Koszul duality and categorifying Jacobi-Trudi

Fan Zhou

Columbia University

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Link

These slides can be found on my webpage:

math.columbia.edu/~fanzhou/files/beamer-KIAS2026.pdf

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Slogan

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This is the case for Soergel, which we discuss here, and Temperley-Lieb.

Today's example

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  - modules over "(cyclotomic) Soergel calculus" (endomorphism algebra of some Soergel/Bott-Samelson (bi)modules).

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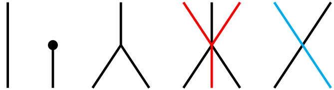

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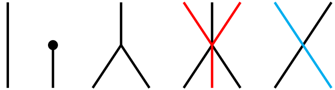

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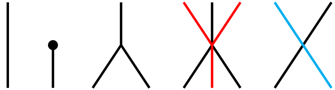

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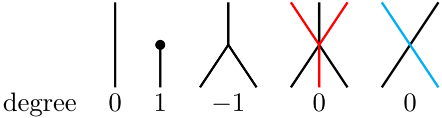

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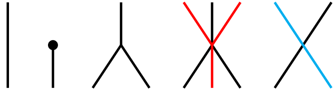

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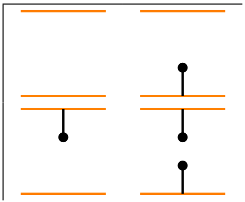

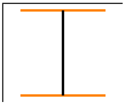

Soergel calculus – relations

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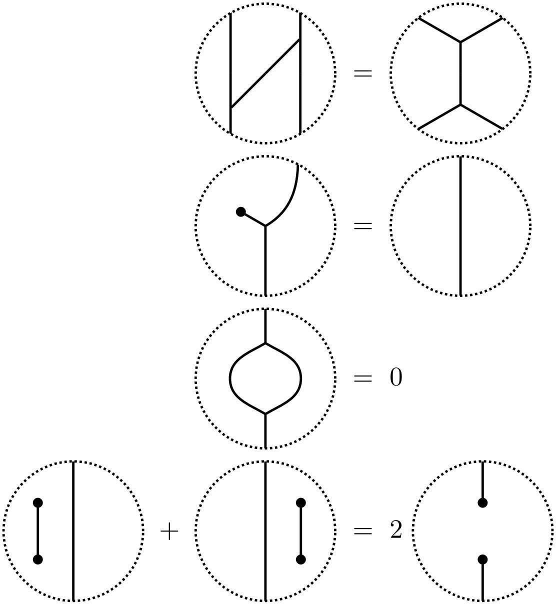

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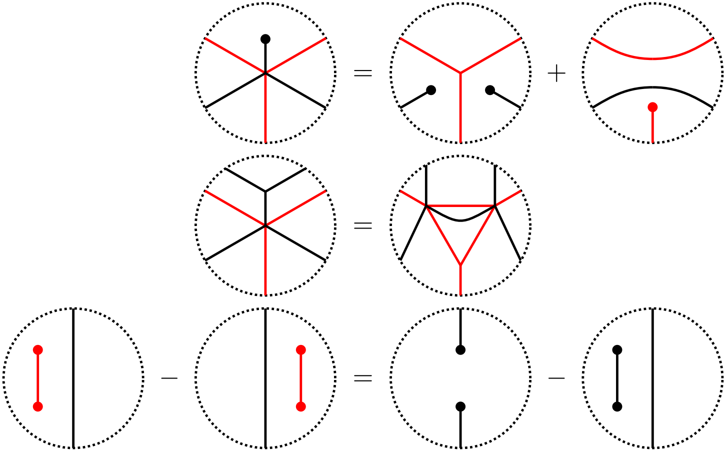

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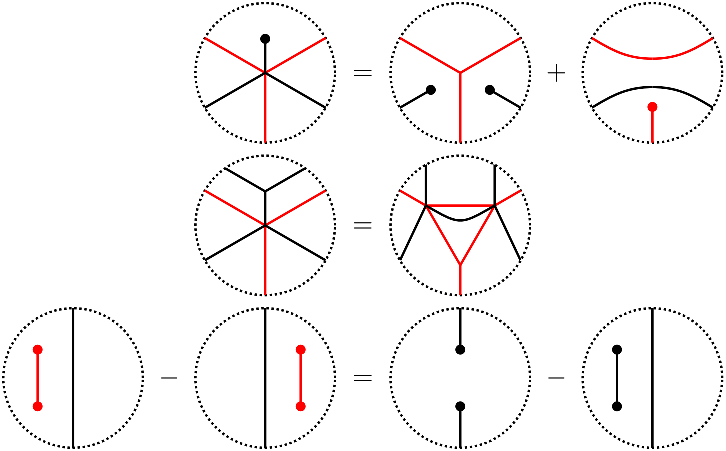

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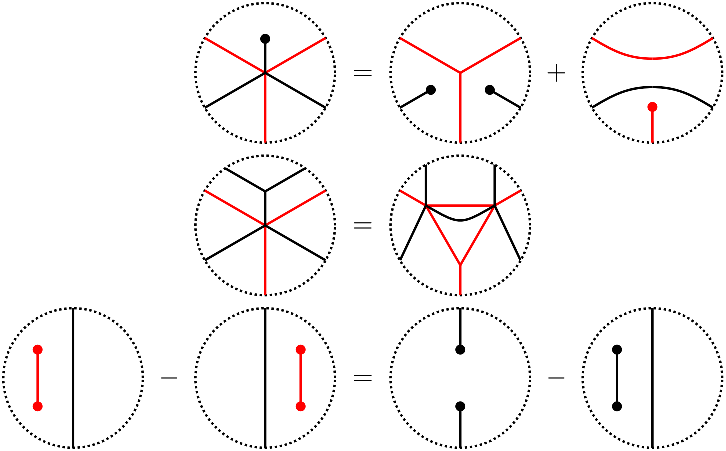

Distant colors pull apart. (There is also a tetrahedral relation.)

Soergel calculus – cyclotomic relation

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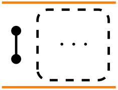
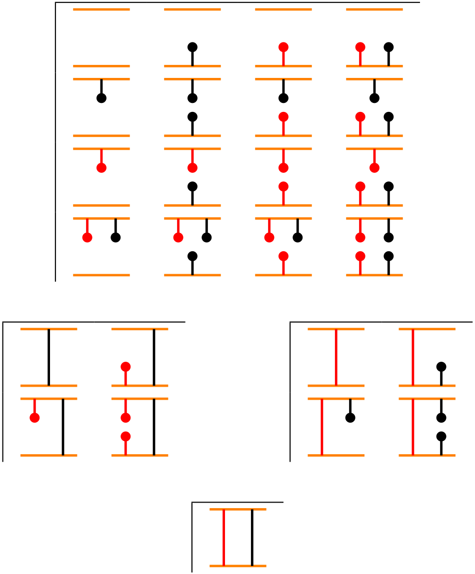


$= 0$

Example: $\mathcal{O}^0(\mathfrak{sl}_3)$ mod simples labeled by $s_1s_2, s_1s_2s_1$



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A graded algebra is "quadratic" if it is generated in degree 1 and relations are generated in degree 2:

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A quadratic graded algebra A ($A_0 = \mathbb{K}$) is “Koszul” if $\mathrm{Ext}_A(\mathbb{K}, \mathbb{K})$ is nonzero only when the homological degree agrees with the Koszul degree.

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Warning: need to be careful about left vs right.

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Comultiplication is

$$x_1 \otimes \cdots \otimes x_n \longmapsto \sum_i (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n).$$

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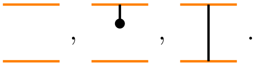
Consider the subalgebra $\mathcal{S}^+$ for $\mathfrak{sl}_2$ spanned by



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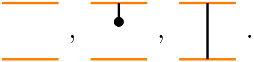

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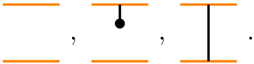

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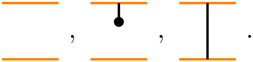

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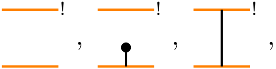

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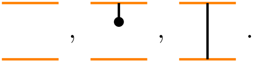

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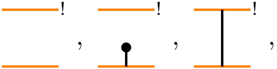

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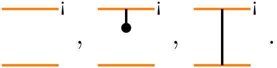

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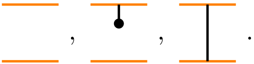

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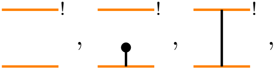

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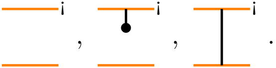

Note: the upside-down flipping is happening because we need to be careful about left vs right.

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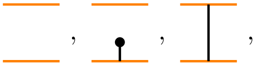

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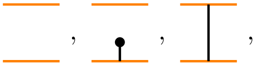

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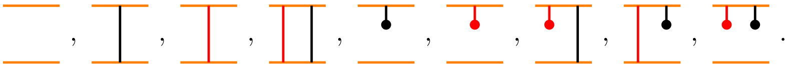
This is simply the “upside-down flipping” anti-involution applied to $\mathcal{S}^+$, i.e. $\mathcal{S}^- = \tau(\mathcal{S}^+)$.

Soergel ($\mathfrak{sl}_3$) example

For the example with 4 cells, consider the subalgebra $\mathcal{S}^+$ spanned by



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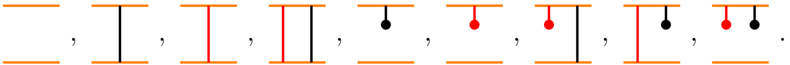

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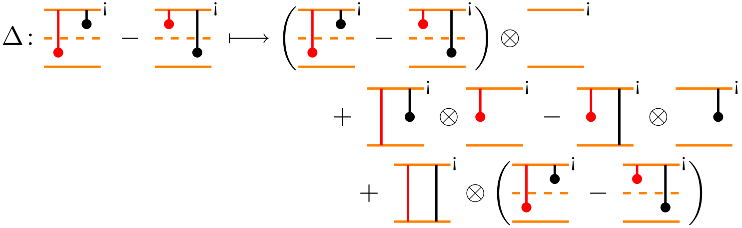

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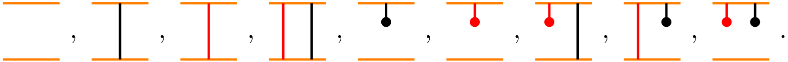

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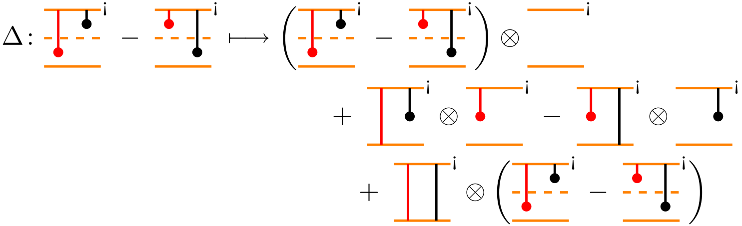
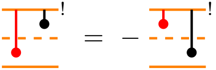


This is secretly saying something about BGG differentials.

Another way to think of this is the relation in $\mathcal{S}^{-,!}$



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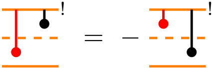

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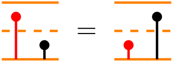

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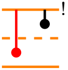
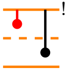

$=$ $-$

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(This is morally the Sym-$\bigwedge$ duality.)

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Then

Theorem (Z.)

$\mathcal{S}$ is nil-Koszul, with $\mathcal{S}^- := \tau(\mathcal{S}^+)$ as the nil-algebra with the Soergel grading.

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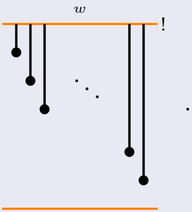

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In this talk we will forgo this in favor of Koszul duality.

Koszul duality

One perspective on Koszul duality, for example in Positselski's work, is that Koszul duality is a functor (equivalence) between derived dg-modules over an algebra and coderived dg-comodules over its Koszul dual coalgebra:

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Here $\otimes^{\tau}$ is a tensor product "twisted" by a "twisting cochain" $\tau\colon A^{\text{!`}} \longrightarrow A$ defined by killing everyone except $A_1^{\text{!`}} \cong A_1$.

The twisting

The differential on $A^{¡} \otimes^{\tau} \square$ is given by

$$\mathrm{d}^{\tau}(c \otimes v) = \mathrm{d}(c) \otimes v + (-1)^{|c|} c \otimes \mathrm{d}(v) + (-1)^{|c_{(1)}|} c_{(1)} \otimes \tau(c_{(2)}) v$$

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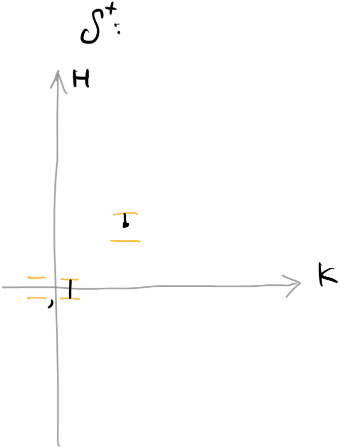
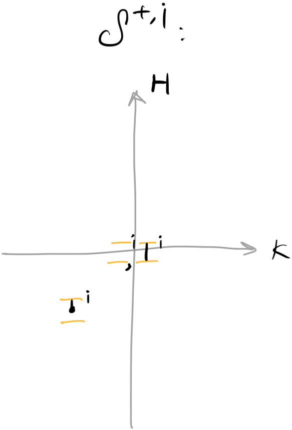
and on $A \otimes^{\tau} \square$ is given by

$$\mathrm{d}^{\tau}(a \otimes u) = \mathrm{d}(a) \otimes u + (-1)^{|a|} a \otimes \mathrm{d}(u) + (-1)^{|a|+1} a \tau(u_{(-1)}) \otimes u_{(0)}.$$

Example: Soergel ($\mathfrak{sl}_2$)





$\mathrm{refl}(M)_j = M_{-j}$, $\mathrm{sh}(M) = M[n]$ if M is in Koszul deg n.

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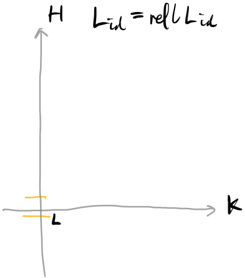

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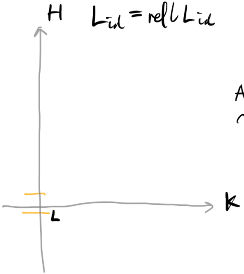

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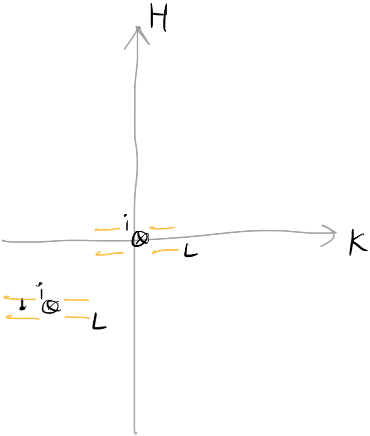

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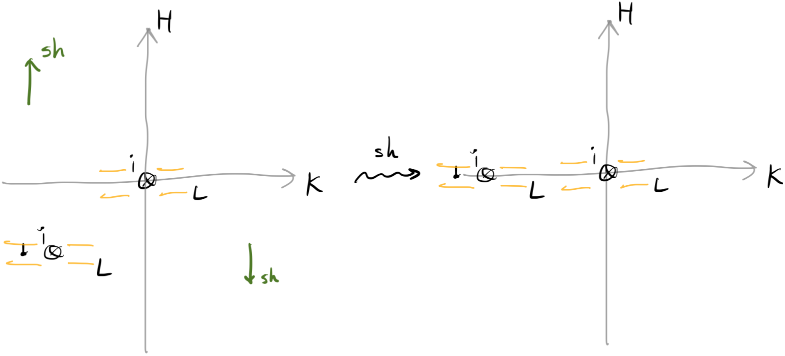

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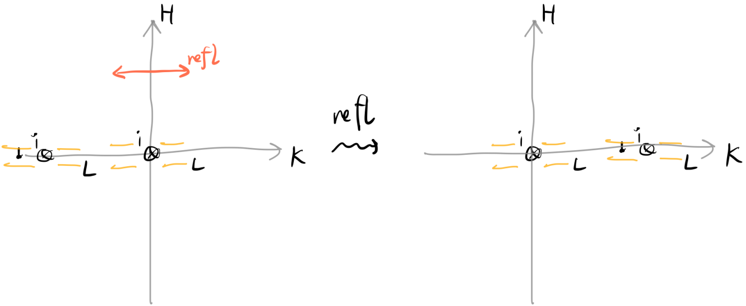

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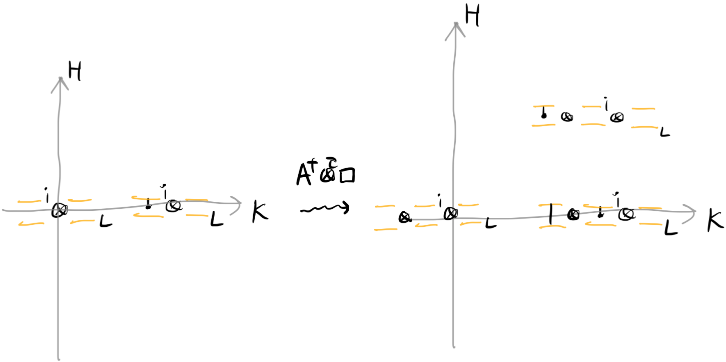

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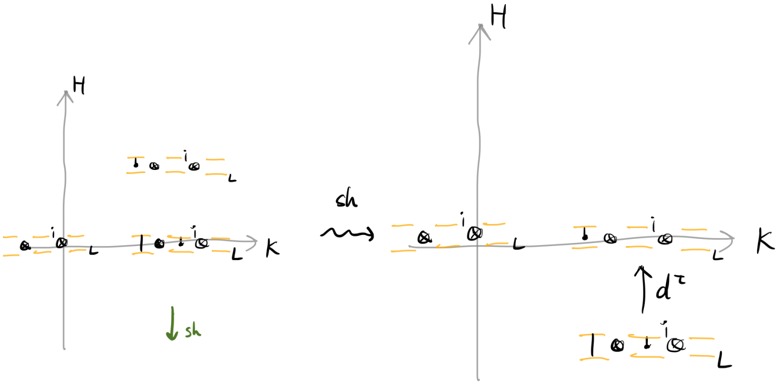

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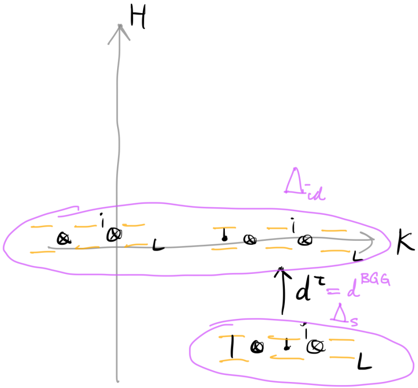

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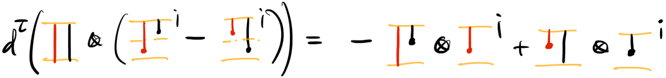
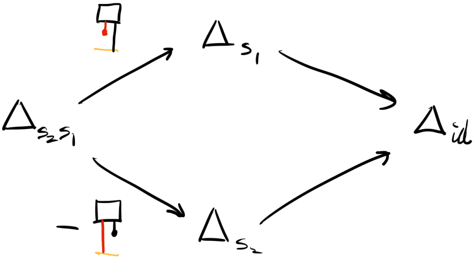


compare to



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$$\mathrm{Id} \cong \mathcal{S} \overset{\mathsf{L}}{\otimes}_{\mathcal{S}^+ \otimes_{\mathbb{K}} \mathcal{S}^-} \mathcal{K}_{\mathcal{S}^+}^{-1} \circ \mathcal{K}_{\mathcal{S}^+} = \mathcal{S} \overset{\mathsf{L}}{\otimes}_{\mathcal{S}^+ \otimes_{\mathbb{K}} \mathcal{S}^-} \mathcal{S}^+ \otimes_{\mathbb{K}}^{\tau} \mathcal{S}^{+,\mathsf{i}} \otimes_{\mathbb{K}}^{\tau} \square.$$

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In particular, applying this to L_{id} recovers the BGG resolution.

Part II: Jacobi-Trudi

A Tale of Two Cities

symmetric functions $\longleftrightarrow$ symmetric group representations

The classical story – symmetric functions

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This is an alternating sum.

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Category $\mathcal{O}$

Compare to the JH filtration of Vermas $\Delta_{w \circ 0}$ in which $L_{w \circ 0}$ appears as the top layer quotient, and

$$[\Delta_w] = [L_w] + \sum_{u > w} m_u [L_u].$$

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- Those works inspired this project.
- We would like to elucidate the highest-weight structure 'natively'.

The KLR algebra – generators

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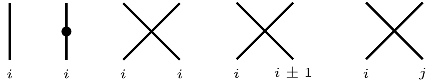
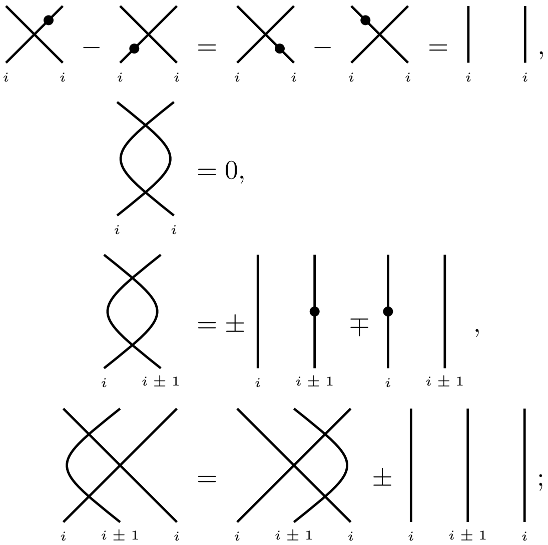


degree 0 2 −2 1 0

where $|j - i| > 1$.

The KLR algebra – relations



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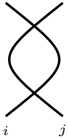

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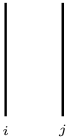
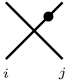


for $|i - j| > 1$,



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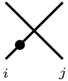

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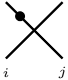

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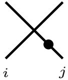

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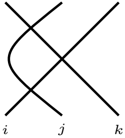

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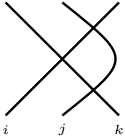

for $(j, k) \neq (i \pm 1, i)$.

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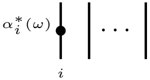
Diagrammatically:



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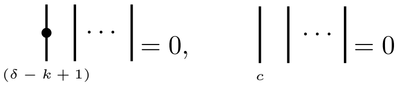

for $1 \leq k \leq \ell(\lambda)$ and $c \notin \{\delta - k + 1\}_{1 \leq k \leq \ell(\lambda)}$.

Dominance order

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For multi-partitions: $\lambda \trianglerighteq \mu$ if

$$\sum_{j=1}^{m-1} |\lambda^{(j)}| + \sum_{i=1}^{k} \lambda_i^{(m)} \geq \sum_{j=1}^{m-1} |\mu^{(j)}| + \sum_{i=1}^{k} \mu_i^{(m)} \quad \forall\, m, k.$$

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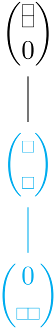

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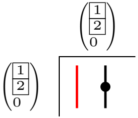

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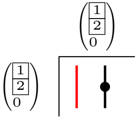

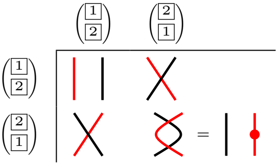

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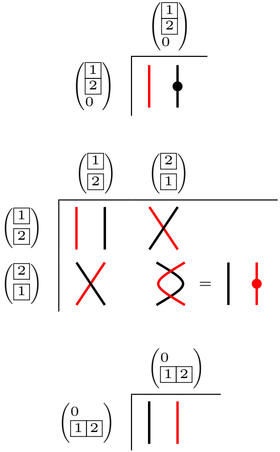

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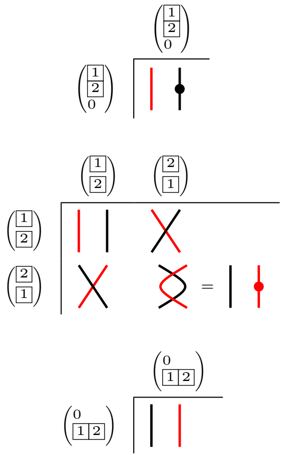
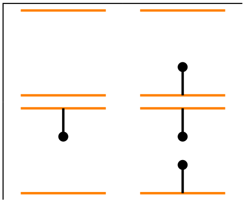
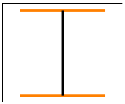


Multiplication only goes upwards.

Compare this to the earlier:





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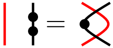

$= 0.$

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We want a quasi-hereditary A, so we want to kill this nilpotent cell.

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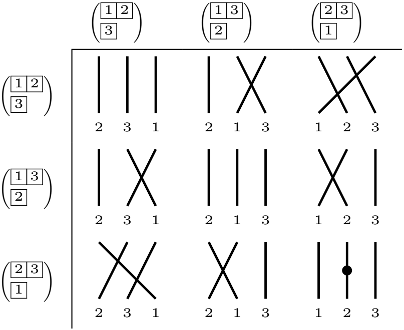
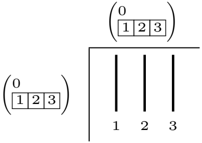
- It is also equivalent to a Soergel calculus depending on W_λ.

Example: $\mathring{\mathcal{R}}_\lambda$ for $\lambda = \square\!\square/\square$, $n = 3$, $\delta = 2$





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There is a positive characteristic version too.

Thank you!

Thank you for coming to my talk!
Questions

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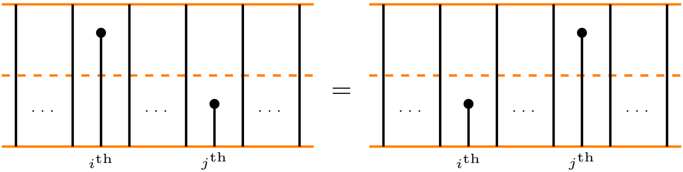
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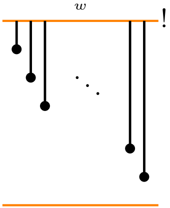
This is the homological information needed to prove the BGG resolution.

The space $e^w \mathcal{S}_\lambda^{-,!} e^1 = \Bbbk[-\ell(w)]$ is spanned by



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- This is a free $\mathcal{S}^-$-resolution, so use it to compute the derived tensor.

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the maps are

$$\begin{aligned} \mathrm{d} \colon \mathcal{S} \otimes_{\mathbb{K}} \mathcal{S}_k^{-,!,\vee} e^w &\longrightarrow \mathcal{S} \otimes_{\mathbb{K}} \mathcal{S}_{k-1}^{-,!,\vee} e^w \\ x \otimes P(-_1 e^w, \cdots, -_n e^w) &\longmapsto \sum_{\text{ways to write } P = -_i \cdot P'} x \cdot -_i \otimes P', \end{aligned}$$

where $P \in \mathcal{S}_k^{-,!,\vee}$ is a degree k anti-commutative polynomial in the lollipops.

Proof strategy

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Slogan

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- Then we can use a naive resolution to compute these Ext groups.
- The spectral sequence is a resolution for modules which are Koszul over half of A.

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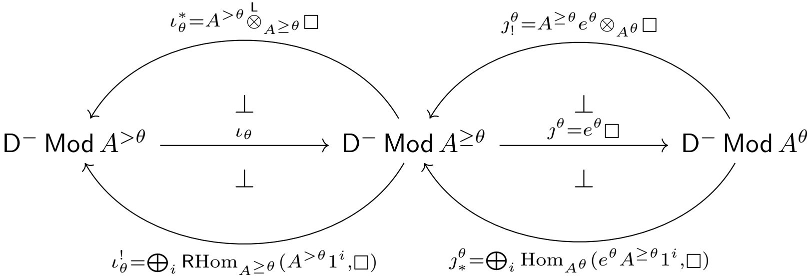
- By the $\mathsf{D}\,\mathsf{Mod}\,A/AeA \longrightarrow \mathsf{D}\,\mathsf{Mod}\,A \longrightarrow \mathsf{D}\,\mathsf{Mod}\,eAe$ setup of [CPS88], set $A = A^{\geq\theta}$ and $e = e^\theta$ to get:



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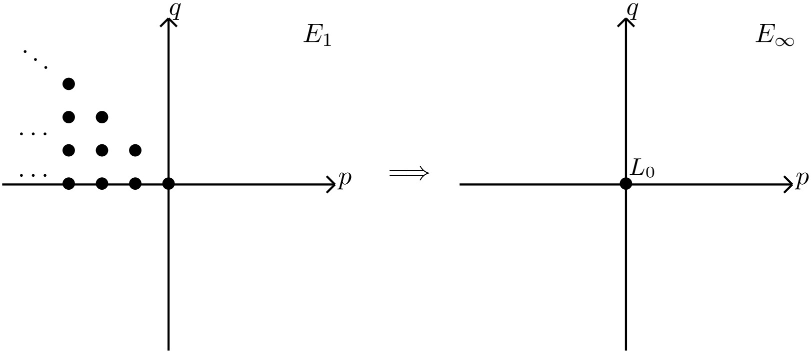

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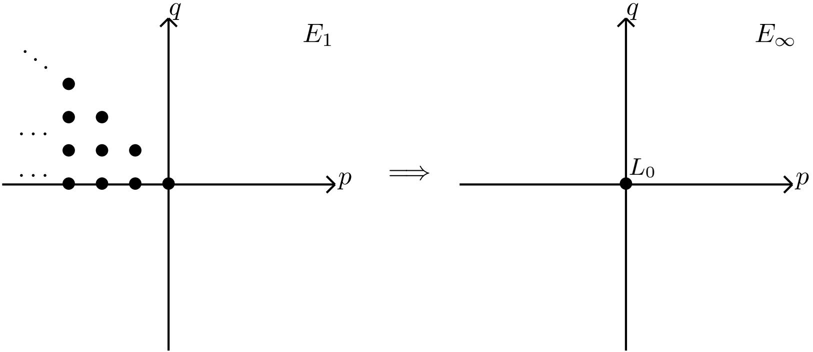

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Definition

A quadratic graded algebra A ($A_0 = \Bbbk$) is "Koszul" if $\mathrm{Ext}_A(\Bbbk, \Bbbk)$ in nonzero only when the homological degree agrees with the Koszul degree.

Example: category $\mathcal{O}$ for $\mathfrak{sl}_2$

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- Then the spectral sequence above exactly recovers the BGG resolution.
- Remark: It can also recover the standard filtration of projectives.

Some predicted questions

- Is there a monoidal product on $\mathring{\mathcal{R}} = \bigoplus_\lambda \mathring{\mathcal{R}}_\lambda$ categorifying the Littlewood-Richardson structure?
- What does this say about positive characteristic?
- Is $\mathring{\mathcal{R}}_\lambda$ itself nil-Koszul?
- What about the skew Jacobi-Trudi identity for skew Schur functions?

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Question

What do the standard modules correspond to?

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- However, this structure alone does not utilize the obvious ordering on the simples of each Cartan.
- By using the stratification of $\widehat{\mathcal{H}}_n$ above, we should obtain finer stratifications.

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Then the spectral sequence looks like

$$\Delta \otimes_{A^{\circ}} \mathcal{K}_{A^+}(\square).$$

The differential in Koszul duality

Let us briefly explain the sign rules for the twisted tensor product. Given $A \circlearrowright M$, and letting $C = A^{¡}$ be the Koszul dual coalgebra, we have that $C \otimes^{\tau} M$ is a dg C-comodule where the coaction is on the first entry and the differential is:

$$\mathrm{d}^{\tau}(c \otimes v) = \mathrm{d}(C) \otimes v + (-1)^{|c|} c \otimes \mathrm{d}(v) + (-1)^{|c_{(1)}|} c_{(1)} \otimes \tau(c_{(2)}) v.$$

Similarly, given a comodule $C \overset{\mathrm{co}}{\circlearrowright} N$, we obtain $A \otimes^{\tau} N$ a dg A-module, where the action is on the first entry and the differential is

$$\mathrm{d}^{\tau}(a \otimes u) = \mathrm{d}(a) \otimes u + (-1)^{|a|} a \otimes \mathrm{d}(u) + (-1)^{|a|+1} a \tau(u_{(-1)}) \otimes u_{(0)}.$$

Let us briefly explain what the coderived category is. The cheap thing to do is to say that it is the localization of the category of (cocomplete, meaning that N is the union of the kernels of $N \longrightarrow \overline{C}^{\otimes n} \otimes N$) dg C-comodules at the class of morphisms which become quasi-isomorphisms under the functor $A \otimes^{\tau} \square$. The longer thing to say is that the coderived category is the quotient category of the homotopy category of dg comodules by the minimal triangulated subcategory closed under infinite direct sums which contains the totalization comodules of all exact triples of C-comodules.

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so that

$$\operatorname{Ext}^\bullet_{\mathcal{S}_\lambda}(\Delta_w, L_1) = \operatorname{Ext}^\bullet_{\mathring{\mathcal{R}}_\lambda}(\Delta_w, L_1).$$