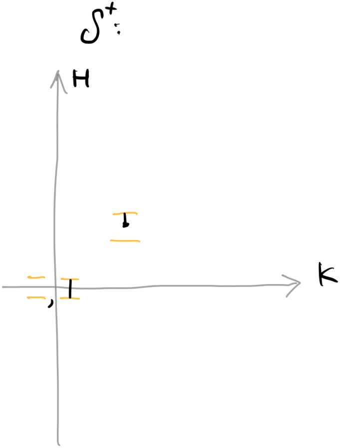
The differential on $A^{¡} \otimes^{\tau} \square$ is given by

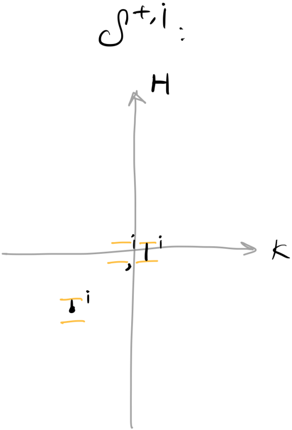
$$\mathrm{d}^{\tau}(c \otimes v) = \mathrm{d}(c) \otimes v + (-1)^{|c|} c \otimes \mathrm{d}(v) + (-1)^{|c_{(1)}|} c_{(1)} \otimes \tau(c_{(2)}) v$$

and on $A \otimes^{\tau} \square$ is given by

$$\mathrm{d}^{\tau}(a \otimes u) = \mathrm{d}(a) \otimes u + (-1)^{|a|} a \otimes \mathrm{d}(u) + (-1)^{|a|+1} a \tau(u_{(-1)}) \otimes u_{(0)}.$$

Example: Soergel ($\mathfrak{sl}_2$)

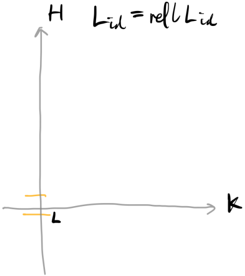




$\mathrm{refl}(M)_j = M_{-j}$, $\mathrm{sh}(M) = M[n]$ if M is in Koszul deg n.

$\mathcal{K}_{A^+} = \mathrm{sh}(A^{+,i} \overset{=}{\underset{\mathbb{K}}{\otimes}} \mathrm{refl}\, \square)$

$L_{id} = \mathrm{refl}\, L_{id}$



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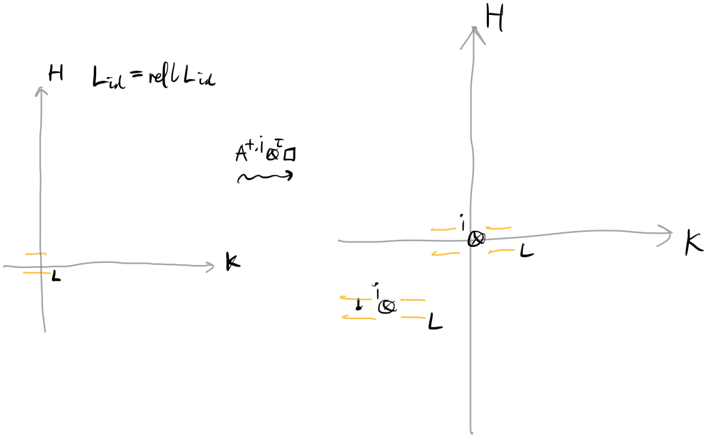
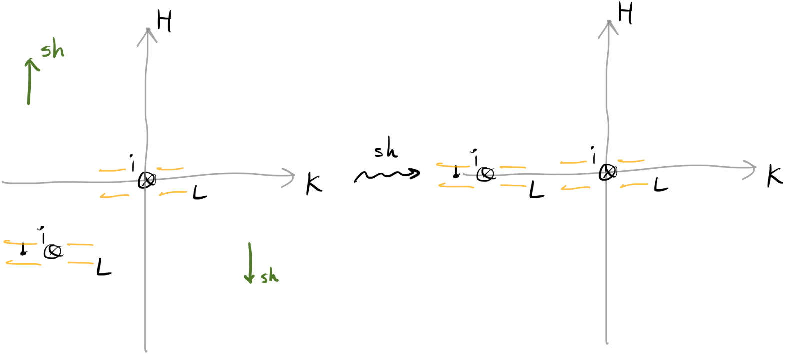
$K_{A^+} = \mathrm{sh}(A^{+,i} \overset{\tau}{\underset{K}{\otimes}} \mathrm{refl}\,\square)$



$\mathrm{refl}(M)_j = M_{-j}$, $\mathrm{sh}(M) = M[n]$ if M is in Koszul deg n.

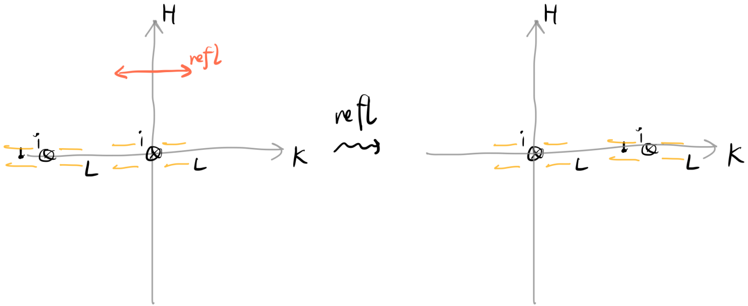
$\mathcal{K}_{A^+} = \mathrm{sh}(A^{+,i} \bar{\otimes}_{\mathbb{K}} \mathrm{refl}\,\square)$



trivial differential d^{τ}

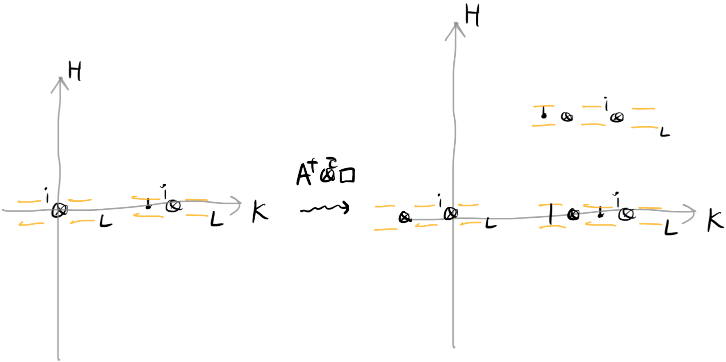
$\mathrm{refl}(M)_j = M_{-j}$, $\mathrm{sh}(M) = M[n]$ if M is in Koszul deg n.

$$\mathbb{K}_{A^+} = \mathrm{sh}(A^{+,i} \bar{\otimes}_{\mathbb{K}} \mathrm{refl}\, \square) \qquad \mathbb{K}^{-1}_{A^+} = \mathrm{sh}(A^+ \tilde{\otimes}_{\mathbb{K}} \mathrm{refl}\, \square)$$



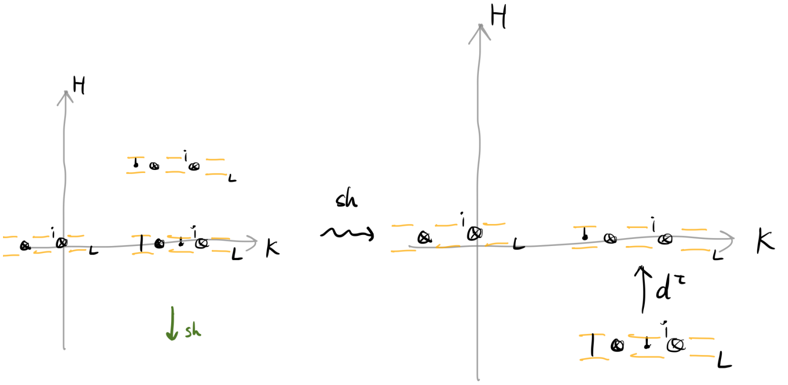
$\mathrm{refl}(M)_j = M_{-j}$, $\mathrm{sh}(M) = M[n]$ if M is in Koszul deg n.

$\mathcal{K}_{A^+} = \mathrm{sh}(A^{+,i} \bar{\otimes}_{\mathbb{K}} \mathrm{refl}\,\square)$ $\quad$ $\mathcal{K}^{-1}_{A^+} = \mathrm{sh}(A^+ \bar{\otimes}_{\mathbb{K}} \mathrm{refl}\,\square)$



$\mathrm{refl}(M)_j = M_{-j}$, $\mathrm{sh}(M) = M[n]$ if M is in Koszul deg n.

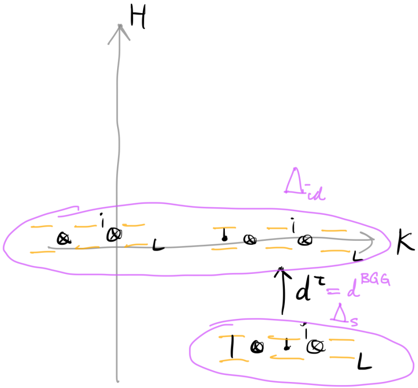
$$\mathcal{K}_{A^+} = \mathrm{sh}(A^{+,i} \otimes_{\mathbb{K}}^{\tau} \mathrm{refl}\,\square) \qquad \mathcal{K}^{-1}_{A^+} = \mathrm{sh}(A^{+} \otimes_{\mathbb{K}}^{\tau} \mathrm{refl}\,\square)$$



$$d^\tau(a \otimes u) = (-1)^{|a|+1} a\tau(u_{(-1)}) \otimes u_{(0)}$$

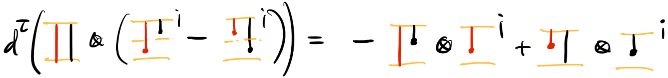
$\mathrm{refl}(M)_j = M_{-j}$, $\mathrm{sh}(M) = M[n]$ if M is in Koszul deg n.

$\mathcal{K}_{A^+} = \mathrm{sh}(A^{+,i} \overset{\tau}{\otimes}_{\mathbb{K}} \mathrm{refl}\,\square)$ $\qquad$ $\mathcal{K}^{-1}_{A^+} = \mathrm{sh}(A^{+} \overset{\tau}{\otimes}_{\mathbb{K}} \mathrm{refl}\,\square)$

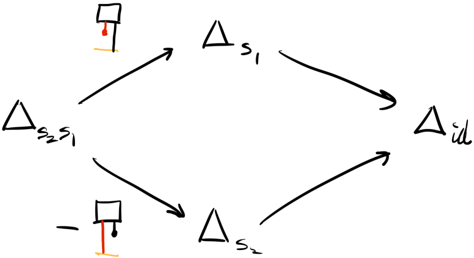


$$d^\tau(a \otimes u) = (-1)^{|a|+1} a\tau(u_{(-1)}) \otimes u_{(0)}$$

Example: $\mathfrak{sl}_3$



compare to



In higher rank

In higher rank, what we have is

$$\mathrm{Id} \cong \mathcal{S} \overset{\mathsf{L}}{\otimes}_{\mathcal{S}^+ \otimes_{\mathbb{K}} \mathcal{S}^-} \mathcal{K}_{\mathcal{S}^+}^{-1} \circ \mathcal{K}_{\mathcal{S}^+} = \mathcal{S} \overset{\mathsf{L}}{\otimes}_{\mathcal{S}^+ \otimes_{\mathbb{K}} \mathcal{S}^-} \mathcal{S}^+ \otimes_{\mathbb{K}}^{\tau} \mathcal{S}^{+,\mathsf{i}} \otimes_{\mathbb{K}}^{\tau} \square.$$

In particular, applying this to L_{id} recovers the BGG resolution.

Part II: Jacobi-Trudi

A Tale of Two Cities

symmetric functions $\longleftrightarrow$ symmetric group representations

The classical story – symmetric functions

We can consider two families of symmetric functions:

- Let Schur functions be s_λ
- and let complete homogeneous functions be $h_\alpha = h_{\alpha_1} \cdots h_{\alpha_k}$, where h_i is the sum of all monomials of degree i.

Jacobi-Trudi

The Jacobi-Trudi determinant identity:

$$s_\lambda = \det(h_{\lambda_i - i + j})_{i,j} = \det \begin{pmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+\ell-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & \vdots \\ & & \ddots & \\ h_{\lambda_\ell-\ell+1} & \cdots & h_{\lambda_\ell-1} & h_{\lambda_\ell} \end{pmatrix},$$

This is an alternating sum.

The classical story – symmetric groups

- Two families of modules over S_n:
- (Over $\mathbb{C}$,) "Specht modules" Σ_λ exhaust irreducibles of S_n.
- Given a composition α and $S_\alpha = S_{\alpha_1} \times \cdots \times S_{\alpha_k}$, the "permutation module" E_α is

$$E_\alpha \coloneqq \mathrm{Ind}_{S_\alpha}^{S_n} \mathrm{triv}\,.$$

  This has dimension $\dim E_\alpha = \binom{n}{\alpha_1, \cdots, \alpha_k}$.
- This decomposes as

$$E_\lambda = \Sigma_\lambda \oplus \bigoplus_{\mu \rhd \lambda} \Sigma_\mu^{\oplus m_\mu}.$$

Category $\mathcal{O}$

Compare to the JH filtration of Vermas $\Delta_{w \circ 0}$ in which $L_{w \circ 0}$ appears as the top layer quotient, and

$$[\Delta_w] = [L_w] + \sum_{u > w} m_u [L_u].$$

The classical story – functions versus groups

- Over $\mathbb{C}$, $\mathsf{Rep}\, S_n$ is equivalent to symmetric functions via the Frobenius character.
- Letting $z_\lambda = \prod_{i \in \mathbb{Z}_+} i^{m_i} m_i!$ where $m_i = \#\{j : \lambda_j = i\}$, $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$,

$$\chi(M) = \sum_{\lambda \vdash n} \mathrm{tra}(\lambda|_M) \frac{p_\lambda}{z_\lambda} = \frac{1}{n!} \sum_{w \in S_n} \mathrm{tra}(w|_M) p_{\lambda(w)}.$$

- This sends

$$\chi \colon \bigoplus_n K_0(\mathsf{Rep}\, S_n) \xrightarrow{\sim} \Lambda$$

$$\Sigma_\lambda \longmapsto s_\lambda$$

$$E_\lambda \longmapsto h_\lambda$$

Question

Is there some highest-weight explanation/elucidation for this (the red boxes)? More precisely:

Question

Find a quasi-hereditary A with a map $\mathbb{C}S_n \longrightarrow A$ such that

$$\text{standard module} \overset{\text{Res}}{\longmapsto} \text{permutation module}$$

$$\text{simple module} \overset{\text{Res}}{\longmapsto} \text{Specht module}$$

Moreover find a BGG resolution over A of simples by standards such that restriction gives a resolution of Spechts by permutations, categorifying

$$s_\lambda = \det(h_{\lambda_i+j-i})_{i,j}.$$

Spoilers

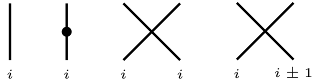
- This is done by considering a quotient of (cyclotomic) KLR.
- $\lambda \rightsquigarrow \alpha, \omega \rightsquigarrow \mathcal{R}_\alpha^\omega$
- Define a quotient $\mathring{\mathcal{R}}_\lambda$ by $\mathcal{R}_\alpha^\omega \twoheadrightarrow \mathring{\mathcal{R}}_\lambda$
- $\mathring{\mathcal{R}}_\lambda$ is quasi-hereditary with weight poset an ideal in $S_{\ell(\lambda)}$.
- The "dominant" simple L_1 will have a BGG resolution by Vermas Δ_w.
- Under $\mathbb{C}S_n \longrightarrow \widehat{\mathcal{H}}_n \longrightarrow \widehat{\mathcal{H}}_\alpha^\omega \xrightarrow{\sim} \mathcal{R}_\alpha^\omega \longrightarrow \mathring{\mathcal{R}}_\lambda$ (Brundan-Kleshchev [BK09]), this becomes a resolution of the Specht module Σ_λ by permutation modules.
- Homological computations are made diagrammatically via Soergel. ("$\mathring{\mathcal{R}}_\lambda$ is Morita-equivalent to a nil-Koszul algebra.")

Previously

- This topic has been explored before in e.g. works of Zelevinsky, Arakawa-Suzuki, Orellana-Ram.
- Their works port the BGG resolution from category $\mathcal{O}$ to some variant of S_n by using an exact ("Arakawa-Suzuki") functor.
- Those works inspired this project.
- We would like to elucidate the highest-weight structure 'natively'.

The KLR algebra – generators

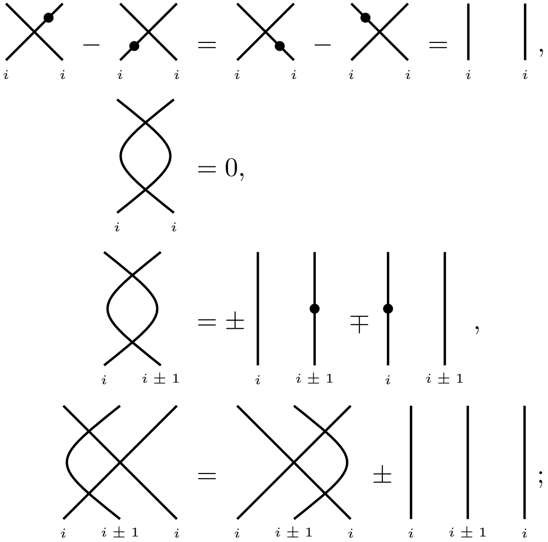
$\mathcal{R} = \bigoplus_\alpha \mathcal{R}_\alpha$. The monoidal generators are



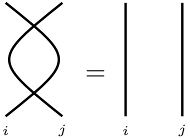
degree 0 2 −2 1 0

where $|j - i| > 1$.

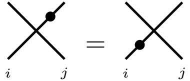
The KLR algebra – relations



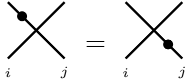
The KLR algebra – relations



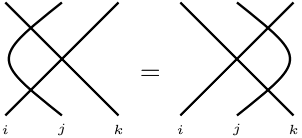
for $|i - j| > 1$,



for $i \neq j$,



for $i \neq j$,



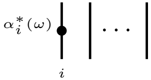
for $(j, k) \neq (i \pm 1, i)$.

The KLR algebra – cyclotomic relation

Given $\omega \in \Lambda_+$,

$$\mathcal{R}_\alpha^\omega := \mathcal{R}_\alpha \big/ \langle y_1^{\alpha^*_{c_1}(\omega)} e_c = 0 \rangle.$$

Diagrammatically:



$= 0.$

$\mathcal{R}_\lambda$

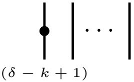
Let ϖ_i be fundamental weights, $\operatorname{cont}\lambda$ be the (multi)set of contents of λ where the top-left box has content δ.
We will let $\mathcal{R}_\lambda = \mathcal{R}_\alpha^\omega$, where

$$\alpha = \sum_{i \in \operatorname{cont}\lambda} \alpha_i$$

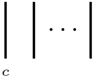
and

$$\omega = \varpi_\delta + \varpi_{\delta-1} + \cdots + \varpi_{\delta-\ell(\lambda)+1}.$$

This is requiring



$= 0,$



$= 0$

for $1 \leq k \leq \ell(\lambda)$ and $c \notin \{\delta - k + 1\}_{1 \leq k \leq \ell(\lambda)}$.

Dominance order

For partitions: $\lambda \trianglerighteq \mu$ if

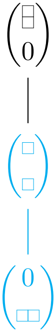
$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i \quad \forall\, k.$$

For multi-partitions: $\lambda \trianglerighteq \mu$ if

$$\sum_{j=1}^{m-1} |\lambda^{(j)}| + \sum_{i=1}^{k} \lambda_i^{(m)} \geq \sum_{j=1}^{m-1} |\mu^{(j)}| + \sum_{i=1}^{k} \mu_i^{(m)} \quad \forall\, m, k.$$

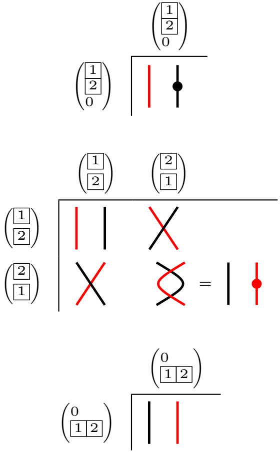
Cellular structure

$\mathcal{R}_\alpha^\omega$ is cellular under the dominance order due to Hu-Mathas [HM10].
Details are too involved.
As an example: let $n = \delta = 2$, $\lambda = \begin{smallmatrix}\square\\ \square\end{smallmatrix}$, so $\alpha = \alpha_1 + \alpha_2$, $\omega = \varpi_2 + \varpi_1$; let 1 be black and 2 be red.



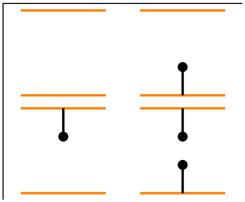
Cellular structure

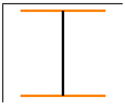
As an example: let $n = \delta = 2$, $\lambda = \square$, so $\alpha = \alpha_1 + \alpha_2$, $\omega = \varpi_2 + \varpi_1$; let 1 be black and 2 be red.



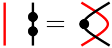
Multiplication only goes upwards.

Compare this to the earlier:





In fact,

 $= 0.$

A cellular algebra is quasi-hereditary iff every cell has an idempotent. We want a quasi-hereditary A, so we want to kill this nilpotent cell.

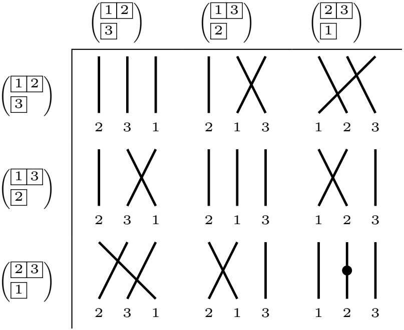
Quotienting the cyclotomic KLR

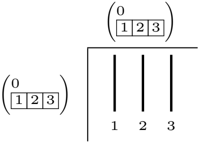
- There is an alternative ordering, the coarsened order of Uglov, on multi-partitions.
- A result of Bowman [Bow17] says the cellular structure of $\mathcal{R}_\lambda$ respects this.
- This justifies the construction

$$\mathring{\mathcal{R}}_\lambda := \mathring{\mathcal{R}} := \mathcal{R}_\lambda \big/ \langle \nu \rangle_{\nu \text{ multi-row}}.$$

- This algebra is very easy to write down explicitly, simply keep the cells labeled by one-row multi-partitions.
- This algebra is quasi-hereditary with poset $W_\lambda = \{w : w \leq w_\lambda\}$.
- It is also equivalent to a Soergel calculus depending on W_λ.

Example: $\mathring{\mathcal{R}}_\lambda$ for $\lambda = \square\!\square / \square$, $n = 3$, $\delta = 2$





The BGG resolution

Theorem (Z.)

$\mathring{\mathcal{R}}_\lambda$ is nil-Koszul. There is a (finite) BGG resolution in $\mathsf{Mod}\,\mathring{\mathcal{R}}_\lambda$, namely a resolution of the simple module L_1 by Vermas,

$$0 \longrightarrow \Delta_{w_\lambda} \longrightarrow \cdots \longrightarrow \bigoplus_{\ell(w)=k} \Delta_w \longrightarrow \cdots \longrightarrow \Delta_1 \longrightarrow L_1 \longrightarrow 0.$$

When restricted to the action of S_n via the map $\mathbb{C}S_n \hookrightarrow \widehat{\mathcal{H}}_n \twoheadrightarrow \widehat{\mathcal{H}}^\omega_\alpha \xrightarrow{\sim} \mathcal{R}^\omega_\alpha \twoheadrightarrow \mathring{\mathcal{R}}_\lambda$, this resolution becomes

$$\cdots \longrightarrow \bigoplus_{\ell(w)=k} E_{w\circ\lambda} \longrightarrow \cdots \longrightarrow E_\lambda \longrightarrow \Sigma_\lambda \longrightarrow 0.$$

Decategorifying this resolution via alternating sum of Frobenius character/cycle index series recovers Jacobi-Trudi:

$$s_\lambda = \det(h_{\lambda_i - i + j})_{i,j}.$$

There is a positive characteristic version too.

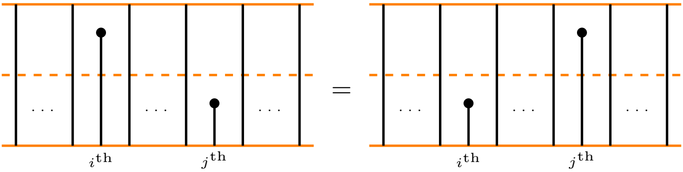
Thank you!

Thank you for coming to my talk!
Questions

Cyclotomic Soergel is nil-Koszul

Theorem (Z.)

There is a “lower-half subalgebra” $\mathcal{S}_\lambda^-$ of $\mathcal{S}_\lambda$ generated by lollipops which is essentially a polynomial ring.



...
=
i^{th}
j^{th}

Cyclotomic Soergel is nil-Koszul

Theorem (Z.)

There is a “lower-half subalgebra” $\mathcal{S}_\lambda^-$ of $\mathcal{S}_\lambda$ generated by lollipops which is essentially a polynomial ring.

$\mathcal{S}_\lambda^-$ is Koszul under the Soergel grading.

Moreover

$$\mathcal{S}_\lambda \overset{\mathsf{L}}{\otimes}_{\mathcal{S}_\lambda^-} \Bbbk e^w = \Delta_w,$$

so that

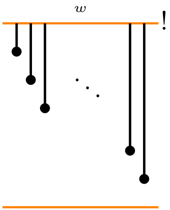
$$\mathsf{RHom}_{\mathcal{S}_\lambda}(\Delta_w, \square) = \mathsf{RHom}_{\mathcal{S}_\lambda^-}(\Bbbk e^w, \square).$$

In particular

$$\mathrm{Ext}^\bullet_{\mathcal{S}_\lambda}(\Delta_w, L_1) = \mathrm{Ext}^\bullet_{\mathcal{S}_\lambda^-}(\Bbbk e^w, \Bbbk e^1) = e^w \mathcal{S}_\lambda^{-,!} e^1 = \Bbbk[-\ell(w)].$$

This is the homological information needed to prove the BGG resolution.

The space $e^w \mathcal{S}_\lambda^{-,!} e^1 = \Bbbk[-\ell(w)]$ is spanned by



Comment on nil-Koszulity

- Last year we showed that nil-Brauer (due to Brundan-Wang-Webster) was nil-Koszul.
- Two algebras (due to Khovanov-Sazdanović), categorifying the Hermite and Chebyshev polynomials, are also nil-Koszul.
- Now one more algebra (cyclotomic Soergel) is on the list.
- It seems lots of algebras appearing in categorification are nil-Koszul.
- Why?

Projective resolutions of Vermas

Incidentally the statement that $\mathcal{S}_\lambda \overset{\mathsf{L}}{\otimes}_{\mathcal{S}_\lambda^-} \Bbbk e^w = \Delta_w$ gives us a projective resolution of Vermas, using that $\mathcal{S}^-$ is essentially a polynomial ring.

- There is an explicit Koszul resolution

$$\mathcal{S}^- \otimes_{\mathbb{K}} \mathcal{S}^{-,!,\vee,\bullet} \simeq \Bbbk e^w.$$

- This is a free $\mathcal{S}^-$-resolution, so use it to compute the derived tensor.

Projective resolutions of Vermas

Corollary

We have a resolution

$$\mathcal{S} \otimes_{\mathbb{K}} \mathcal{S}^{-,!,\vee,\bullet} e^w \simeq \Delta_w;$$

the maps are

$$\begin{aligned} \mathrm{d} \colon \mathcal{S} \otimes_{\mathbb{K}} \mathcal{S}_k^{-,!,\vee} e^w &\longrightarrow \mathcal{S} \otimes_{\mathbb{K}} \mathcal{S}_{k-1}^{-,!,\vee} e^w \\ x \otimes P(-_1 e^w, \cdots, -_n e^w) &\longmapsto \sum_{\text{ways to write } P = -_i \cdot P'} x \cdot -_i \otimes P', \end{aligned}$$

where $P \in \mathcal{S}_k^{-,!,\vee}$ is a degree k anti-commutative polynomial in the lollipops.

Proof strategy

1. First key idea: “weight theory” $\longrightarrow$ stratification of the module category $\longrightarrow$ “filtration” of the identity functor $\longrightarrow$ spectral sequence converging to any object.
    - In particular, we can apply this to simple objects.
    - Terms of this spectral sequence involve homological information, in the form of certain Ext groups.
    - Concentration of these Ext groups (cf. “Kostant modules”) imply a “BGG resolution”.
    - This idea is due to Gaitsgory, Ayala-Mazel-Gee-Rozenblyum [AMGR22], and Dhillon [Dhi19].
2. Second key idea: This homological information can be computed using Koszul methods.

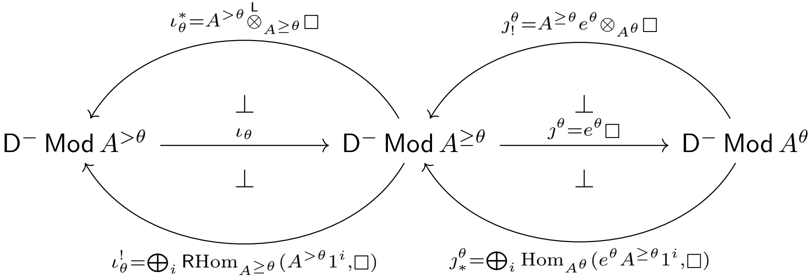
Slogan

Koszulity of half of A is intimately connected to BGG resolutions.

- Then we can use a naive resolution to compute these Ext groups.
- The spectral sequence is a resolution for modules which are Koszul over half of A.

Algebraic recollement

- The stratification will have recollements on each level. Details unimportant.
- By the $\mathsf{D}\,\mathsf{Mod}\,A/AeA \longrightarrow \mathsf{D}\,\mathsf{Mod}\,A \longrightarrow \mathsf{D}\,\mathsf{Mod}\,eAe$ setup of [CPS88], set $A = A^{\geq\theta}$ and $e = e^{\theta}$ to get:

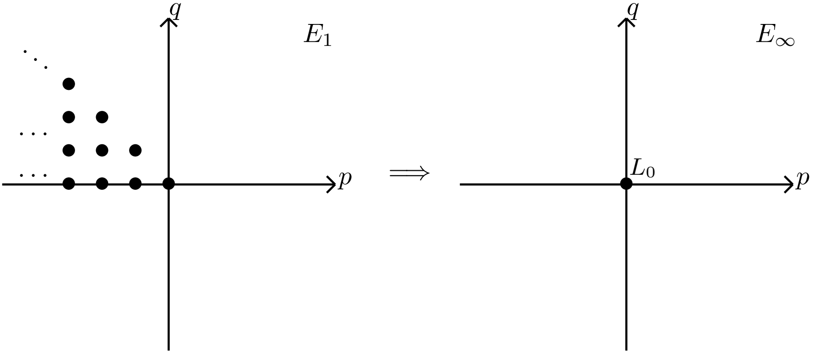


Stratification and spectral sequences

- Let Θ be sufficiently nice.
- There is a spectral sequence (functorial in the input $\square$)

$$E_1^{p,q} = \bigoplus_{\ell(\theta)=-p} \Delta(\theta)\otimes_{A^\theta}\mathrm{Ext}_A^{-(p+q)}(\Delta(\theta),\square^\dagger)^* \implies E_\infty^{p,q} = \mathrm{gr}^{-p} H^{p+q}(\square),$$

where $\Delta(\theta) \coloneqq \bigoplus_{\lambda\in\theta} \Delta_\lambda^{\overline{l_\lambda(\theta)}} = A^{\geq\theta}e^\theta$. $\deg \mathrm{d}_r = (r, 1-r)$:



Remark: Cf. Koszul duality.

- To obtain a resolution, we need the Ext groups to be concentrated in certain degrees.
- Idea: Koszul objects have good Ext concentration properties.

Definition

A quadratic graded algebra A ($A_0 = \Bbbk$) is "Koszul" if $\mathrm{Ext}_A(\Bbbk, \Bbbk)$ in nonzero only when the homological degree agrees with the Koszul degree.

Example: category $\mathcal{O}$ for $\mathfrak{sl}_2$

- It is classical that L_0 has concentrated Ext groups:

$$\operatorname{Ext}^0(\Delta_0, L_0) = \mathbb{C}, \qquad \operatorname{Ext}^0(\Delta_{-2}, L_0) = 0$$

$$\operatorname{Ext}^1(\Delta_0, L_0) = 0, \qquad \operatorname{Ext}^1(\Delta_{-2}, L_0) = \mathbb{C}.$$

- Then the spectral sequence above exactly recovers the BGG resolution.
- Remark: It can also recover the standard filtration of projectives.

Some predicted questions

- Is there a monoidal product on $\mathring{\mathcal{R}} = \bigoplus_\lambda \mathring{\mathcal{R}}_\lambda$ categorifying the Littlewood-Richardson structure?
- What does this say about positive characteristic?
- Is $\mathring{\mathcal{R}}_\lambda$ itself nil-Koszul?
- What about the skew Jacobi-Trudi identity for skew Schur functions?

Non-split decomposition

- Lauda's $\mathcal{U}_q(\mathfrak{sl}_2)$ is triangular-based.
- However, it does not have a subalgebra like $\mathcal{U}_q(\mathfrak{sl}_2)^-$.
- Instead, one needs to work with a bigger object $\mathcal{U}_q(\mathfrak{sl}_2)^\flat$, which actually is a subalgebra.
- Categorification: projectives categorify Lusztig's canonical basis.

Question

What do the standard modules correspond to?

Stratifications within stratifications

- The affine oriented Brauer category is triangular-based.
- However, this structure alone does not utilize the obvious ordering on the simples of each Cartan.
- By using the stratification of $\widehat{\mathcal{H}}_n$ above, we should obtain finer stratifications.

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Koszul duality

We are using the *inverse* Koszul duality of [BGS96], and we flip the axes and swap the roles of A and $A^!$.

$$\mathcal{K}_{A^+} \colon \mathsf{D}^{\nwarrow} \operatorname{\mathsf{Mod}} A^+ \longrightarrow \mathsf{D}^{\searrow} \operatorname{\mathsf{Mod}} A^{+,!},$$

where

$$\mathcal{K}_{A^+} = \operatorname{sh}(\mathbb{K} \overset{\mathsf{L}}{\otimes}_{A^+} \operatorname{refl} \square) = \operatorname{sh} \mathsf{R}\mathrm{Hom}_{A^-}(\mathbb{K}, \operatorname{refl} \square^{\dagger})^*.$$

Here $\operatorname{sh} M = M[n]$ if M is concentrated in Koszul degree n, and $\operatorname{refl}(M)_j = M_{-j}$.
Then the spectral sequence looks like

$$\Delta \otimes_{A^\circ} \mathcal{K}_{A^+}(\square).$$

The differential in Koszul duality

Let us briefly explain the sign rules for the twisted tensor product. Given $A \circlearrowright M$, and letting $C = A^{¡}$ be the Koszul dual coalgebra, we have that $C \otimes^{\tau} M$ is a dg C-comodule where the coaction is on the first entry and the differential is:

$$\mathrm{d}^{\tau}(c \otimes v) = \mathrm{d}(C) \otimes v + (-1)^{|c|} c \otimes \mathrm{d}(v) + (-1)^{|c_{(1)}|} c_{(1)} \otimes \tau(c_{(2)}) v.$$

Similarly, given a comodule $C \overset{\mathrm{co}}{\circlearrowright} N$, we obtain $A \otimes^{\tau} N$ a dg A-module, where the action is on the first entry and the differential is

$$\mathrm{d}^{\tau}(a \otimes u) = \mathrm{d}(a) \otimes u + (-1)^{|a|} a \otimes \mathrm{d}(u) + (-1)^{|a|+1} a \tau(u_{(-1)}) \otimes u_{(0)}.$$

Let us briefly explain what the coderived category is. The cheap thing to do is to say that it is the localization of the category of (cocomplete, meaning that N is the union of the kernels of $N \longrightarrow \overline{C}^{\otimes n} \otimes N$) dg C-comodules at the class of morphisms which become quasi-isomorphisms under the functor $A \otimes^{\tau} \square$. The longer thing to say is that the coderived category is the quotient category of the homotopy category of dg comodules by the minimal triangulated subcategory closed under infinite direct sums which contains the totalization comodules of all exact triples of C-comodules.

Our choice for Soergel

Let

$S_\lambda = \{\underline{w} : \ell(\underline{w}) \leq \ell(w_\lambda),\ \underline{w} \text{ is a subword of some reduced word for } w_\lambda\}.$

We will consider

$$\mathcal{S}_\lambda := \mathbb{C} \otimes_R \operatorname{End}\left(\bigoplus_{\underline{w} \in S_\lambda} \mathrm{BS}_{\underline{w}}\right).$$

In other words we take cyclotomic Soergel calculus with endpoints $\underline{w} \in S_\lambda$.
This is a cellular algebra via the light leaves basis of Elias-Williamson [EW16].
It is Morita-equivalent to $\mathring{\mathcal{R}}_\lambda$:

$$\mathsf{Mod}\, \mathring{\mathcal{R}}_\lambda \cong \mathsf{Mod}\, \mathcal{S}_\lambda,$$

so that

$$\operatorname{Ext}^\bullet_{\mathcal{S}_\lambda}(\Delta_w, L_1) = \operatorname{Ext}^\bullet_{\mathring{\mathcal{R}}_\lambda}(\Delta_w, L_1).$$