THE CLASSICAL AND THE FUNCTORIAL BGG RESOLUTIONS

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This senior thesis, advised by Professor Dennis Gaitsgory and presented in the Spring of 2021, consists of two parts. In Part I, we give an exposition on the classical proof due to Bernstein-Gelfand-Gelfand [2] of the BGG resolution. In Part II, we give an exposition on a "modern" categorical viewpoint/proof of the BGG resolution. As I understand it, the mathematics presented in Part II is far from novel – it is considered common knowledge in the field. This story was communicated to me by Professor Gaitsgory, Charles Fu, and Kevin Lin; I am incredibly grateful for their help and patience. In particular, I was entirely a listener. However, I could not find a reference where this story was committed to paper, and so I decided to write it down. My role is only that of a scribe.

Part I presents the classical proof of the BGG resolution, which is a categorification of the celebrated Weyl character formula. In this section, we assume of the reader basic familiarity with the representation theory of Lie algebras, for example to the extent of one who has read a dense subset of Kirillov's book [6].

Part II presents a modern categorical approach to the BGG resolution. In this section, we assume of the reader slightly more familiarity with the representation theory of Lie algebras than in Part I. For example, we assume the reader knows what category $\mathcal{O}$ is. We also assume basic familiarity with the theory of triangulated categories and category theory at large; a working knowledge of stable ∞-categories would be ideal, but not crucial.

The two parts are more-or-less independent of one another, and the reader can choose to read whichever part they like, be it the former, the latter, both, or neither. We expect that a reader who is equipped to read Part I will be able to read Part II after reading Part I.

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与君歌一曲 请君为我倾耳听！

– 李白，《将进酒》

“With thee a song I sing, pray thee lend thine ears for me!”

– Li Bai, *Ode to Wine*

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Part I

The Classical BGG Resolution

1. A Short Introduction to Part I

1.1. **Foreword to Part I.** In this expository section we will give[2] the classical proof of the Bernstein-Gelfand-Gelfand (BGG) resolution, following the original paper of BGG [2], and use it to prove the celebrated Weyl character formula (in some sense the BGG resolution is a categorification of the Weyl character formula). I have tried my best to flesh out details which BGG omitted in their original paper, as well as organize the proof and the flow of logic in a manner which I find most motivated and easily understood.

Structurally we will tend to assume facts and prove goals, proving the facts later, i.e., logically this exposition should be read backwards; for example we will begin by proving Weyl assuming BGG. We do this for the sake of clarity and motivation. More specifically, we will prove BGG in this order: we will first show how Three Lemmas imply BGG; then we will show how Weak BGG[3] implies Three Lemmas; then we will prove Weak BGG. Along the way we will assume some theorems not to be proven in this exposition; these facts are enumerated in Section 1.

Some notes about labelling: we will label theorems by their names when appropriate, and we will label statements from BGG by just their number, e.g. writing [10.5] rather than [BGG10.5]. When we need to refer to facts from e.g. Kirillov [6] or Humphreys[4] [5], we will for example say [K8.27] or [H4.2].

1.2. **Notations/Conventions.** Some notations/conventions: We will work over $\mathbb{C}$ throughout. Unless otherwise stated, $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ will be a semisimple Lie algebra. We will write Σ for the set of simple roots; if α_i is a simple root, it will be said that $\alpha_i \in \Sigma$, and otherwise α_i denotes any indexed set of roots. In this spirit, we will denote by $I(\Sigma)$ the index set of the simple roots, i.e. $i \in I(\Sigma) \iff \alpha_i \in \Sigma$. Let ω_i be the fundamental weights. The Verma module is denoted M_λ. An irreducible representation of highest weight λ is commonly called L_λ, and sometimes to emphasize that it is finite-dimensional we may write Π_λ instead. Let $\Lambda = \mathbb{Z}\{\omega_i\}_\Sigma$ be the weight lattice, and let $Q = \mathbb{Z}\{\alpha_i\}_\Sigma$ be the root lattice; similarly denote $\Lambda_+ = \mathbb{N}\{\omega_i\}_\Sigma$ and $Q_+ = \mathbb{N}\{\alpha_i\}_\Sigma$ (our convention is that $\mathbb{N}$ contains 0). Recall the notion, for $\lambda, \mu \in \Lambda$, of

$$\mu \leq \lambda \iff \lambda - \mu \in Q_+.$$

We will write W for the Weyl group and

$$W_k \coloneqq \{w \in W : \ell(W) = k\}.$$

We will let

$$w \circ \lambda \coloneqq w(\lambda + \varrho) - \varrho$$

define the affine action of the Weyl group W on $\mathfrak{h}^*$, where recall $\varrho \coloneqq \frac{1}{2}\sum_{\Phi_+} \alpha = \sum_\Sigma \alpha$, where $\Sigma \subseteq \Phi_+$ denotes the set of simple roots. We will also write

$$\lambda \sim \mu \iff \exists\, w : \lambda = w(\mu),$$

[2] Though we will blackbox some facts in the interest of length, e.g. the conditions for embeddings to exist between Verma modules.

[3] This is Theorem 9.9 in BGG's paper, and they did not call it this; but this seems an appropriate name, as BGG appears to strictly improve upon it.

[4] Unfortunately Humphreys does not label his theorems/propositions/lemmas, instead relying on the fact that there is a unique theorem/proposition/lemma per section; we will rely on context to tell what we mean.

and

$$\lambda \overset{\circ}{\sim} \mu \iff \exists\, w : \lambda = w \circ \mu.$$

In line with the notation of $\mathfrak{g}^\alpha$, we will also write

$$M^\mu := \{v \in M : \mathfrak{h} v = \mu(\mathfrak{h}) v\}$$

for the μ-weight space of M; for example for Verma modules this would be written ${M_\lambda}^\mu$ with staggered indices.

Full disclosure: **here are the facts we will be blackboxing** (in addition to some standard homological algebra facts/constructions, such as the Jordan-Holder theorem, which is for example covered in the first two pages of Benson's *Representations and Cohomology I*) in the interest of length:

(1) Verma modules admit finite Jordan-Holder composition series; in fact, the category $\mathcal{O}$ consists of Artinian objects.
(2) Moreover, for $\lambda \in \Lambda_+$, the Jordan-Holder decomposition of a Verma module $M_{w\circ\lambda}$ contains irreducibles of form

$$L_{w'\circ\lambda} \in \mathrm{JH}(M_{w\circ\lambda}), \quad w' \geq w.$$

In fact there are more precise conditions (which we won't need), which we will give later.
(3) Maps between Verma modules have

$$\mathrm{Hom}_{\mathfrak{g}}(M_\lambda, M_\mu) = \begin{cases} 0 & \\ \mathbb{C} & \end{cases};$$

moreover, for $\lambda \in \Lambda_+$,

$$\mathrm{Hom}_{\mathfrak{g}}(M_{w_1\circ\lambda}, M_{w_2\circ\lambda}) = \mathbb{C} \iff w_1 \geq w_2$$

where $w_1 \geq w_2$ refers to the Bruhat order (more on this later). In fact there are more precise conditions for these homs, too long to be appropriate for this preamble, which we will state later.
(4) The Harish-Chandra theorem, which states that central characters[5] $\vartheta_\lambda = \vartheta_\mu$ are equal iff $\lambda = w \circ \mu$ for some w, i.e. iff $\lambda + \varrho \sim \mu + \varrho$.
(5) Various facts about central characters, such as the exactness of the functor $\square^\vartheta$.
(6) Various purely combinatorial facts about Weyl groups, e.g. the existence of squares and the existence of a choice of signs attached to arrows so that the product of signs in each square is -1.

(1) is proved in chapter 1 of Humphreys's $\mathcal{O}$ book, whereas (3) is proved in chapter 4. (6) is proved in section 11 of BGG, but as it is purely combinatorial we omit it for the sake of length here.

Interestingly I could not find any classical (non-generalization) sources on the BGG resolution aside from the original BGG paper, which we (try to) follow here. The original BGG paper was surprisingly difficult to obtain, and in it BGG uses some conventions/notations which are different from those most use (as noted in Humphreys's book on the category $\mathcal{O}$). Moreover, there were some points in BGG's proofs which I found rather difficult to follow (for example due to omitted details). In this exposition I will try to flesh out these details to the best of my ability and organize the material in a way which is most motivated and easily understood, as well as switch the conventions/notations of BGG to something more familiar. Any errors are, of course, entirely my own.

Among the odd notations of BGG, the most notable is that BGG writes $M_{\lambda+\varrho}$ where we would write M_λ for Verma modules. Though changing BGG's $M_{\lambda+\varrho}$ notation to M_λ is nothing more than an index shift, I can only hope I've made no errors. Here are some others: it seems to be common for central characters to be denoted χ, whereas BGG uses ϑ; this is a convention I will keep. BGG also writes

$$M_\vartheta = \mathrm{Ker}^\infty \mathrm{Ker}\, \vartheta$$

for the eventual kernel of $\mathrm{Ker}\, \vartheta \subseteq Z(U\mathfrak{g})$, which I will instead denote by

$$M^\vartheta$$

[5]More of this later.

since in some sense M admits a "weight decomposition" in this way (more later). Interestingly BGG also writes $w_1 \geq w_2$ implies $\ell(w_1) \leq \ell(w_2)$, so that instead of a unique maximal element in the Weyl group there is a unique minimal element. We will stick to the unique maximal element convention.

2. The Setting: Category $\mathcal{O}$

For completeness let us describe the setting we work in. We will only be story-telling and won't prove any of the details in this section. This is described in section 8 of BGG, where BGG mostly describes and cites things, and also elaborated upon in chapter 1 of Humphreys, where he proves e.g. $\mathcal{O}$ is Artinian.

Let us first define $\mathcal{O}$:

Definition. Let $\mathcal{O}$ be the full subcategory of the category $\mathsf{Mod}_{U\mathfrak{g}}$ of left $U\mathfrak{g}$-modules, whose objects are all M such that:

(1) M is $U\mathfrak{g}$-finitely-generated.
(2) M is $\mathfrak{h}$-semisimple (i.e. $M = \bigoplus_{\lambda\in\mathfrak{h}^*} M^\lambda$ has a weight basis).
(3) M is locally $U\mathfrak{n}^+$-finite, i.e.

$$\dim \operatorname{Span}_{U\mathfrak{n}^+}(v) < \infty \quad \forall\, v \in M.$$

Recall that Verma modules lie in this category.

Before going on to give more facts about $\mathcal{O}$, let us describe the notion of central characters:

Definition. For any $M \in \mathsf{Rep}\,\mathfrak{g}$, if $v \in M$ is an eigenvector with respect to all of $Z(U\mathfrak{g})$, then we can find a

$$\vartheta \in \operatorname{Hom}_{\mathsf{Alg}_{\mathbb{C}}}(Z(U\mathfrak{g}), \mathbb{C})$$

such that

$$zv = \vartheta(z)v \quad \forall\, z \in Z(U\mathfrak{g}).$$

These ϑ are called "central characters"; more generally we may refer to any such homomorphism as a central character.

Let us also write

$$\Theta(M) := \{\text{such } \vartheta\}$$

for the set of central characters of a module.

That this is a homomorphism of algebras is clear since for example $(z_1 + z_2)v = z_1v + z_2v$ so ϑ respects addition, and similarly for multiplication.

For completeness let us cite some facts about $\mathcal{O}$ (see Humphreys):

Proposition. Let $M \in \mathcal{O}$.

(1) All weight spaces of M are finite-dimensional:
$$\dim M^\lambda < \infty.$$
Moreover, the set of weights of M is contained in a finite union of cones $\lambda_i - Q_+$, $\lambda_i \in \mathfrak{h}^*$:
$$\operatorname{Wt} M \subseteq \bigcup_{i \text{ fnt}: \lambda_i \in \mathfrak{h}^*} (\lambda_i - Q_+).$$
(2) $\mathcal{O}$ is both Noetherian and Artinian, i.e. every $M \in \mathcal{O}$ is both Noetherian and Artinian as a $U\mathfrak{g}$-module. In particular this means every $M \in \mathcal{O}$ admits a finite Jordan-Holder decomposition series.
(3) $\mathcal{O}$ is closed under taking submodules, quotients, and finite direct sums.
(4) $\mathcal{O}$ is abelian.
(5) For $M \in \mathcal{O}$ and $\dim V < \infty$,
$$M \otimes V \in \mathcal{O}.$$
In particular
$$\dim V < \infty \implies \square \otimes V : \mathcal{O} \xrightarrow{\text{exact}} \mathcal{O}.$$
(6) M is $U\mathfrak{n}^-$-finitely-generated.
(7) M is $Z(U\mathfrak{g})$-finite:
$$\dim \operatorname{Span}_{Z(U\mathfrak{g})}(v) < \infty \quad \forall\, v \in M.$$
(8) Every irreducible module in $\mathcal{O}$ is of form L_λ, the quotient of M_λ by the maximal submodule ($\lambda \in \mathfrak{h}^*$).

Next some facts regarding the central characters:

(9) For $\lambda \in \mathfrak{h}^*$ and a Verma module M_λ, there is exactly one central character, which we will call ϑ_λ:
$$\Theta(M_\lambda) = \{\vartheta_\lambda\}.$$
(10) For any $M \in \mathcal{O}$,
$$|\Theta(M)| < \infty.$$
(11) For any $\vartheta \in \operatorname{Hom}(Z(U\mathfrak{g}), \mathbb{C})$, its kernel is an ideal $\operatorname{Ker} \vartheta \subseteq Z(U\mathfrak{g})$ which has stabilizing eventual kernel:
$$M \supseteq \{v \in M : (\operatorname{Ker} \vartheta)^n v = 0\} \text{ stabilizes for large } n.$$
This will be denoted
$$M^\vartheta := \operatorname{Ker}^\infty \operatorname{Ker} \vartheta \overset{\mathsf{Rep}}{\subseteq} M,$$
which is a subrepresentation of M.
(12) Moreover,
$$\Theta(M^\vartheta) = \{\vartheta\}$$
and
$$M = \bigoplus_{\vartheta \in \Theta(M)} M^\vartheta$$
and
$$\square^\vartheta : \mathcal{O} \xrightarrow{\text{exact}} \mathcal{O}$$
is exact.

I think the proof[6] of most of these are not too difficult and can in fact be found in chapter 1.1 of Humphreys, except maybe for showing $\mathcal{O}$ is Artinian (requires citing Harish-Chandra), which is in chapter 1.11 of Humphreys.

There is another theorem about these central characters which we will need to cite:

[6] A note to myself: as Humphreys remarks, it is very easy to for example see M^ϑ is a subrep of M, since $\operatorname{Ker}\vartheta \subseteq Z(U\mathfrak{g})$ and so $(\operatorname{Ker}\vartheta)^n v = 0 \implies (\operatorname{Ker}\vartheta)^n \mathfrak{g} v = \mathfrak{g}(\operatorname{Ker}\vartheta)^n v = 0$.

Theorem (Harish-Chandra)**.** For $\lambda, \mu \in \mathfrak{h}^*$,

$$\begin{aligned} \vartheta_\lambda = \vartheta_\mu &\iff \lambda \overset{\circ}{\sim} \mu \\ &\iff \lambda + \varrho \sim \mu + \varrho, \end{aligned}$$

where the second line is the definition of $\overset{\circ}{\sim}$.

Interestingly Harish-Chandra is a single person and not the last name of two authors.

3. The BGG Resolution and The Weyl Character Formula

In this section we will state the main results of this exposition: the BGG resolution and the Weyl character formula. We will use the former to prove the latter.

3.1. **The BGG Setup.** Unfortunately the full statement of the BGG resolution takes a bit of setup. Rather than state a partial result now and use it to prove Weyl, only to give a full statement later, we will begin with this setup and give the full statement right away. Most of the proofs implicit in this setup will be omitted in the interest of length.

Recall that Verma modules were defined as the "universal" highest weight modules,

$$M_\lambda := U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}^\lambda,$$

where $\lambda \in \mathfrak{h}^*$. One may ask what homs between such spaces look like, and there is a theorem of Verma and BGG which characterizes this (I will give the statement as it appears in BGG, adjusting for index shifts):

Theorem (Verma)**.** For $\lambda, \mu \in \mathfrak{h}^*$,

$$\mathrm{Hom}_{\mathfrak{g}}(M_\mu, M_\lambda) = \begin{cases} 0 \\ \mathbb{C} \end{cases},$$

and any nonzero map between two Verma modules is an injection.

Moreover,

$$\begin{gathered} \mathrm{Hom}_{\mathfrak{g}}(M_\mu, M_\lambda) = \mathbb{C} \\ \Updownarrow \\ \begin{aligned} &\exists\, \alpha_1, \cdots, \alpha_k \in \Phi_+ \\ &: \mu + \varrho = s_{\alpha_k} \cdots s_{\alpha_1}(\lambda + \varrho), \\ &\quad s_{\alpha_{i-1}} \cdots s_{\alpha_1}(\lambda + \varrho) - s_{\alpha_i} \cdots s_{\alpha_1}(\lambda + \varrho) \in \mathbb{Z}_{0+}\alpha_i. \end{aligned} \end{gathered}$$

In the case that $\lambda \in \Lambda_+$, these conditions simplify to

$$\begin{gathered} \mathrm{Hom}_{\mathfrak{g}}(M_{w_1 \circ \lambda}, M_{w_2 \circ \lambda}) = \mathbb{C} \\ \Updownarrow \\ w_1 \geq w_2. \end{gathered}$$

The proof of this theorem is a long journey through chapter 4 of Humphreys; as one might expect, it it much easier to prove that $\dim \mathrm{Hom}_{\mathfrak{g}}(M_\mu, M_\lambda) \leq 1$ and is injective when nonzero than it is to prove the full criterion.

For our purposes the $w_1 \geq w_2$ will be the relevant condition. As noted earlier, $w_1 \geq w_2$ refers to the Bruhat order on the Weyl group: i.e. meaning there exists a chain

$$w_1 \xrightarrow{s_{i_1}} u_1 \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{i_{n-1}}} u_{n-1} \xrightarrow{s_{i_n}} w_2$$

such that

$$u_k = s_{i_{k+1}} u_{k+1},$$
$$\ell(u_k) = \ell(u_{k+1}) + 1,$$

where we set $u_0 = w_1$ and $u_n = w_2$. We may sometimes suppress the arrow labels s in this notation and instead just write e.g.

$$w_1 \longrightarrow u_1.$$

In view of this theorem, since all nonzero maps are injective, when appropriate, there is a Verma submodule M_μ inside M_λ. Hence, for $w_1 \geq w_2$ let us write

$$\iota_{w_1 \to w_2} \colon M_{w_1 \circ \lambda} \hookrightarrow M_{w_2 \circ \lambda}$$

for the canonical embedding.

There are two other combinatorial facts about the Weyl group which we will quote; proofs may be found in section 11 of BGG. Here we will refer to them by their numbers in BGG.

Consider the (finite) directed graph $\Gamma(W)$ whose vertices are elements of W and whose arrows are precisely those such that $w_1 \xrightarrow{s} w_2$. We will call (w_1, w_2, w_3, w_4) a "square" if

$$\begin{tikzcd} & w_2 \arrow[dr] & \\ w_1 \arrow[ur] \arrow[dr] & & w_4 \\ & w_3 \arrow[ur] & \end{tikzcd} .$$

Lemma (10.3,10.4)**.** For $w_1, w_4 \in W$ with $\ell(w_1) = \ell(w_4) + 2$, there are either zero or two vertices that fit in arrows between:

$$\#\{w : w_1 \to w \to w_4\} = \begin{cases} 0 \\ 2 \end{cases} .$$

Moreover, to each arrow $w_1 \to w_2$ of $\Gamma(W)$ we may assign a sign

$$\operatorname{sgn}(w_1, w_2) = \pm 1$$

such that, for all squares (w_1, w_2, w_3, w_4),

$$\prod_{\text{4 arrows in square}} \operatorname{sgn}(w, w') = -1.$$

3.2. **The BGG Resolution.** Now we are in a position to set up the BGG resolution. This resolution will be constructed as follows for $\lambda \in \Lambda_+$: grade the graph $\Gamma(W)$ by length $\ell(w)$; place a Verma module $M_{w \circ \lambda}$ at each vertex w of $\Gamma(W)$; define maps in the resolution by putting a map $\operatorname{sgn}(w_1, w_2)\iota_{w_1 \to w_2}$ between $M_{w_1 \circ \lambda}$ and $M_{w_2 \circ \lambda}$ for each arrow $w_1 \to w_2$ (recall that maps between these two is $\mathbb{C}$ since $w_1 \to w_2$, so any map is given by a multiple of the canonical embedding); and lastly direct sum all modules in the same grading (i.e. of same length $\ell(w)$), appropriately combining the maps $\operatorname{sgn}(w_1, w_2)\iota_{w_1 \to w_2}$ to obtain d. Note well that, since each ι is a map of representations, d so defined is also a map of representations.

Theorem (BGG)**.** For $\Pi_\lambda \in \mathsf{irRep}_{\mathrm{fd}}\,\mathfrak{g}$ a finite-dimensional irrep of highest weight $\lambda \in \Lambda_+$, there is a resolution by $\mathfrak{g}$-modules of Π_λ:

$$0 \longrightarrow M_{w_0 \circ \lambda} \xrightarrow{\mathrm{d}_{|\Phi_+|}} \cdots \longrightarrow \bigoplus_{w \in W_k} M_{w \circ \lambda} \xrightarrow{\mathrm{d}_k} \cdots \longrightarrow \bigoplus_{i \in I(\Sigma)} M_{s_i \circ \lambda} \xrightarrow{\mathrm{d}_1} M_\lambda \xrightarrow{\mathrm{d}_0} \Pi_\lambda \longrightarrow 0.$$

Note that each term of the complex is given by ($|\Phi_+| \geq k \geq 0$)

$$C_k = \bigoplus_{w \in W_k} M_{w \circ \lambda},$$

out of which d_k ($k \geq 1$) is defined as

$$\mathrm{d}_k\big|_{M_{w\circ\lambda}} = \big(\operatorname{sgn}(w, w')\iota_{w \to w'}\big)_{w' \in W_{k-1}};$$

d_0 is defined as

$$\mathrm{d}_0 \coloneqq \pi \colon M_\lambda \twoheadrightarrow \Pi_\lambda$$

the projection.

Note also that, as $\ell(w_0) = |\Phi_+| = \dim \mathfrak{n}^-$, $M_{w_0 \circ \lambda}$ belongs to the ($|\Phi_+| = \dim \mathfrak{n}^-$)-th term of the sequence (where we take Π_λ to be the -1-th term).

3.3. **The Weyl Character Formula.** We will now prove the Weyl character formula. First some background: recall we had defined

Definition.

$$\mathbb{C}[P] \coloneqq \mathbb{C}\langle e^\lambda : \lambda \in \Lambda,\ e^\lambda e^\mu = e^{\lambda+\mu},\ e^0 = 1\rangle,$$

in which lives a character (for $V \in \mathsf{Rep}_{\mathrm{fd}}\,\mathfrak{g}$ finite-dimensional)

$$\chi_V \coloneqq \sum_{\lambda \in \mathrm{Wt}\,V} \dim(V^\lambda) e^\lambda.$$

This character enjoys some basic properties, e.g.

Fact.

$$\begin{aligned} \chi_{\mathbb{C}} &= 1, \\ \chi_{V \oplus W} &= \chi_V + \chi_W, \\ \chi_{V \otimes W} &= \chi_V \chi_W, \\ \chi_{V^*} &= \overline{\chi}_V, \end{aligned}$$

where $\overline{\square}$ is defined by

$$\overline{e^\lambda} \coloneqq e^{-\lambda}.$$

For finite-dimensional representations we know that the above sum must be finite (since there are only finitely many weights of a finite-dimensional V), so there are no issues. In the case of an infinite-dimensional representation, however, we must be more careful; to this end define

Definition.

$$\widehat{\mathbb{C}[P]} \coloneqq \left\{ \sum_{\lambda \in \Lambda} c_\lambda e^\lambda : \{\lambda : c_\lambda \neq 0\} \subseteq \bigcup_{\substack{i \text{ fnt} \\ \lambda_i \in \Lambda}} (\lambda_i - Q_+) \right\}$$

where we allow infinite sums as long as all nonzero terms lie in a finite union of cones of form $\lambda_i - Q_+$.

Since highest weight representations (those generated by a single $v \in \operatorname{Ker} \mathfrak{n}^+$) have weights $\lambda - Q_+$, a character χ_{V_λ} for a highest weight representation V_λ of highest weight λ will live in $\widehat{\mathbb{C}[P]}$; in fact, the nonzero terms lie in a single cone $\lambda - Q_+$.

Since $\chi_{V \oplus W} = \chi_V + \chi_W$, one may wonder what the characters of finite-dimensional irreducibles are. This is the theorem of Weyl:

Theorem (Weyl Character)**.** For $\Pi_\lambda \in \mathsf{irRep}_{\mathrm{fd}}\, \mathfrak{g}$ the finite-dimensional irreducible of highest weight $\lambda \in \Lambda_+$, the character of Π_λ is given by

$$\chi_{\Pi_\lambda} = \sum_{w \in W} \operatorname{sgn}(w) e^{w \circ \lambda} \prod_{\alpha \in \Phi_+} \frac{1}{1 - e^{-\alpha}},$$

where

$$\operatorname{sgn}(w) := (-1)^{\ell(w)}$$

and $\frac{1}{1-e^{-\alpha}}$ represents a formal series

$$\frac{1}{1 - e^{-\alpha}} = 1 + e^{-\alpha} + e^{-2\alpha} + \cdots .$$

This will follow directly from computing the characters of Verma modules and applying the BGG resolution. Indeed, the Verma modules have character:

Proposition. For $\lambda \in \mathfrak{h}^*$, the Verma module M_λ has character

$$\chi_{M_\lambda} = e^\lambda \prod_{\alpha \in \Phi_+} \frac{1}{1 - e^{-\alpha}},$$

where $\frac{1}{1-e^{-\alpha}}$ represents a formal series

$$\frac{1}{1 - e^{-\alpha}} = 1 + e^{-\alpha} + e^{-2\alpha} + \cdots .$$

Proof. Recall that the M_λ has weights $\mu \in \lambda - Q_+$, each of which is finite-dimensional. Recall moreover

$$U\mathfrak{n}^- \overset{\mathsf{Vec}}{\cong} M_\lambda$$

via

$$x \longmapsto x v^\lambda.$$

Recall also from a computation[7] on pset 10 that, for $\alpha \in \Phi_+$, f_α brings a vector in the μ-weight space to the $\mu - \alpha$-weight space, for example

$$f_\alpha v^\lambda \in {M_\lambda}^{\lambda - \alpha}. \tag{$*$}$$

Then, since the Verma module is the free $U\mathfrak{n}^-$ module generated on one vector, if linearly independent $x \neq y \in U\mathfrak{n}^-$ both have $xv^\lambda, yv^\lambda \in {M_\lambda}^{\lambda-\delta}$, then xv^λ, yv^λ are linearly independent. Therefore, to compute the dimensional of each weight space, it suffices to enumerate the number of linearly independent elements of $U\mathfrak{n}^-$ which bring v^λ to the right weight space. By the PBW theorem, since $\mathfrak{n}^-$ has an additive basis $\{f_\alpha\}_{\alpha \in R^+}$, we know $U\mathfrak{n}^-$ has a basis given by $\{\prod_\alpha f_\alpha^{n_\alpha}\}_n$; therefore, for $\delta \in Q_+$ the dimension of each weight space is given by

$$\dim {M_\lambda}^{\lambda - \delta} = \#\left\{ \sum_{\alpha \in \Phi_+} n_\alpha \alpha : \sum_{\Phi_+} n_\alpha \alpha = \delta \right\}.$$

[7]For completeness, this is saying

$$\mathfrak{h} f_\alpha v^\lambda = [\mathfrak{h}, f_\alpha] v^\lambda + f_\alpha \mathfrak{h} v^\lambda = -\alpha(\mathfrak{h}) f_\alpha v^\lambda + f_\alpha \lambda(\mathfrak{h}) v^\lambda = (\lambda - \alpha)(\mathfrak{h}) f_\alpha v^\lambda.$$

By elementary combinatorics this is the coefficient

$$\dim {M_\lambda}^{\lambda-\delta} = [e^{-\delta}] \prod_{\alpha\in\Phi_+} (1 + e^{-\alpha} + e^{-2\alpha} + \cdots).$$

Then we have

$$\begin{aligned}
\chi_{M_\lambda} &= \sum_{\mu\in\lambda-Q_+} \dim({M_\lambda}^\mu) e^\mu \\
&= \sum_{\delta\in Q_+} \dim {M_\lambda}^{\lambda-\delta} e^{\lambda-\delta} \\
&= \sum_{\delta\in Q_+} e^{\lambda-\delta}[e^{-\delta}] \prod_{\alpha\in\Phi_+} (1 + e^{-\alpha} + e^{-2\alpha} + \cdots) \\
&= e^\lambda \sum_{\delta\in Q_+} e^{-\delta}[e^{-\delta}] \prod_{\alpha\in\Phi_+} \frac{1}{1-e^{-\alpha}} \\
&= e^\lambda \prod_{\alpha\in\Phi_+} \frac{1}{1-e^{-\alpha}},
\end{aligned}$$

where in the last line we recall Q_+ is precisely the set of all nonnegative spans of positive roots. This is as advertised. ■

We are now in a position to prove Weyl.

Proof of Weyl Character. Recall from linear algebra that the alternating sum of dimensions in an exact sequence of vector spaces is zero. In fact, for $\varphi\colon M \longrightarrow N$ a map of representations, we have

$$\mathfrak{h}\varphi(v^\lambda) = \varphi\mathfrak{h}(v^\lambda) = \varphi\lambda(\mathfrak{h})(v^\lambda) = \lambda(\mathfrak{h})\varphi(v^\lambda),$$

so that

$$\varphi(M^\lambda) \subseteq N^\lambda \tag{$*$}$$

maps of representations preserve weight spaces. Therefore, in an exact sequence of representations we may restrict attention to each weight space and find that, for

$$0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0,$$

the alternating sum of the dimensions of each weight spaces is also zero:

$$\sum_{i=1}^n (-1)^i \dim({V_i}^\lambda) = 0; \tag{$*$}$$

applying this for each weight space gives

$$\sum_{i=1}^n (-1)^i \chi_{V_i} = 0. \tag{$*$}$$

Apply this now to the BGG resolution

$$0 \longrightarrow M_{w_0\circ\lambda} \longrightarrow \cdots \longrightarrow \bigoplus_{w\in W_k} M_{w\circ\lambda} \longrightarrow \cdots \longrightarrow \bigoplus_{I(\Sigma)} M_{s_i\circ\lambda} \longrightarrow M_\lambda \longrightarrow \Pi_\lambda \longrightarrow 0,$$

whereupon we obtain

$$\begin{aligned}
\chi_{\Pi_\lambda} &= \sum_{k=1}^{|\Phi_+|} (-1)^k \sum_{w:\ell(w)=k} \chi_{M_{w\circ\lambda}} \\
&= \sum_{w\in W} \operatorname{sgn}(w) \chi_{M_{w\circ\lambda}} \\
&= \sum_{w\in W} \operatorname{sgn}(w) e^{w\circ\lambda} \prod_{\alpha\in\Phi_+} \frac{1}{1-e^{-\alpha}},
\end{aligned}$$

as desired. ∎

In some sense, BGG, being a resolution which turns this alternating sum of formal things into an exact sequence of actual representations, is a "categorification" of the Weyl character formula.

4. Proving BGG from Weak BGG: Three Steps

Now comes the daunting task of proving the BGG resolution. First, some brief words on how we will do this: we will first prove BGG assuming three key lemmas (the three main chunks of the proof), 10.5, 10.6, and 10.7 in the paper. We will then prove these three lemmas, assuming (which we will prove in a later section) two statements from section 9 of BGG (Theorem 9.9 and its Corollary (unnumbered)), which may be thought of as a weaker version of the full BGG theorem.

Along the way we will need to assume some facts (e.g. Harish-Chandra) which will not be proven; we will try to make it clear when we do so, and give precise statements of what the claims are.

As mentioned in the beginning, we will prove BGG in a way which is logically backwards, assuming facts and proving them later, for the sake of clarity and motivation.

4.1. **Lemmas imply BGG.**

We will prove BGG assuming Lemmas 10.5, 10.6, and 10.7 of BGG. Other facts along the way, being easier to see, will be appropriately proved. For ease of reading let us reproduce a statement of BGG here:

Theorem (BGG)**.** For $\Pi_\lambda \in \mathsf{irRep}_{\mathrm{fd}}\,\mathfrak{g}$ with $\lambda \in \Lambda_+$, there is a resolution by $\mathfrak{g}$-modules of Π_λ:

$$0 \longrightarrow M_{w_0 \circ \lambda} \xrightarrow{\mathrm{d}_{|\Phi_+|}} \cdots \longrightarrow \bigoplus_{w \in W_k} M_{w \circ \lambda} \xrightarrow{\mathrm{d}_k} \cdots \longrightarrow \bigoplus_{i \in I(\Sigma)} M_{s_i \circ \lambda} \xrightarrow{\mathrm{d}_1} M_\lambda \xrightarrow{\mathrm{d}_0} \Pi_\lambda \longrightarrow 0,$$

where d_k ($k \geq 1$) is defined as

$$\mathrm{d}_k\big|_{M_{w\circ\lambda}} = \big(\operatorname{sgn}(w, w')\iota_{w \to w'}\big)_{w' \in W_{k-1}}$$

and d_0 is defined as

$$\mathrm{d}_0 \coloneqq \pi \colon M_\lambda \twoheadrightarrow \Pi_\lambda.$$

Proof of BGG. Step 1: Complex: Let us first see BGG is a complex. We have already described in the last section the "geometry" of the BGG resolution as well as what the maps are. From the way we have defined the maps, it is immediate that BGG gives a complex, i.e. $\mathrm{dd} = 0$: indeed, by the combinatorial lemmas, given two terms in BGG which are two apart, i.e. $\ell(w_1) = \ell(w_4) + 2$, either there are is no path in $\Gamma(W)$ from w_1 to w_4, in which case $\mathrm{d}^2\big|_{w_1 \to w_4} = 0$ trivially, or we can write a square with attached signs

$$\begin{tikzcd}
 & w_2 \arrow[dr, "\operatorname{sgn}(w_2,w_4)"] & \\
w_1 \arrow[ur, "\operatorname{sgn}(w_1,w_2)"] \arrow[dr, "\operatorname{sgn}(w_1,w_3)"'] & & w_4 \\
 & w_3 \arrow[ur, "\operatorname{sgn}(w_3,w_4)"'] &
\end{tikzcd}$$

such that the product of the four signs is -1; for this to happen, either three are $+1$ and one is -1 or three are -1 and one is $+1$. It's then pretty clear that in any case either the top path is $+1$ and the bottom path is -1 or vice versa, in which case the sum (which is how d is defined) is zero, so that $\mathrm{dd} = 0$ still.

Step 2: Exact Beginning: Next let us see why this sequence is exact[8] at M_λ (position 0) and at Π_λ (position -1).

First some facts:

[8]BGG claims that this follows from Harish-Chandra's theorem on ideals, but I could not quite understand this or locate precisely what statement they were referring to; instead we will give here an alternate and no doubt clumsier (though hopefully not incorrect) argument.

Fact (K8.27)**.** For $v^\mu \in {M_\lambda}^\mu$,

$$\mathfrak{n}^+ v^\mu = 0 \implies M_\lambda \supseteq \operatorname{Span}_{U\mathfrak{g}}(v^\mu) = \operatorname{Span}_{U\mathfrak{n}^-}(v^\mu) \cong M_\mu$$

is a Verma module.

Proof of K8.27. If $\mathfrak{n}^+$ acts by zero, then clearly instead of acting by all of $U\mathfrak{g}$ it suffices to act on by $U\mathfrak{n}^-$ ($\mathfrak{h}$ will of course act by a character), so that $\operatorname{Span}_{U\mathfrak{g}} v^\mu = \operatorname{Span}_{U\mathfrak{n}^-} v^\mu$.

$\mathfrak{n}^+ v^\mu = 0$ implies the submodule generated[9] by v^μ, $\operatorname{Span}_{U\mathfrak{g}} v^\mu$, is a highest weight representation of highest weight μ, so that it is a quotient of M_μ. Since $U\mathfrak{n}^- \cong M_\mu$ as vector spaces via action, to show $\operatorname{Span}_{U\mathfrak{g}} v^\mu \cong M_\mu$ is suffices to show the map

$$\begin{aligned} U\mathfrak{n}^- &\longmapsto \operatorname{Span}_{U\mathfrak{g}} v^\mu \\ x &\longmapsto xv^\mu \end{aligned}$$

is injective (it is automatically surjective since $\mathfrak{n}^+ v^\mu = 0$). AFSOC $xv^\mu = 0$ for some x; since $v^\mu \in M_\lambda$, there is some $y \in U\mathfrak{n}^-$ such that $yv^\lambda = v^\mu$, so this is saying $xyv^\lambda = 0$, which can only happen if $xy = 0$ since $M_\lambda \cong U\mathfrak{n}^-$; but by PBW

Theorem (PBW)**.** Any Lie algebra $\mathfrak{g}$ with ordered basis

$$\mathfrak{g} = \mathbb{C}(\xi_1, \cdots, \xi_n)$$

has that $U^k\mathfrak{g}$ has basis

$$U^k\mathfrak{g} = \mathbb{C}\{\xi_1^{k_1} \cdots \xi_n^{k_n}\}_{\sum k_i \leq k}.$$

we have $U\mathfrak{n}^-$ has no zero divisors, so this implies $x = 0$ (since $v^\mu \neq 0$, so that $y \neq 0$), which implies the map is injective and therefore surjective. ∎

Recall that, for $\alpha_i \in \Sigma$ and $\lambda \in \Lambda_+$,

$$f_i^{\lambda(h_i)+1}(\widetilde{v}^\lambda) = 0 \in L_\lambda.$$

In fact this can be made more precise:

Lemma (K8.28a)**.** For $\lambda \in \Lambda_+$ and $\alpha_i \in \Sigma$, the submodule inside M_λ generated by $f_i^{\lambda(h_i)+1} v^\lambda$

$$M_\lambda \supseteq \operatorname{Span}_{U\mathfrak{g}}\left(f_i^{\lambda(h_i)+1} v^\lambda\right) = \operatorname{Span}_{U\mathfrak{n}^-}\left(f_i^{\lambda(h_i)+1} v^\lambda\right) \cong M_{s_i \circ \lambda}$$

is isomorphic to a Verma module $M_{s_i \circ \lambda}$.

Proof of K8.28a. We saw earlier that

$$f_\alpha v^\lambda \in {M_\lambda}^{\lambda - \alpha},$$

so in particular

$$f_i^{\lambda(h_i)+1} v^\lambda \in {M_\lambda}^{\lambda - \left(\lambda(h_i)+1\right)\alpha_i} = {M_\lambda}^{s_i \circ \lambda},$$

where

$$s_i \circ \lambda = s_i(\lambda + \varrho) - \varrho = s_i\lambda + s_i\varrho - \varrho = s_i\lambda - \alpha_i = \lambda - \left(\alpha_i^*(\lambda) + 1\right)\alpha_i,$$

where we recall that $\varrho - s_i\varrho = \alpha_i$, and more generally

$$\varrho - w(\varrho) = \sum_{\alpha \in \Phi_+ : w^{-1}\alpha \in \Phi_-} \alpha.. \tag{$*$}$$

Since

$$f_i^{\lambda(h_i)+1} v^\lambda \in {M_\lambda}^{s_i \circ \lambda},$$

[9] It feels like $U\mathfrak{g} \cdot v^\mu$ would refer to the free module generated by v^μ, so to emphasize that this is a *sub*module generated by v^μ, we will write $\operatorname{Span}_{U\mathfrak{g}}^{M_\lambda} v^\mu$, or just $\operatorname{Span}_{U\mathfrak{g}} v^\mu$ for short.

to show the desired by Fact K8.27 it suffices to show $\mathfrak{n}^+$ brings this to zero. Recalling still that $\mathfrak{n}^+$ is generated by e_i for $i \in I(\Sigma)$, it suffices to show this is killed by all e_j for $j \in I(\Sigma)$. For $j \neq i$ this is easy, as $[e_j, f_i] = 0$ and so

$$e_j \cdot f_i^{\lambda(h_i)+1} v^\lambda = f_i^{\lambda(h_i)+1} e_j v^\lambda = 0$$

since, being a highest weight vector, $\mathfrak{n}^+ v^\lambda = 0$. For $i = j$ we must[10] embark on a slightly lengthier computation:

$$\begin{aligned}
e_i f_i^{\lambda(h_i)+1} v^\lambda &= \big([e_i, f_i] + f_i e_i\big) f_i^{\lambda(h_i)} v^\lambda \\
&= h_i f_i^{\lambda(h_i)} v^\lambda \quad + \quad f_i e_i f_i^{\lambda(h_i)} v^\lambda \\
&= h_i f_i^{\lambda(h_i)} v^\lambda \quad + \quad f_i \big([e_i, f_i] + f_i e_i\big) f_i^{\lambda(h_i)-1} v^\lambda \\
&= h_i f_i^{\lambda(h_i)} v^\lambda + f_i h_i f_i^{\lambda(h_i)-1} v^\lambda \quad + \quad f_i^2 e_i f_i^{\lambda(h_i)-1} v^\lambda \\
&\vdots \\
&= h_i f_i^{\lambda(h_i)} v^\lambda + f_i h_i f_i^{\lambda(h_i)-1} v^\lambda + \cdots + f_i^{\lambda(h_i)} h_i v^\lambda \quad + \quad f_i^{\lambda(h_i)+1} \cancel{e_i v^\lambda} \\
&= h_i f_i^{\lambda(h_i)} v^\lambda + f_i h_i f_i^{\lambda(h_i)-1} v^\lambda + \cdots + f_i^{\lambda(h_i)} h_i v^\lambda \\
&= (\lambda - \lambda(h_i)\alpha_i)(h_i) f_i^{\lambda(h_i)} v^\lambda \\
&\quad + \big(\lambda - (\lambda(h_i) - 1)\alpha_i\big)(h_i) f_i^{\lambda(h_i)} v^\lambda \\
&\quad\quad + \cdots \\
&= \sum_{k=0}^{\lambda(h_i)} \big(\lambda - (\lambda(h_i) - k)\alpha_i\big)(h_i) \cdot f_i^{\lambda(h_i)} v^\lambda \\
&= f_i^{\lambda(h_i)} v^\lambda \sum_{k=0}^{\lambda(h_i)} \lambda(h_i) - \lambda(h_i)\alpha_i(h_i) + k\alpha_i(h_i) \\
&= f_i^{\lambda(h_i)} v^\lambda \sum_{k=0}^{\lambda(h_i)} \lambda(h_i) - 2\lambda(h_i) + 2k \\
&= f_i^{\lambda(h_i)} v^\lambda \left(-\lambda(h_i)(\lambda(h_i) + 1) + 2 \cdot \frac{1}{2} \lambda(h_i)(\lambda(h_i) + 1) \right) \\
&= 0,
\end{aligned}$$

completing our check that $\mathfrak{n}^+ f_i^{\lambda(h_i)+1} v^\lambda$. Hence, by Fact K8.27, the submodule generated by this vector is a Verma module of the form advertised. ■

Recall that L_λ is a quotient of a Verma module by a maximal submodule not containing the highest weight vector; in fact, this can also be "refined" to

Lemma (K8.28b)**.** For $\lambda \in \Lambda_+$,

$$\Pi_\lambda = M_\lambda \Big/ \sum_{i \in I(\Sigma)} M_{s_i \circ \lambda},$$

where by "$M_{s_i \circ \lambda}$" we mean the submodule $\mathrm{Span}_{U\mathfrak{n}^-}(f_i^{\lambda(h_i)+1} v^\lambda) \subset M_\lambda$ constructed in the previous lemma which is isomorphic to $M_{s_i \circ \lambda}$.

Proof of K8.28b. We will show the RHS is a finite-dimensional representation and then appeal to complete reducibility.

[10] Probably I am missing some easier way to do this, but this certainly works.

To see that

$$\dim_{\mathbb{C}} M_\lambda \Big/ \sum_{i\in I(\Sigma)} M_{s_i\circ\lambda} < \infty,$$

note

$$M_\lambda \Big/ \sum_{i\in I(\Sigma)} M_{s_i\circ\lambda} \overset{\mathsf{Vec}}{\cong} U\mathfrak{n}^- \Big/ U\mathfrak{n}^- \langle f_i^{\lambda(h_i)+1} \rangle_{i\in I(\Sigma)}$$

where we have used $M_\lambda \cong U\mathfrak{n}^-$ again, as well as using that $M_{s_i\circ\lambda} \subset M_\lambda$ is generated by $f_i^{\lambda(h_i)+1} v^\lambda$. But recall that an ideal of form $U\mathfrak{n}^-\langle f_i^{\lambda(h_i)+1}\rangle_{i\in I(\Sigma)}$ has finite codimension

$$\operatorname{codim}_{U\mathfrak{n}^-} U\mathfrak{n}^-\langle f_i^{\lambda(h_i)+1}\rangle_{i\in I(\Sigma)} < \infty,$$

which shows finite-dimensionality.

Then, by complete reducibility, we may write

$$M_\lambda \Big/ \sum_{i\in I(\Sigma)} M_{s_i\circ\lambda} \cong \bigoplus_{\mu\leq\lambda} m_\mu \Pi_\mu,$$

where we know $\mu \leq \lambda$ since the left hand side is a quotient of a Verma module and so its weights must be of form $\lambda - Q_+$. Moreover, by looking at the dimension of the λ-space, since the LHS has $\dim(M_\lambda / \sum_i M_{s_i\circ\lambda})^\lambda = 1$ (the quotient not containing the highest weight vector), we force $m_\lambda = 1$:

$$M_\lambda \Big/ \sum_{i\in I(\Sigma)} M_{s_i\circ\lambda} \cong \Pi_\lambda \oplus \bigoplus_{\mu<\lambda} m_\mu \Pi_\mu.$$

In particular, this means the highest weight vector $\widetilde{v}^\lambda$ of $M_\lambda/\sum_i M_{s_i\circ\lambda}$ lies inside the factor Π_λ, so that the submodule inside generated by $\widetilde{v}^\lambda$ must be Π_λ (by irreducibility of Π_λ). On the other hand, as v^λ generates M_λ, we have $\widetilde{v}^\lambda$ generates all of $M_\lambda/\sum_i M_{s_i\circ\lambda}$. Hence we have $M_\lambda/\sum_i M_{s_i\circ\lambda} \cong \Pi_\lambda$, as claimed. ■

Indeed, the failure of the $\sum_{i\in I(\Sigma)}$ to be a direct sum is why the rest of the terms in BGG are necessary. That is, K8.28b allows us to describe Π_λ as a cokernel

$$\bigoplus_{i\in I(\Sigma)} M_{s_i\circ\lambda} \longrightarrow M_\lambda \longrightarrow \Pi_\lambda \longrightarrow 0,$$

but says nothing about what the kernel is. BGG will do this for us. Note well that the map $\bigoplus_{i\in I(\Sigma)} M_{s_i\circ\lambda} \longrightarrow M_\lambda$ is given by the canonical embeddings[11] $M_{s_i\circ\lambda} \subset M_\lambda$, as prescribed by Verma (Section 2 above); this agrees with the differential maps we described in the statement of BGG.

Step 3: Exact Everywhere: Now that we know this is a complex which is exact at degrees 0 and -1, we will cite three key lemmas (to be proved later this section) to show BGG is exact everywhere. The proof will be by induction:

$$\text{by induction, assume BGG is exact at degrees } -1, 0, 1, \cdots, k-1.$$

We wish to show exactness at degree k also. Since this is a complex, we already have $\mathrm{d}_{k+1}(C_{k+1}) \subseteq \operatorname{Ker}\mathrm{d}_k \subseteq C_k$; we wish only to show

$$\mathrm{d}_{k+1}\colon C_{k+1} \overset{?}{\twoheadrightarrow} \operatorname{Ker}\mathrm{d}_k$$

is surjective, i.e. $\operatorname{Img}\mathrm{d}_{k+1} = \operatorname{Ker}\mathrm{d}_k$.

In the below, by $U\mathfrak{n}^-$-free, we mean free as a $U\mathfrak{n}^-$-module.

[11]Here we have chosen $\operatorname{sgn}(s_i, \mathrm{id}) = +1$ for all $i \in I(\Sigma)$, which might seem to be an issue given that in our statement of Lemma 10.4 we do not have any choice on what the signs are; however, by looking at the proof of Lemma 10.4 in section 11 of BGG, one sees that the proof goes by induction on $\ell(w)$, where at the base case (i.e. in the case of $\operatorname{sgn}(s_i, \mathrm{id})$) we are free to choose whatever signs we fancy; so in particular we may take them all to be positive signs, as we do here.

Lemma (10.5)**.** For $M, N \in \mathcal{O}$ such that M is $U\mathfrak{n}^-$-free with generators $v_1, \cdots, v_n$,

$$M = \operatorname{Span}_{U\mathfrak{n}^-}\{v_1, \cdots, v_n\}$$

and

$$\varphi \colon M \xrightarrow{U\mathfrak{n}^-} N$$

a map of $U\mathfrak{n}^-$-modules such that

$$\varphi(v_i) \text{ is a weight vector,}$$

we have

$$\varphi \colon M \twoheadrightarrow N \ \text{ surj } \iff \widetilde{\varphi} \colon M\big/\mathfrak{n}^- M \twoheadrightarrow N\big/\mathfrak{n}^- N \ \text{ surj.}$$

In particular we will be interested in applying 10.5 for $M = C_{k+1} = \bigoplus_{w \in W_{k+1}} M_{w \circ \lambda}$ and $N = \operatorname{Ker} \mathrm{d}_k$.

Lemma (10.6)**.**

$$\widetilde{\mathrm{d}}_{k+1} \colon C_{k+1}\big/\mathfrak{n}^- C_{k+1} \hookrightarrow \operatorname{Ker} \mathrm{d}_k\big/\mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k \ \text{ inj}$$

is injective.

In fact, since we are inducting on k (i.e. to get to the k-th step we have proved this lemma for every $i < k$), this will be true of all $\widetilde{d}_i$ for $i \leq k+1$ (sort of like strong induction).

Lemma (10.7)**.**

$$\dim_{\mathbb{C}} C_{k+1}\big/\mathfrak{n}^- C_{k+1} = \dim_{\mathbb{C}} \operatorname{Ker} \mathrm{d}_k\big/\mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k < \infty.$$

Assuming these lemmas, it is clear we can now show exactness at k, completing induction. Indeed, 10.6 gives an injection

$$\widetilde{\mathrm{d}}_{k+1} \colon C_{k+1}\big/\mathfrak{n}^- C_{k+1} \hookrightarrow \operatorname{Ker} \mathrm{d}_k\big/\mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k$$

which by 10.7 is an injection between two finite-dimensional vector spaces of the same dimension; therefore $\widetilde{\mathrm{d}}_{k+1}$ is also surjective

$$\widetilde{\mathrm{d}}_{k+1} \colon C_{k+1}\big/\mathfrak{n}^- C_{k+1} \hookrightarrow\!\!\!\!\rightarrow \operatorname{Ker} \mathrm{d}_k\big/\mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k,$$

which by 10.5 implies

$$\mathrm{d}_{k+1} \colon C_{k+1} \twoheadrightarrow \operatorname{Ker} \mathrm{d}_k$$

is also surjective, completing the proof of exactness. This concludes the proof of BGG (Lemmas to be proved later). ■

4.2. **Weak BGG implies Lemmas.** In this subsection we will prove the Lemmas 10.5, 10.6, 10.7 cited above. The title is slightly misleading; 10.5 does not require Weak BGG, 10.6 does, and 10.7 requires Bott's Theorem, a corollary of Weak BGG.

We will restate the Lemmas each time so the viewer does not have to scroll up.

4.2.1. *Lemma 10.5.* First let us show 10.5.

Lemma (10.5)**.** For $M, N \in \mathcal{O}$ such that M is $U\mathfrak{n}^-$-free with generators $v_1, \cdots, v_n$,

$$M = \operatorname{Span}_{U\mathfrak{n}^-}\{v_1, \cdots, v_n\}$$

and

$$\varphi \colon M \xrightarrow{U\mathfrak{n}^-} N$$

a map of $U\mathfrak{n}^-$-modules such that

$$\varphi(v_i) \text{ is a weight vector,}$$

we have

$$\varphi \colon M \twoheadrightarrow N \ \text{ surj } \iff \widetilde{\varphi} \colon M\big/\mathfrak{n}^- M \twoheadrightarrow N\big/\mathfrak{n}^- N \ \text{ surj.}$$

Proof of 10.5. $\implies$ It is clear that φ surjective implies $\widetilde{\varphi}$ surjective (since φ commutes with $\mathfrak{n}^-$, it is not possible for a nonzero person in $N/\mathfrak{n}^- N$ to have preimage in $\mathfrak{n}^- M$; indeed, the image of $\mathfrak{n}^- M$ lies inside $\mathfrak{n}^- N$).

$\impliedby$ Now let us see that $\widetilde{\varphi}$ surjective implies φ surjective. We will show that any weight vector in N is actually in the image of φ, which implies the desired[12]; that is, we claim

$$u \in N^\mu \implies u \in \operatorname{Img}\varphi \qquad \forall\, \mu.$$

The idea is to proceed by induction/infinite descent (not sure what to call this) on the weights of N. That is, since $N \in \mathcal{O}$, we know the set of weights of N lie in a finite union of cones (see Section 2):

$$\operatorname{Wt} N \subseteq \bigcup_{i \text{ fnt}:\lambda_i \in \mathfrak{h}^*} (\lambda_i - Q_+).$$

We will start by showing $N^{\lambda_i} \subseteq \operatorname{Img}\varphi$ and work our way downwards. In fact, the base case of the highest weights λ_i is a formal consequence of the inductive step, as we will highlight below.

Hence, for the inductive step let us pick a weight μ so that all weights above μ are contained in $\operatorname{Img}\varphi$:

$$\implies \text{pick } \ u \in N^\mu : N^{>\mu} \subseteq \operatorname{Img}\varphi.$$

Under projection by $\mathfrak{n}^-$ this goes to

$$\begin{aligned} \pi\colon N &\twoheadrightarrow N/\mathfrak{n}^- N \\ u &\longmapsto \widetilde{u}. \end{aligned}$$

Then

$$\begin{aligned} \widetilde{\varphi} \ \text{ surj} &\implies \widetilde{\varphi}(v_1), \cdots, \widetilde{\varphi}(v_n) \ \text{ generate } N/\mathfrak{n}^- N \\ &\implies \widetilde{u} = \sum_i c_i \widetilde{\varphi}(v_i) \end{aligned}$$

for some coefficients c_i. Note[13] that $\mathfrak{h}$ acts on $N/\mathfrak{n}^- N$, with[14]

$$\pi(u) = \widetilde{u} \in (N/\mathfrak{n}^- N)^\mu,$$

i.e.

$$\sum_i c_i \widetilde{\varphi}(v_i) \in (N/\mathfrak{n}^- N)^\mu.$$

However, from linear algebra we know the sum of eigenvectors with differing eigenvalues is not an eigenvector[15], which forces

$$c_i \neq 0 \implies \operatorname{Wt}(\widetilde{\varphi}(v_i)) = \mu.$$

Consider $u - \sum_i c_i \varphi(v_i)$. We've noted in an earlier footnote that quotienting (which a priori commutes with $\mathfrak{n}^-$) commutes with $\mathfrak{h}$, so that

$$\operatorname{Wt}(\widetilde{\varphi}(v_i)) = \mu \implies \operatorname{Wt}(\varphi(v_i)) = \mu$$

and therefore

$$u - \sum_i c_i \varphi(v_i) \in N^\mu$$

since each u and $\varphi(v_i)$ is in N^μ. Since $\pi(u - \sum_i c_i \varphi(v_i)) = 0$, we also have

$$u - \sum_i c_i \varphi(v_i) \in \mathfrak{n}^- N,$$

[12] Since weight vectors form an eigenbasis with respect to $\mathfrak{h}$, by definition.

[13] This is since

$$\mathfrak{h}\mathfrak{n}^- N = [\mathfrak{h}, \mathfrak{n}^-]N + \mathfrak{n}^-\mathfrak{h}N \subseteq \mathfrak{n}^- N + \mathfrak{n}^- N = \mathfrak{n}^- N,$$

so that $\mathfrak{h}$ acts on $\mathfrak{n}^- N$ (i.e. $\mathfrak{n}^- N$ is closed under $\mathfrak{h}$), so that $\mathfrak{h}$ acts on $N/\mathfrak{n}^- N$.

[14] We know $\mathfrak{h}\pi(u) = \pi(\mathfrak{h}u)$ since $\mathfrak{h}\colon \mathfrak{n}^- N \longrightarrow \mathfrak{n}^- N$; then

$$\mathfrak{h}\pi(u) = \pi(\mathfrak{h}u) = \mu(\mathfrak{h})\pi(u) \implies \pi(u) \in (N/\mathfrak{n}^- N)^\mu.$$

[15] In other words, eigenvectors of differing eigevalues are linearly independent.

so

$$\implies u - \sum_i c_i\varphi(v_i) \in N^\mu \cap \mathfrak{n}^- N.$$

Hence we can write $u - \sum_i c_i\varphi(v_i)$ as

$$u - \sum_i c_i\varphi(v_i) = \sum_{\alpha\in\Phi_+} c_\alpha f_\alpha w^{\mu+\alpha}$$

for some weight vectors $w^{\mu+\alpha} \in N^{\mu+\alpha}$ and constants c_α. Here we know we have factors of f_α since $u - \sum_i c_i\varphi(v_i) \in \mathfrak{n}^- N$, and we know the weights on w must be $\mu+\alpha$ since $u - \sum_i c_i\varphi(v_i) \in N^\mu$ and f_α drops the weight by α.

But

$$w^{\mu+\alpha} \in N^{>\mu} \subseteq \operatorname{Img}\varphi,$$

which is a submodule of N, so $f_\alpha w^{\mu+\alpha} \in \operatorname{Img}\varphi$, and

$$\implies u - \sum_i c_i\varphi(v_i) = \sum_{\alpha\in\Phi_+} c_\alpha f_\alpha w^{\mu+\alpha} \in \operatorname{Img}\varphi$$

$$\implies u \in \operatorname{Img}\varphi,$$

completing the proof.

Note well that, in the above argument, if $\mu = \lambda_i$ is a maximal weight, then $w^{\mu+\alpha} = 0$ and we immediately have $u - \sum_i c_i\varphi(v_i) = 0 \implies u = \sum_i c_i\varphi(v_i) \in \operatorname{Img}\varphi$, so that indeed the base case is subsumed by the inductive step. ■

4.2.2. *Lemma 10.6.* Next let us do 10.6. This lemma will be broken up into two parts, 10.6a and 10.6b (as they are named in BGG). To prove this lemma we will need to cite Weak BGG as well as some facts about the Jordan-Holder decomposition of Verma modules. First let me recall what Jordan-Holder is:

Definition. A "decomposition series" (or composition series) of $M \in \mathsf{Mod}_R$ is a filtration

$$0 = M_0 \subseteq \cdots \subseteq M_n = M$$

such that

$$M_i/M_{i-1} \text{ is simple.}$$

We will denote the set of such simple quotients by

$$\mathrm{JH}(M) := \{M_i/M_{i-1}\}_i,$$

which we will call the "Jordan-Holder factors".

It is a theorem that this $\mathrm{JH}(M)$ is well-defined. Concretely one can think about this in two ways: to get a decomposition series, one may either keep taking maximal submodules[16] to obtain the desired filtration, or one may do the following process: take a maximal simple submodule of M, Π_1, and pass to M/Π_1; then take a maximal simple submodule Π_2 of V/Π_1, and pass to $(M/\Pi_1)/\Pi_2$; etc.. The filtration is then $F_0 = 0$, $F_1 = \Pi_1$, $F_2 = \pi_1^{-1}(\Pi_2)$, etc..

There are three key things we must cite for this subsubsection. The first is the Jordan-Holder Theorem (a standard fact from homological algebra which takes about a page and a half to prove), the second is the Jordan-Holder factors of a Verma module, and the third is Weak BGG. The former two will not be proved in this exposition, while the third will be proved in a later section.

First, the Jordan-Holder Theorem says[17]

[16]If N is a maximal submodule of M, then M/N is simple since the existence of a nontrivial submodule of M/N contradicts the maximality of N.

[17]This statement is taken from the second page of Benson.

Theorem (Jordan-Holder)**.** Given two filtrations of M of possibly different length

$$0 = M_0 \subseteq \cdots \subseteq M_n = M,$$
$$0 = M_0' \subseteq \cdots \subseteq M_m' = M,$$

we may refine them (i.e. stick in extra terms) to obtain two filtrations of equal length

$$0 = N_0 \subseteq \cdots \subseteq N_k = M,$$
$$0 = N_0' \subseteq \cdots \subseteq N_k' = M$$

such that

$$\{N_i/N_{i-1}\}_i \simeq \{N_j'/N_{j-1}'\}_j,$$

where by $\simeq$ we mean the N_i/N_{i-1} are a permutation of the N_j'/N_{j-1}' up to isomorphism.

Moreover, the following are equivalent:

- M admits a decomposition series;
- Every filtration of M can be refined to a decomposition series;
- M is both Noetherian and Artinian.

Here recall from commutative algebra that Noetherian is "ascending chains terminate" and Artinian is "descending chains terminate". Some additional easy facts about JH factors:

Fact. For $M, N \in \mathsf{Mod}_R$ which admit decomposition series,

$$\begin{aligned} \mathrm{JH}(M \oplus N) &= \mathrm{JH}(M) \sqcup \mathrm{JH}(N), \\ \mathrm{JH}(M) &= \mathrm{JH}(M/N) \sqcup \mathrm{JH}(N), \\ \mathrm{JH}(M) &\supseteq \mathrm{JH}(N \subseteq M), \\ \mathrm{JH}(M) &= \bigsqcup_i \mathrm{JH}(M_i/M_{i-1}), \end{aligned}$$

where in the middle two $N \subseteq M$ is a submodule and in the last fact

$$0 = M_0 \subseteq \cdots \subseteq M_n = M$$

is any filtration (not necessarily a decomposition series).

These facts are pretty easy to exhibit[18].

Second, we must also cite another fact about the Jordan-Holder factors of Verma modules:

[18] The first fact follows by taking as filtration

$$0 = \underbrace{M_0 \oplus N_0 \subseteq \cdots \subseteq M_0 \oplus N_n}_{\mathrm{JH}\,N} = \underbrace{M_0 \oplus N \subseteq \cdots \subseteq M_m \oplus N}_{\mathrm{JH}\,M} = M \oplus N,$$

the second follows by taking

$$0 = \underbrace{N_0 \subseteq \cdots \subseteq N_n}_{\mathrm{JH}\,N} = N = \underbrace{M_0 \subseteq \cdots \subseteq M_m}_{\mathrm{JH}(M/N)} = M,$$

the third follows formally from the second, and the fourth follows by using

$$0 = \widetilde{N}_0 \subseteq \cdots \subseteq \widetilde{N}_n = M_i/M_{i-1}$$

to refine

$$0 = M_0 \subseteq \cdots \subseteq M_m = M$$

to

$$\cdots \subseteq M_{i-1} = \underbrace{N_0 \subseteq \cdots \subseteq N_n}_{\mathrm{JH}(M_i/M_{i-1})} = M_i \subseteq \cdots .$$

Theorem (Jordan-Holder of Verma, 8.12)**.** For $\lambda, \mu \in \mathfrak{h}^*$,

$$L_\mu \in \mathrm{JH}(M_\lambda)$$
$$\Updownarrow$$
$$\begin{aligned} &\exists\, \alpha_1, \cdots, \alpha_k \in \Phi_+ \\ &: \mu + \varrho = s_{\alpha_k} \cdots s_{\alpha_1}(\lambda + \varrho), \\ &\; s_{\alpha_{i-1}} \cdots s_{\alpha_1}(\lambda+\varrho) - s_{\alpha_i} \cdots s_{\alpha_1}(\lambda + \varrho) \in \mathbb{Z}_{0+}\alpha_i. \end{aligned}$$

For $\lambda \in \Lambda_+$,

$$L \in \mathrm{JH}(M_{w\circ\lambda}) \implies L = L_{u\circ\lambda},\; u \geq w,$$

with $L_{w\circ\lambda}$ appearing exactly once.

Lastly, we must also cite Weak BGG, to be proved in a later section. To set up Weak BGG we must first define the notion of "type":

Definition. For $M \in \mathcal{O}$ and $\Psi = \{\psi_i\}$ a finite set of weights (with multiplicity, i.e. possibly with repetition), we say M is of type Ψ,

$$\operatorname{Typ} M = \Psi,$$

if there exists a filtration of M

$$0 = M_0 \subseteq \cdots \subseteq M_n = M$$

such that the factors

$$M_i/M_{i-1} \cong M_{\psi_i}$$

precisely exhibit all of Ψ.

Now we may state Weak BGG:

Theorem (Weak BGG, 9.9)**.** For $\lambda \in \Lambda_+$ and $\Pi_\lambda \in \mathsf{irRep}_{\mathrm{fd}}\,\mathfrak{g}$, there is a resolution of $\mathfrak{g}$-modules

$$0 \longrightarrow B_{|\Phi_+|} \longrightarrow \cdots \longrightarrow B_1 \longrightarrow B_0 \longrightarrow \Pi_\lambda \longrightarrow 0$$

such that

$$\operatorname{Typ} B_k = \{w \circ \lambda : w \in W_k\}.$$

Let us remark that the BGG resolution (as one might expect, being stronger than Weak BGG) satisfies this, having terms $C_k = \bigoplus_{w \in W_k} M_{w\circ\lambda}$, so that we may take a filtration consisting e.g. of $M_{w_1\circ\lambda} \subseteq M_{w_1\circ\lambda} \oplus M_{w_2\circ\lambda} \subseteq M_{w_1\circ\lambda} \oplus M_{w_2\circ\lambda} \oplus M_{w_3\circ\lambda} \subseteq \cdots$ realizing $\operatorname{Typ} C_k = \{w\circ\lambda\}_{w\in W_k}$.

In the spirit of working backwards, we will state two lemmas, 10.6a and 10.6b in BGG, and show how they imply 10.6, and prove the two smaller lemmas later this section.

Lemma (10.6a)**.**

$$L \in \mathrm{JH}(\operatorname{Ker} \mathrm{d}_k) \implies L = L_{w\circ\lambda},\; \ell(w) > k.$$

Lemma (10.6b)**.** Let $w_0 \in W$, $M \in \mathcal{O}$ be such that

$$\ell(w) \geq \ell(w_0) \quad \forall\, L_{w\circ\lambda} \in \mathrm{JH}(M).$$

Then, for

$$\varphi \colon M_{w_0\circ\lambda} \xrightarrow{U\mathfrak{g}} M$$

a map of representations, we have

$$\varphi(v^{w_0\circ\lambda}) \neq 0 \implies \widetilde{\varphi(v^{w_0\circ\lambda})} \neq 0 \in M/\mathfrak{n}^- M.$$

Note that the statement in 10.6b can also be written

$$\varphi(v^{w_0\circ\lambda}) \neq 0 \implies \varphi(v^{w_0\circ\lambda}) \notin \mathfrak{n}^- M.$$

Let us see how this implies Lemma 10.6:

Lemma (10.6)**.**

$$\widetilde{\mathrm{d}}_{k+1}\colon C_{k+1}\big/\mathfrak{n}^- C_{k+1} \hookrightarrow \operatorname{Ker}\mathrm{d}_k\big/\mathfrak{n}^- \operatorname{Ker}\mathrm{d}_k \quad \text{inj}$$

is injective.

Proof of 10.6, assuming 10.6a/b. Since each term C_{k+1} is $C_{k+1} = \bigoplus_{w\in W_{k+1}} M_{W\circ\lambda}$ with $M_{w\circ\lambda}$ generated over $U\mathfrak{n}^-$ by $v^{w\circ\lambda}$ the highest weight vector of weight $w\circ\lambda$, we can write

$$C_{k+1} = U\mathfrak{n}^-\{v^{w\circ\lambda}\}_{w\in W_{k+1}}$$

where by not writing Span we are indicating that this is the free[19] module generated over $U\mathfrak{n}^-$. Then, modding out by the ideal $\mathfrak{n}^- \overset{\bullet}{\subset} U\mathfrak{n}^-$ gives

$$\implies C_{k+1}\big/\mathfrak{n}^- C_{k+1} \overset{\mathsf{Vec}}{\cong} \mathbb{C}\{\widetilde{v}^{w\circ\lambda}\}_{w\in W_{k+1}}$$

is isomorphic to a vector space with basis $\{\widetilde{v}^{w\circ\lambda}\}_{w\in W_{k+1}}$.

The claim is that, to show $\operatorname{Ker}\widetilde{\mathrm{d}}_{k+1} = 0$, it suffices to show $\widetilde{\mathrm{d}}_{k+1}(\widetilde{v}^{w\circ\lambda}) \neq 0$; that is,

$$\widetilde{\mathrm{d}}_{k+1}(\widetilde{v}^{w\circ\lambda}) \neq 0 \ \ \forall\, w \in W_{k+1} \implies \widetilde{\mathrm{d}}_{k+1}(\widetilde{v}) \neq 0 \ \ \forall\, \widetilde{v} \neq 0.$$

To see this, note[20] that $\widetilde{\mathrm{d}}_{k+1}$ commutes with $\mathfrak{h}$, and that the basis of the domain of $\widetilde{\mathrm{d}}_{k+1}$, $\{\widetilde{v}^{w\circ\lambda}\}$, consists of eigenvectors of different[21] weights; then, since $\widetilde{\mathrm{d}}_{k+1}$ commutes with $\mathfrak{h}$, the nonzero vectors $\{\widetilde{\mathrm{d}}_{k+1}(\widetilde{v}^{w\circ\lambda})\}$ are still eigenvectors in $\operatorname{Ker}\mathrm{d}_k/\mathfrak{n}^- \operatorname{Ker}\mathrm{d}_k$ of differing eigenvalues.

AFSOC some nonzero $\widetilde{v} = \sum_w c_w \widetilde{v}^{w\circ\lambda} \neq 0$ had $\widetilde{\mathrm{d}}_{k+1}(\widetilde{v}) = 0$; then

$$\begin{aligned} \widetilde{\mathrm{d}}_{k+1}(\widetilde{v}) &= 0 \\ \widetilde{\mathrm{d}}_{k+1}\left(\sum_w c_w \widetilde{v}^{w\circ\lambda}\right) &= \\ \sum_w c_w \widetilde{\mathrm{d}}_{k+1}(\widetilde{v}^{w\circ\lambda}) &= \end{aligned}$$

i.e. a nontrivial linear combination of $\widetilde{\mathrm{d}}_{k+1}(\widetilde{v}^{w\circ\lambda})$ vanishes, contradicting that eigenvectors of differing eigenvalues are linearly independent. Hence we see that $\widetilde{\mathrm{d}}_{k+1}(\widetilde{v}^{w\circ\lambda}) \neq 0 \implies \widetilde{\mathrm{d}}_{k+1}(\neq 0) \neq 0$, so it suffices to show $\widetilde{\mathrm{d}}_{k+1}$ does not vanish on weight vectors.

Now let us appeal to 10.6a and 10.6b. Take $M = \operatorname{Ker}\mathrm{d}_k$ with any $w_0 \in W_{k+1}$ in the setup of 10.6b; this satisfies the problem conditions by 10.6a. To be more explicit, by 10.6a, $L_{w\circ\lambda} \in \mathrm{JH}(\operatorname{Ker}\mathrm{d}_k) \implies \ell(w) \geq k+1 = \ell(w_0)$.

By our work at the beginning of this proof, it suffices to show that $\widetilde{\mathrm{d}}_{k+1}(\widetilde{v}^{w_0\circ\lambda}) \neq 0$ for any $w_0 \in W_{k+1}$. Since $C_{k+1} = \bigoplus_{w_0\in W_{k+1}} M_{w_0\circ\lambda}$, it suffices to show the differential restricted to each summand

$$\mathrm{d}_{k+1}\big|_{w_0}\colon M_{w_0\circ\lambda} \longrightarrow \operatorname{Ker}\mathrm{d}_k$$

---

[19] The different Verma modules don't talk to each other.

[20] Indeed, more generally for a map

$$\varphi\colon M \xrightarrow{U\mathfrak{n}^-} N$$

which commutes with $\mathfrak{h}$, we have

$$\widetilde{\varphi}\colon M/\mathfrak{n}^- M \longrightarrow N/\mathfrak{n}^- N$$

also commutes with $\mathfrak{h}$ since $\mathfrak{h}\widetilde{\varphi}(\widetilde{v}) = \mathfrak{h}((\varphi(v) + \mathfrak{n}^- N) = \mathfrak{h}\varphi(v) + \mathfrak{h}\mathfrak{n}^- N = \varphi(\mathfrak{h}v) + \mathfrak{h}\mathfrak{n}^- N \subseteq \varphi(\mathfrak{h}v) + \mathfrak{n}^- N = \widetilde{\varphi}(\mathfrak{h}\widetilde{v})$, where we recall from an earlier footnote (I think it's footnote 11) that $\mathfrak{h}\mathfrak{n}^- N \subseteq \mathfrak{n}^- N$.

In this case d_{k+1} is certainly a map of representations and so commutes with $\mathfrak{h}$, and therefore so does $\widetilde{\mathrm{d}}_{k+1}$.

[21] $w \circ \lambda = w' \circ \lambda \implies w = w'$.

has

$$\mathrm{d}_{k+1}(v^{w_0\circ\lambda}) \notin \mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k.$$

But this follows from 10.6b, since by construction $\mathrm{d}_{k+1}(v^{w_0\circ\lambda}) \neq 0$, as recall $\mathrm{d}_{k+1}|_{w_0}$ is defined by the direct sum of the canonical embeddings of Verma modules.

This concludes 10.6. ■

And now we must prove 10.6a and 10.6b. For these two we shall require the Jordan-Holder stuff, and for 10.6a we shall moreover need Weak

Proof of 10.6a. Since $C_i = \bigoplus_{W_i} M_{w\circ\lambda}$, the JH factors of C_i is

$$\operatorname{JH} C_i = \operatorname{JH} \bigoplus_{w\in W_i} M_{w\circ\lambda} = \bigsqcup_{w\in W_i} \operatorname{JH} M_{w\circ\lambda}.$$

By Weak BGG, we get a resolution

$$0 \longleftarrow \Pi_\lambda \longleftarrow B_0 \longleftarrow \cdots \longleftarrow B_{|\Phi_+|} \longleftarrow 0,$$

where since $\operatorname{Typ} B_k = \{w\circ\lambda\}_{w\in W_k}$, there exists a filtration of B_k whose quotients are $M_{w\circ\lambda}$, so by a previously noted fact about JH factors we have

$$\operatorname{JH} B_i = \bigsqcup_j \operatorname{JH}(B_i^j/B_i^{j-1}) = \bigsqcup_{w\in W_i} \operatorname{JH} M_{w\circ\lambda},$$

whereupon

$$\implies \operatorname{JH} C_i = \operatorname{JH} B_i.$$

The idea is sort of like DNA/RNA[22], where we only know $C_\bullet$ is exact in degrees $\leq k-1$, whereas $B_\bullet$ is exact everywhere. Let us begin:

$$\begin{aligned}
\Pi = B_0/\operatorname{Ker}\mathrm{d}_0^B = C_0/\operatorname{Ker}\mathrm{d}_0 &\implies \operatorname{JH}(B_0/\operatorname{Ker}\mathrm{d}_0^B) = \operatorname{JH}(C_0/\operatorname{Ker}\mathrm{d}_0)\\
(\operatorname{JH} B_0 = \operatorname{JH} C_0) &\implies \operatorname{JH}(\operatorname{Ker}\mathrm{d}_0^B) = \operatorname{JH}(\operatorname{Ker}\mathrm{d}_0)\\
(\text{exact at } 0) &\implies \operatorname{JH}(\operatorname{Img}\mathrm{d}_1^B) = \operatorname{JH}(\operatorname{Img}\mathrm{d}_1)\\
&\implies \operatorname{JH}(B_1/\operatorname{Ker}\mathrm{d}_1^B) = \operatorname{JH}(C_1/\operatorname{Ker}\mathrm{d}_1)\\
(\operatorname{JH} B_1 = \operatorname{JH} C_1) &\implies \operatorname{JH}(\operatorname{Ker}\mathrm{d}_1^B) = \operatorname{JH}(\operatorname{Ker}\mathrm{d}_1)\\
(\text{exact at } 1) &\implies \operatorname{JH}(\operatorname{Img}\mathrm{d}_2^B) = \operatorname{JH}(\operatorname{Img}\mathrm{d}_2)\\
&\implies \operatorname{JH}(B_2/\operatorname{Ker}\mathrm{d}_2^B) = \operatorname{JH}(C_2/\operatorname{Ker}\mathrm{d}_2)\\
(\operatorname{JH} B_2 = \operatorname{JH} C_2) &\implies \cdots\\
&\vdots\\
(\text{exact at } k-1) &\implies \operatorname{JH}(\operatorname{Img}\mathrm{d}_k^B) = \operatorname{JH}(\operatorname{Img}\mathrm{d}_k)\\
&\implies \operatorname{JH}(B_k/\operatorname{Ker}\mathrm{d}_k^B) = \operatorname{JH}(C_k/\operatorname{Ker}\mathrm{d}_k)\\
(\operatorname{JH} B_k = \operatorname{JH} C_k) &\implies \operatorname{JH}(\operatorname{Ker}\mathrm{d}_k^B) = \operatorname{JH}(\operatorname{Ker}\mathrm{d}_k).
\end{aligned}$$

[22] I don't remember any high school biology, but I think I remember the word "polymerase" or something...

At this point we cannot proceed any further since we do not know $C_\bullet$ is exact at k. However, $B_\bullet$ *is* exact at k, so we get

$$\begin{aligned}
\mathrm{JH}(\operatorname{Ker} \mathrm{d}_k) &= \mathrm{JH}(\operatorname{Ker} \mathrm{d}_k^B) \\
&= \mathrm{JH}(\operatorname{Img} \mathrm{d}_{k+1}^B) \\
&= \mathrm{JH}(B_{k+1}/ \operatorname{Ker} \mathrm{d}_{k+1}^B) \\
&\subseteq \mathrm{JH}\, B_{k+1} \\
&= \bigsqcup_{w \in W_{k+1}} \mathrm{JH}\, M_{w \circ \lambda},
\end{aligned}$$

i.e.

$$\implies \mathrm{JH}(\operatorname{Ker} \mathrm{d}_k) \subseteq \bigsqcup_{w \in W_{k+1}} \mathrm{JH}\, M_{w \circ \lambda}.$$

By theorem 8.12, we know that $\mathrm{JH}\, M_{w\circ\lambda}$ contains things of form $L_{u\circ\lambda}$, where $u \geq w$; hence

$$L \in \mathrm{JH}(\operatorname{Ker} \mathrm{d}_k) \implies L = L_{u\circ\lambda},\ u \geq w \text{ for some } w \in W_{k+1},$$

which in particular means

$$L \in \mathrm{JH}(\operatorname{Ker} \mathrm{d}_k) \implies L = L_{u\circ\lambda},\ \ell(u) \geq k+1,$$

which is the statement of 10.6a. ■

10.6b, while also involving Jordan-Holder, will not involve Weak BGG.

Proof of 10.6b. We will prove this by induction on the number of JH factors, i.e. induction on $|\mathrm{JH}(M)|$.

Since $M \in \mathcal{O}$, the set of weights of M is contained in some finite union of cones $\lambda_i - Q_+$, so we may pick a vector of maximal weight:

$$\text{pick } \ v \in M^\mu : \mathfrak{n}^+ v = 0.$$

Denote by N the submodule inside M generated by v:

$$N := \operatorname{Span}_{U\mathfrak{g}}(v) \subseteq M;$$

as this is a submodule generated by a single highest weight vector, this is a highest weight representation which must therefore be a quotient of a Verma module

$$\implies N \cong M_\mu/\text{smth}.$$

In particular

$$\mathrm{JH}\, N \subseteq \mathrm{JH}\, M_\mu.$$

Since $v \in M^\mu$ is a[23] highest weight vector, it (or rather, its image under an appropriate quotient) also generates an irreducible L_μ "inside" N; that is, we can quotient $N \cong M_\mu/\text{smth}$ by something else to get to the quotient by the maximal submodule not containing the highest weight vector, which is L_μ. Hence

$$\implies L_\mu \in \mathrm{JH}\, N.$$

We will break into two cases, the second of which is induction/reduction. Consider

$$\varphi(v^{w_0\circ\lambda});$$

either this is contained in N or it isn't.

Case 1: Suppose it is

$$\varphi(v^{w_0\circ\lambda}) \in N.$$

Since φ is a map of representations, it preserves weights so that we have $\varphi(v^{w_0\circ\lambda}) \in M^{w_0\circ\lambda}$; moreover, it commutes with $\mathfrak{n}^+$, so that $\mathfrak{n}^+\varphi(v^{w_0\circ\lambda}) = \varphi(\mathfrak{n}^+ v^{w_0\circ\lambda}) = 0$. Therefore $\varphi(v^{w_0\circ\lambda})$ generates a submodule in N which is a highest weight module and therefore isomorphic to a quotient of $M_{w_0\circ\lambda}$. This quotient of $M_{w_0\circ\lambda}$ can be further quotiented to obtain $L_{w_0\circ\lambda}$, so that

$$L_{w_0\circ\lambda} \in \mathrm{JH}\, N \subseteq \mathrm{JH}\, M_\mu.$$

[23] Not *the*, but *a*.

By Theorem 8.12 of BGG (JH factors of Verma), this implies

$$\implies \mu = w \circ \lambda \quad \text{for some } w \leq w_0.$$

But we saw earlier that $L_{w\circ\lambda} = L_\mu \in \mathrm{JH}\, N \subseteq \mathrm{JH}\, M$. By the conditions of 10.6b, this implies

$$\implies \ell(w) \geq \ell(w_0).$$

Combined with $w \leq w_0$, this forces

$$\implies w = w_0, \text{ and } \mu = w_0 \circ \lambda.$$

Then

$$\varphi(v^{w_0\circ\lambda}) \in M^{w_0\circ\lambda} = M^\mu$$

is a vector inside a weight space of maximal weight, which implies

$$\implies \varphi(v^{w_0\circ\lambda}) \notin \mathfrak{n}^- M.$$

This concludes Case 1.

Case 2: We will reduce to a smaller JH size. Suppose

$$\varphi(v^{w_0\circ\lambda}) \notin N.$$

Then

$$\widetilde{\varphi(v^{w_0\circ\lambda})} \neq 0 \in M/N$$

where the tilda refers to the equivalence class under quotient $\pi\colon M \to M/N$. Since $\mathrm{JH}(M/N) \subset \mathrm{JH}(M)$ (clearly N is a nontrivial submodule), we have

$$\implies |\mathrm{JH}(M/N)| < |\mathrm{JH}(M)|,$$

so by applying the inductive hypothesis to

$$\pi \circ \varphi\colon M_{w_0\circ\lambda} \xrightarrow{U\mathfrak{g}} M/N$$

we have $\pi \circ \varphi(v^{w_0\circ\lambda}) \neq 0 \in M/N$ implies

$$\implies \pi\varphi(v^{w_0\circ\lambda}) \notin \mathfrak{n}^-(M/N)$$

$$\implies \varphi(v^{w_0\circ\lambda}) \notin \mathfrak{n}^- M.$$

This completes the induction and therefore the lemma. ■

Having paid off our debts to 10.6a and 10.6b, we may continue to the third lemma in our path.

4.2.3. *Lemma 10.7.* The proof of this lemma will not directly involve Weak BGG, but will require Bott's Theorem (unnamed Corollary of Theorem 9.9 in BGG) on cohomology, which is a corollary of Weak BGG. First let us state this corollary:

Theorem (Bott)**.** For $\Pi \in \mathsf{irRep}_{\mathrm{fd}}\, \mathfrak{g}$ a finite-dimensional irrep,

$$\dim H^k(\mathfrak{n}^- : \Pi) = |W_k|.$$

Here recall that, for any $\mathfrak{g} \in \mathsf{LieAlg}$, Lie algebra cohomology was secretly the same as Ext, i.e.

$$H^k(\mathfrak{g} : M) = \mathrm{Ext}^k_{U\mathfrak{g}}(\mathbb{C}, M) \tag{$*$}$$

are canonically isomorphic. We will show how this follows from Weak BGG later.

We are now in a position to prove

Lemma (10.7)**.**

$$\dim_{\mathbb{C}} C_{k+1}\big/\mathfrak{n}^- C_{k+1} = \dim_{\mathbb{C}} \mathrm{Ker}\, \mathrm{d}_k \big/ \mathfrak{n}^- \,\mathrm{Ker}\, \mathrm{d}_k < \infty.$$

Proof of 10.7. We noted in the beginning of the proof of 10.6 that

$$C_{k+1}\big/\mathfrak{n}^- C_{k+1} \overset{\mathsf{Vec}}{\cong} \mathbb{C}\{\widetilde{v}^{w\circ\lambda}\}_{w\in W_{k+1}}$$

so that $C_{k+1}/\mathfrak{n}^- C_{k+1}$ is in particular finite-dimensional. Similarly, $\operatorname{Ker} \mathrm{d}_k \subseteq C_k = \bigoplus_{w\in W_k} M_{w\circ\lambda}$ is a submodule of a finitely-generated module over $U\mathfrak{n}^-$, which is a Noetherian ring; therefore[24][25], $\operatorname{Ker}\mathrm{d}_k$ is also finitely-generated over $U\mathfrak{n}^-$, which implies

$$\operatorname{Ker}\mathrm{d}_k/\mathfrak{n}^- \operatorname{Ker}\mathrm{d}_k \quad \text{is a finite-dimensional vector space;}$$

let us take weight vector generators $v_1, \cdots, v_n \in \operatorname{Ker}\mathrm{d}_k$ so that

$$\operatorname{Ker}\mathrm{d}_k\big/\mathfrak{n}^- \operatorname{Ker}\mathrm{d}_k \overset{\mathsf{Vec}}{\cong} \mathbb{C}\{\widetilde{v}_1, \cdots, \widetilde{v}_n\}.$$

In summary this gives

$$\implies \dim_{\mathbb{C}} C_{k+1}\big/\mathfrak{n}^- C_{k+1}, \dim_{\mathbb{C}} \operatorname{Ker}\mathrm{d}_k\big/\mathfrak{n}^- \operatorname{Ker}\mathrm{d}_k < \infty.$$

Now we will show these dimensions are moreover equal by passing through $\dim \operatorname{Tor}_{k+1}^{U\mathfrak{n}^-}(\mathbb{C}, \Pi_\lambda)$ and using Bott's Theorem. Define

$$D_{k+1} \coloneqq U\mathfrak{n}^-\{g_1, \cdots, g_n\}$$

the free module generated over $U\mathfrak{n}^-$ with a map

$$\begin{aligned} \delta_{k+1} \colon D_{k+1} &\longrightarrow \operatorname{Ker}\mathrm{d}_k \\ g_i &\longmapsto v_i. \end{aligned}$$

Then, since

$$D_{k+1}\big/\mathfrak{n}^- D_{k+1} \overset{\mathsf{Vec}}{\cong} \mathbb{C}\{\widetilde{g}_1, \cdots, \widetilde{g}_n\}$$

has the same dimension as $\operatorname{Ker}\mathrm{d}_k/\mathfrak{n}^- \operatorname{Ker}\mathrm{d}_k$, we have

$$\widetilde{\delta}_{k+1} \colon D_{k+1}\big/\mathfrak{n}^- D_{k+1} \overset{\sim}{\hookrightarrow\!\!\!\!\rightarrow} \operatorname{Ker}\mathrm{d}_k\big/\mathfrak{n}^- \operatorname{Ker}\mathrm{d}_k$$

is in particular surjective. By 10.5[26] this implies

$$\implies \delta_{k+1} \colon D_{k+1} \twoheadrightarrow \operatorname{Ker}\mathrm{d}_k \quad \text{surj},$$

so that the sequence

$$D_{k+1} \xrightarrow{\delta_{k+1}} C_k \longrightarrow \cdots \longrightarrow C_0 \longrightarrow \Pi_\lambda \longrightarrow 0$$

is exact[27].

The idea is to extend this exact sequence even further. Now let us take a free resolution (in the category of $\mathfrak{n}^-$-modules, i.e. the terms are free over $U\mathfrak{n}^-$) of $\operatorname{Ker}\delta_{k+1}$:

$$\cdots \longrightarrow D_{k+3} \longrightarrow D_{k+2} \xrightarrow{\delta_{k+2}} \operatorname{Ker}\delta_{k+1} \longrightarrow 0$$

so that we may extend

$$\cdots \longrightarrow D_{k+2} \xrightarrow{\delta_{k+2}} D_{k+1} \xrightarrow{\delta_{k+1}} C_k \longrightarrow \cdots \longrightarrow C_0 \longrightarrow \Pi_\lambda \longrightarrow 0 \qquad \text{exact.}$$

Note well that this resolution has terms which are $U\mathfrak{n}^-$-free modules: the $D_\bullet$ are $U\mathfrak{n}^-$-free by construction, and the $C_\bullet$ are $U\mathfrak{n}^-$-free since they are direct sums of Vermas, which are $U\mathfrak{n}^-$-free.

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[24] Being a submodule of a Noetherian module; recall Noetherian module is equivalent to all submodules being finitely-generated.

[25] Alternatively, $C_k \in \mathcal{O}$ since it is a direct sum of modules in $\mathcal{O}$, and $\operatorname{Ker}\mathrm{d}_k$, being a submodule of an object in $\mathcal{O}$, is also in $\mathcal{O}$, which implies it is $U\mathfrak{g}$-finitely-generated; moreover it is locally $U\mathfrak{n}^+$-finite, so that when we mod out by the ideal action by $\mathfrak{n}^-$, we obtain a finite-dimensional space.

[26] This clearly satisfies the conditions of 10.5, as D_{k+1} is a free $U\mathfrak{n}^-$-module with the image of each generator $\delta_{k+1}(g_i) = v_i$ a weight vector.

[27] Recall it is exact from -1 to $k-1$ by induction.

Recall that in general $M \otimes_R R/I \cong M/IM$; in particular, since $\mathbb{C} \cong U\mathfrak{n}^-/(\mathfrak{n}^- U\mathfrak{n}^-)$ as $\mathfrak{n}^-$-modules, we have

$$\mathbb{C} \otimes_{U\mathfrak{n}^-} M \cong M\big/\mathfrak{n}^- M.$$

Let us compute

$$\operatorname{Tor}_{k+1}^{U\mathfrak{n}^-}(\mathbb{C}, \Pi_\lambda)$$

by resolving the second term like we did above. Then by definition Tor is the homology of the complex

$$\cdots \longrightarrow \mathbb{C} \otimes_{\mathfrak{n}^-} D_{k+2} \longrightarrow \mathbb{C} \otimes_{\mathfrak{n}^-} D_{k+1} \longrightarrow \mathbb{C} \otimes_{\mathfrak{n}^-} C_k \longrightarrow \cdots \longrightarrow \mathbb{C} \otimes_{\mathfrak{n}^-} C_0 \longrightarrow 0,$$

which is

$$\cdots \longrightarrow D_{k+2}\big/\mathfrak{n}^- D_{k+2} \xrightarrow{\widetilde{\delta}_{k+2}} D_{k+1}\big/\mathfrak{n}^- D_{k+1} \xrightarrow{\widetilde{\delta}_{k+1}} C_k\big/\mathfrak{n}^- C_k \longrightarrow \cdots \longrightarrow C_0\big/\mathfrak{n}^- C_0 \longrightarrow 0.$$

Hence

$$\begin{aligned} \Longrightarrow \quad \operatorname{Tor}_{k+1}^{U\mathfrak{n}^-}(\mathbb{C}, \Pi_\lambda) &= H_{k+1}(\to \mathbb{C} \otimes_{\mathfrak{n}^-} D_{k+1} \to) \\ &= H_{k+1}(\to D_{k+1}/\mathfrak{n}^- D_{k+1} \to) \\ &= \operatorname{Ker} \widetilde{\delta}_{k+1}\big/\operatorname{Img} \widetilde{\delta}_{k+2}. \end{aligned}$$

We claim that

$$\widetilde{\delta}_{k+1} = \widetilde{\delta}_{k+2} = 0.$$

To see the second one, apply $\mathbb{C} \otimes \square$ to the exact sequence

$$\begin{gathered} D_{k+2} \xrightarrow{\delta_{k+2}} D_{k+1} \xrightarrow{\delta_{k+1}} \operatorname{Ker} \mathrm{d}_k \longrightarrow 0 \qquad \text{exact} \\ \Big\Downarrow \mathbb{C} \otimes_{\mathfrak{n}^-} \square \\ D_{k+2}\big/\mathfrak{n}^- D_{k+2} \xrightarrow{\widetilde{\delta}_{k+2}} D_{k+1}\big/\mathfrak{n}^- D_{k+1} \xrightarrow{\widetilde{\delta}_{k+1}} \operatorname{Ker} \mathrm{d}_k\big/\mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k \longrightarrow 0 \qquad \text{exact,} \end{gathered}$$

where we know the result is still exact since in general tensor products are right-exact. But we saw earlier that $\widetilde{\delta}_{k+1}$ is an isomorphism, which forces

$$\Longrightarrow \quad \widetilde{\delta}_{k+2} = 0.$$

To see the first one, similarly apply $\mathbb{C} \otimes_{\mathfrak{n}^-} \square$ to

$$\begin{gathered} D_{k+1} \xrightarrow{\delta_{k+1}} C_k \xrightarrow{\mathrm{d}_k} \operatorname{Ker} \mathrm{d}_{k-1} \longrightarrow 0 \qquad \text{exact} \\ \Big\Downarrow \mathbb{C} \otimes_{\mathfrak{n}^-} \square \\ D_{k+1}\big/\mathfrak{n}^- D_{k+1} \xrightarrow{\widetilde{\delta}_{k+1}} C_k\big/\mathfrak{n}^- C_k \xrightarrow{\widetilde{\mathrm{d}}_k} \operatorname{Ker} \mathrm{d}_{k-1}\big/\mathfrak{n}^- \operatorname{Ker} \mathrm{d}_{k-1} \longrightarrow 0 \qquad \text{exact.} \end{gathered}$$

Strong induction (that is, to get to the point where we know exactness at $-1, \cdots, k-1$, we must have along the way shown Lemma 10.6[28], that $\widetilde{\mathrm{d}}_{i+1}$ is injective, for all $i < k$) on Lemma 10.6 tells us $\widetilde{\mathrm{d}}_k$ is injective, which also forces

$$\Longrightarrow \quad \widetilde{\delta}_{k+1} = 0.$$

That $\widetilde{\delta}_{k+1} = \widetilde{\delta}_{k+2} = 0$ implies

$$\operatorname{Tor}_{k+1}^{U\mathfrak{n}^-}(\mathbb{C}, \Pi_\lambda) = \operatorname{Ker} 0/\operatorname{Img} 0 = D_{k+1}\big/\mathfrak{n}^- D_{k+1},$$

which, since (recall from the beginning of this proof) $D_{k+1}/\mathfrak{n}^- D_{k+1} \cong \operatorname{Ker} \mathrm{d}_k/\mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k$, implies

$$\Longrightarrow \quad \operatorname{Ker} \mathrm{d}_k\big/\mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k \cong \operatorname{Tor}_{k+1}^{U\mathfrak{n}^-}(\mathbb{C}, \Pi_\lambda).$$

Now we need to cite[29] a fact from homological algebra:

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[28] And 10.5 and 10.7 as well, but 10.6 is the relevant one here.

[29] I think the most satisfying way to see this is to use that Tor and Ext, being derived functors, are universal delta functors; it then suffices to check these natural isomorphisms in degree 0, where it is very easy.

Fact. For any Lie algebra $\mathfrak{g}$ and M, N left $U\mathfrak{g}$-modules, let $M^\dagger$ be the right $U\mathfrak{g}$-module defined by $v \cdot \xi \coloneqq -\xi \cdot v$ (for $\xi \in \mathfrak{g}$, extended to $U\mathfrak{g}$ appropriately) and $M^* = \operatorname{Hom}(M, \mathbb{C})$ be the right $U\mathfrak{g}$-module which is the dual representation. Then

$$\operatorname{Ext}^n_{U\mathfrak{g}}(M, N) \cong \operatorname{Tor}_n^{U\mathfrak{g}}(N^*, M)^* \cong \operatorname{Tor}_n^{U\mathfrak{g}}(M^\dagger, N^{\dagger,*})^*.$$

In particular, let us apply this to $M = \mathbb{C}$ and $N = \Pi$. Since $\mathbb{C}$ is the trivial representation, we have $\mathbb{C} = \mathbb{C}^* = \mathbb{C}^\dagger$. Meanwhile, since Π^*_λ is the representation on which ξ acts by $-\rho_\Pi(\xi)$, we have $\Pi^{*,\dagger} \cong \Pi^*$ as vector spaces is also irreducible (that Π^*_λ is irreducible requires complete reducibility, so that $\operatorname{End}(\Pi^*_\lambda) = \mathbb{C} \implies \Pi^*_\lambda \in \mathsf{irRep}\,\mathfrak{g}$). Then we have

$$\begin{aligned}\operatorname{Tor}^{U\mathfrak{n}^-}_{k+1}(\mathbb{C}, \Pi_\lambda) &\cong \operatorname{Ext}^{k+1}_{U\mathfrak{n}^-}(\mathbb{C}, \Pi^{*,\dagger}_\lambda)^* \\ &\cong H^{k+1}(\mathfrak{n}^- : \Pi^{*,\dagger}_\lambda)^*,\end{aligned}$$

whereupon by Bott's Theorem

$$\implies \dim \operatorname{Tor}^{U\mathfrak{n}^-}_{k+1}(\mathbb{C}, \Pi_\lambda) = \dim H^{k+1}(\mathfrak{n}^- : \Pi^{*,\dagger}_\lambda) = |W_{k+1}| = \dim C_{k+1}\big/\mathfrak{n}^- C_{k+1},$$

which combined with the previous $\operatorname{Ker} \mathrm{d}_k / \mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k \cong \operatorname{Tor}^{U\mathfrak{n}^-}_{k+1}(\mathbb{C}, \Pi_\lambda)$ implies

$$\implies \dim \operatorname{Ker} \mathrm{d}_k\big/\mathfrak{n}^- \operatorname{Ker} \mathrm{d}_k = \dim \operatorname{Tor}^{U\mathfrak{n}^-}_{k+1}(\mathbb{C}, \Pi_\lambda) = |W_{k+1}| = \dim C_{k+1}\big/\mathfrak{n}^- C_{k+1},$$

precisely as claimed by 10.7. This concludes. ■

At last, having proved 10.5, 10.6, and 10.7, we have shown how Weak BGG implies the full BGG. It lastly remains to show Weak BGG.

5. Proving Weak BGG

In this section we shall prove Weak BGG and also derive its corollary, Bott's Theorem (stated and used in the last section).

Recall that the Weak BGG theorem claimed the existence of a resolution of form

$$0 \longrightarrow B_{|\Phi_+|} \longrightarrow \cdots \longrightarrow B_1 \longrightarrow B_0 \longrightarrow \Pi_\lambda \longrightarrow 0$$

such that

$$\operatorname{Typ} B_k = \{w \circ \lambda : w \in W_k\}.$$

We will first exhibit this resolution for $\Pi_0 = \mathbb{C}$ the trivial (irreducible) representation, then use this exhibition to obtain this resolution for all other irreps.

5.1. General Lemmas.

First some generalities for arbitrary $\mathfrak{g}$. Recall the construction of a resolution of $\mathbb{C}$ by

$$\mathbb{C}_n \coloneqq U\mathfrak{g} \otimes \mathfrak{g}^{\wedge n}.$$

There is a variant of this which says

Theorem (9.1)**.** For any Lie algebra $\mathfrak{g}$ and $\mathfrak{a} \subseteq \mathfrak{g}$ a subalgebra, there is a resolution of the trivial representation

$$\cdots \longrightarrow \mathbb{C}_2 \xrightarrow{\mathrm{d}_2} \mathbb{C}_1 \xrightarrow{\mathrm{d}_1} \mathbb{C}_0 \longrightarrow \mathbb{C} \longrightarrow 0$$

whose terms are

$$\mathbb{C}_n \coloneqq U\mathfrak{g} \otimes_{U\mathfrak{a}} (\mathfrak{g}/\mathfrak{a})^{\wedge n}$$

and whose differentials are

$$\begin{aligned}\mathrm{d}_n \colon U\mathfrak{g} \otimes_{U\mathfrak{a}} (\mathfrak{g}/\mathfrak{a})^{\wedge n} &\longrightarrow U\mathfrak{g} \otimes_{U\mathfrak{a}} (\mathfrak{g}/\mathfrak{a})^{\wedge n-1} \\ \alpha \otimes \bigwedge \widetilde{\xi} &\longmapsto \sum_{i=1}^{n} (-1)^{i+1} \alpha \xi_i \otimes \bigwedge_{\setminus i} \widetilde{\xi} + \sum_{1 \le i < j \le n} (-1)^{i+j} \alpha \otimes [\xi_i, \xi_j] \wedge \bigwedge_{\setminus i,j} \widetilde{\xi}.\end{aligned}$$

Part of the theorem is that this is well-defined.

The well-defined-ness of the differential is an immediate check, and the proof of exactness is also standard, so we skip it here.

The idea is that we will use this resolution of $\mathbb{C}$ to build the weak BGG resolution of $\mathbb{C}$. First let us note

Fact (9.3)**.**

$$U\mathfrak{g} \otimes_{U\mathfrak{b}} \square \colon \mathsf{Rep}\,\mathfrak{b} \overset{\text{exact}}{\longrightarrow} \mathsf{Rep}\,\mathfrak{g}$$

is an exact functor. If $V \in \mathsf{Rep}\,\mathfrak{b}$ moreover has $\dim V = 1$, $\mathfrak{h}v = \lambda v$, and $\mathfrak{n}^+ v = 0$, then

$$U\mathfrak{g} \otimes_{U\mathfrak{b}} V = M_\lambda.$$

This is pretty obvious since $U\mathfrak{g}$ is $U\mathfrak{b}$-free (recall PBW), and the second bit is definitional (we include it for completeness since BGG did). Next let us establish some facts about types in moving to construct weak BGG for $\mathbb{C}$.

Lemma (9.5)**.** For $N \in \mathsf{Rep}\,\mathfrak{b}$ such that $\dim N < \infty$ and N is $\mathfrak{h}$-semisimple, we have

$$\mathrm{Typ}(U\mathfrak{g} \otimes_{U\mathfrak{b}} N) = \mathrm{Wt}\, N.$$

Proof. Since $\mathfrak{b}$ is solvable, by Lie's theorem there exists a filtration of $\mathfrak{b}$-modules

$$\exists\, 0 = N_0 \subseteq \cdots \subseteq N_n = N$$

such that

$$\dim N_i/N_{i-1} = 1.$$

To exhibit $\mathrm{Typ}(U\mathfrak{g} \otimes_{U\mathfrak{b}} N) = \mathrm{Wt}\, N$, let us give a filtration

$$0 = U\mathfrak{g} \otimes_{U\mathfrak{b}} N_0 \subseteq \cdots \subseteq U\mathfrak{g} \otimes_{U\mathfrak{b}} N_n = U\mathfrak{g} \otimes_{U\mathfrak{b}} N,$$

where the quotients are

$$\implies U\mathfrak{g}\otimes_{U\mathfrak{b}} N_i \big/ U\mathfrak{g}\otimes_{U\mathfrak{b}} N_{i-1} \cong U\mathfrak{g} \otimes_{U\mathfrak{b}} N_i\big/N_{i-1}$$

since $U\mathfrak{g}\otimes_{U\mathfrak{b}}$ is exact and therefore preserves quotients:

$$0 \longrightarrow N_{i-1} \longrightarrow N_i \longrightarrow N_i\big/N_{i-1} \longrightarrow 0$$

$$\Big\Downarrow U\mathfrak{g} \otimes_{U\mathfrak{b}} \square$$

$$0 \longrightarrow U\mathfrak{g} \otimes_{U\mathfrak{b}} N_{i-1} \longrightarrow U\mathfrak{g} \otimes_{U\mathfrak{b}} N_i \longrightarrow U\mathfrak{g} \otimes_{U\mathfrak{b}} N_i\big/N_{i-1} \longrightarrow 0$$

which forces the last term of the second sequence to be $(U\mathfrak{g} \otimes_{U\mathfrak{b}} N_i)/(U\mathfrak{g} \otimes_{U\mathfrak{b}} N_{i-1})$.

Recall the way Lie was proved was by induction[30] on $\dim V$ and using

$$\exists\, v : \mathfrak{b}v = \chi(\mathfrak{b})v$$

(where it is a formal consequence that $\chi|_{[\mathfrak{b},\mathfrak{b}]=\mathfrak{n}^+} = 0$), so that N_i/N_{i-1} is a 1-dimensional space on which $\mathfrak{h}$ acts by a character and on which $\mathfrak{n}^+$ acts by zero. Therefore

$$\implies U\mathfrak{g} \otimes_{U\mathfrak{b}} N_i\big/N_{i-1} \cong M_\mu \qquad \mu \in \mathrm{Wt}\, N,$$

where $\mu = \chi \in \mathrm{Wt}\, N$ is a weight of N since there was a vector $\widehat{v} \in N_i \subseteq N$ on whom $\mathfrak{h}$ (through which the character must pass since it vanishes on $[\mathfrak{b},\mathfrak{b}] = \mathfrak{n}^+$) acts by a character μ. It is clear that, as we run across i, the weights will precisely run across and exhaust all of $\mathrm{Wt}\, N$, exhibiting the desired claim. ■

Next, we will establish a fact about the type of M^ϑ (recall the bit about central characters earlier in Section 2).

[30] I.e., we exhibited a desired filtration by using this fact.

Lemma (9.7)**.** For ϑ a central character of M,

$$\operatorname{Typ} M^{\vartheta} = \{\psi \in \operatorname{Typ} M : \vartheta_\psi = \vartheta\} := \operatorname{Typ}^{\vartheta} M,$$

where the second equality sign is a definition of the third object and by ϑ_ψ we mean the central character of the Verma M_ψ.

Proof. Let us take a filtration[31] exhibiting $\operatorname{Typ} M$:

$$0 = M_0 \subseteq \cdots \subseteq M_n = M$$

such that

$$M_i/M_{i-1} \cong M_\psi, \qquad \psi \in \operatorname{Typ} M.$$

Recall that the functor $\square^{\vartheta}$ is exact. Let us then apply it to

$$0 \longrightarrow M_{i-1} \hookrightarrow M_i \twoheadrightarrow M_i/M_{i-1} \longrightarrow 0$$

$$\Big\Downarrow \square^{\vartheta}$$

$$0 \longrightarrow {M_{i-1}}^{\vartheta} \hookrightarrow {M_i}^{\vartheta} \twoheadrightarrow \left({M_i}/{M_{i-1}}\right)^{\vartheta} \longrightarrow 0,$$

whereupon we obtain a filtration

$$0 = {M_0}^{\vartheta} \subseteq \cdots \subseteq {M_n}^{\vartheta} = M^{\vartheta}$$

with quotients

$$\implies {M_i}^{\vartheta}/{M_{i-1}}^{\vartheta} \cong \left({M_i}/{M_{i-1}}\right)^{\vartheta} \cong {M_\psi}^{\vartheta}.$$

However, recall from Section 2 that

$$\Theta(M_\psi) = \{\vartheta_\psi\},$$

so that ${M_\psi}^{\vartheta}$ is either all of M_ψ if $\vartheta = \vartheta_\psi$ or 0 if $\vartheta \neq \vartheta_\psi$:

$$\implies {M_i}^{\vartheta}/{M_{i-1}}^{\vartheta} \cong {M_\psi}^{\vartheta} = \begin{cases} M_\psi & \vartheta = \vartheta_\psi \\ 0 & \vartheta \neq \vartheta_\psi \end{cases}.$$

Note that in the case of 0, we have ${M_i}^{\vartheta} = {M_{i-1}}^{\vartheta}$, so there is sort of a redundant term.

Then we can appropriately delete all such redundant terms in the filtration

$$0 = {M_0}^{\vartheta} \subseteq \cdots \subseteq {M_n}^{\vartheta} = M^{\vartheta}$$

to obtain a filtration exhibiting the claimed $\operatorname{Typ} M^{\vartheta} = \operatorname{Typ}^{\vartheta} M$, as desired. ■

Lastly, let us compute the type of the tensor product of a finite-dimensional representation and a Verma module. Note in particular by taking $V = \mathbb{C}$ the trivial representation we obtain the type of a Verma module (not that this is needed for us).

Lemma (9.10)**.** For $V \in \mathsf{Rep}_{\mathrm{fd}}\, \mathfrak{g}$, $\psi \in \mathfrak{h}^*$, we have

$$\operatorname{Typ}(M_\psi \otimes_{\mathbb{C}} V) = \{\lambda + \psi\}_{\lambda \in \operatorname{Wt} V} = \psi + \operatorname{Wt} V.$$

Proof. Let us take a weight basis of V

$$V = \mathbb{C}\{v_1, \cdots, v_n\}$$

ordered so that their corresponding weights are ordered

$$\lambda_1 \geq \cdots \geq \lambda_n.$$

Let v^{ψ} be the highest weight vector of M_ψ.

Consider then the set of vectors

$$\{v^{\psi} \otimes v_i\}_i.$$

[31] Unfortunately our index for the notation here will collide with the place where the highest weight of a Verma module would be; but hopefully it is clear upon context whether we refer to a filtration index or a highest weight. I guess this collision was more or less unavoidable, since highest weights are labelled below and central characters are placed on top.

The claim is that these guys are weight vectors. Indeed, compute

$$\begin{aligned}\mathfrak{h}(v^\psi \otimes v_i) &= \mathfrak{h}v^\psi \otimes v_i + v^\psi \otimes \mathfrak{h}v_i \\ &= \psi(\mathfrak{h})v^\psi \otimes v_i + \lambda_i(\mathfrak{h})v^\psi \otimes v_i \\ &= (\psi + \lambda_i)(\mathfrak{h})(v^\psi \otimes v_i),\end{aligned}$$

so that in fact

$$\implies v^\psi \otimes v_i \in (M_\psi \otimes V)^{\psi+\lambda_i}.$$

Moreover,

$$\begin{aligned}\mathfrak{n}^+(v^\psi \otimes v_i) &= \cancel{\mathfrak{n}^+ v^\psi} \otimes v_i + v^\psi \otimes \mathfrak{n}^+ v_i \\ &= v^\psi \otimes \mathfrak{n}^+ v_i,\end{aligned}$$

where $\mathfrak{n}^+ v_i \in \mathrm{Span}\{v_1, \cdots, v_{i-1}\}$ since $\mathfrak{n}^+$ raises the eigenvalue.

To exhibited the claimed type, consider now the filtration

$$0 = N_0 \subseteq \cdots \subseteq N_n = M_\psi \otimes V$$

where

$$N_k := \mathrm{Span}_{U\mathfrak{g}}\{v^\psi \otimes v_1, \cdots, v^\psi \otimes v_k\},$$

which is clearly a filtration. It is not yet obvious that

$$N_n = M_\psi \otimes V;$$

we will show this. We will also claim that

$$N_k/N_{k-1} = M_{\psi+\lambda_k}.$$

It is obvious that

$$N_k/N_{k-1} = \mathrm{Span}_{U\mathfrak{g}}\{v^\psi \otimes v_k\}$$

is generated by a single vector (here we really mean the equivalence class). Moreover, by the above computations of the action of $\mathfrak{h}$ and $\mathfrak{n}^+$, we have

$$v^\psi \otimes v_k \in \left(N_k \big/ N_{k-1}\right)^{\psi+\lambda_k}$$

which is sent under $\mathfrak{n}^+$ to[32]

$$\mathfrak{n}^+ \colon v^\psi \otimes v_k \longmapsto 0 \in N_k \big/ N_{k-1}.$$

Therefore

$$\implies N_k = \mathrm{Span}_{U\mathfrak{n}^-}\{v^\psi \otimes v_1, \cdots, v^\psi \otimes v_k\}.$$

We claim that this is moreover free over $U\mathfrak{n}^-$. To see this, let $\sum\prod\xi \in U\mathfrak{n}^-$ denote any arbitrary element in the universal enveloping algebra, and suppose there was a linear dependence relation:

$$\sum_{i=1}^k (\sum\prod\xi)(v^\psi \otimes v_i) = 0.$$

Then, exchanging the sums out and distributing according to Leibniz, we obtain

$$\begin{aligned}0 = &\sum\sum_{i=1}^k((\prod\xi)v^\psi) \otimes v_i \\ &+ \sum\sum_{i=1}^k(\prod_{\text{smaller}}\xi)v^\psi \otimes \underbrace{(\prod_{\text{smaller}}\xi)v_i}_{\text{absorb into } \mathrm{Span}\{v_{i+1},\cdots,v_n\}} + \cdots + \sum\sum_{i=1}^k v^\psi \otimes \underbrace{(\prod\xi)v_i}_{\text{absorb into } \mathrm{Span}\{v_{i+1},\cdots,v_n\}}.\end{aligned}$$

But now, after converting each instance of an element of $U\mathfrak{n}^-$ acting on v_i to a linear combination of v_i's, the second line in the above is a sum of pure tensors where the second factor is some v_i and the first factor is some element of $U\mathfrak{n}^-$ acting on v^ψ, where the element is of filtration degree strictly less than that of

[32]This is since $\mathfrak{n}^+(v^\psi \otimes v_k) \in v^\psi \otimes \mathrm{Span}\{v_1, \cdots, v_{k-1}\} \subseteq N_{k-1}$.

the element in the first line. Hence the first line cannot be cancelled out (recall that $v^\psi \in M_\psi$ belongs to a module which is $U\mathfrak{n}^-$-free), and we have contradiction. Hence N_k is moreover $U\mathfrak{n}^-$-free,

$$\implies N_k = U\mathfrak{n}^-\{v^\psi \otimes v_1, \cdots, v^\psi \otimes v_k\}.$$

But then this implies N_k/N_{k-1} is free also,

$$\implies N_k\big/N_{k-1} = U\mathfrak{n}^-\{v^\psi \otimes v_k\} = M_{\psi+\lambda_k},$$

where we recall the weight of this vector is $\mathrm{Wt}(v^\psi \otimes v_k) = \psi + \lambda_k$.

Moreover, at $k = n$, this gives $N_n = U\mathfrak{n}^-\{v^\psi \otimes v_1, \cdots, v^\psi \otimes v_n\}$, i.e. (since $U\mathfrak{n}^-$ acting on v^ψ generates all of M_ψ)

$$\implies N_n = M_\psi \otimes V.$$

This completes showing that the filtration we constructed exhibits the claimed type, so we are done. ■

5.2. **Base Case of Weak BGG.** In particular, let us apply these facts (9.10 is not necessary) to the resolution of $\mathbb{C}$ given in 9.1.

Lemma. Weak BGG holds for $\Pi_0 = \mathbb{C}$, i.e. there is a resolution of $\mathbb{C}$ with terms in $\mathcal{O}$

$$B_k(\mathbb{C}) = \left(U\mathfrak{g} \otimes_{U\mathfrak{b}} (\mathfrak{g}/\mathfrak{b})^{\wedge k}\right)^{\vartheta_0}$$

of type

$$\operatorname{Typ} B_k(\mathbb{C}) = \{w \circ 0\}_{w \in W_k}.$$

Proof. Recall the resolution of $\mathbb{C}$ given at the beginning of this section. In particular, we will take $\mathfrak{g}$ to be semisimple and $\mathfrak{a} = \mathfrak{b}$ to be the Borel subalgebra. This is a resolution of form

$$\cdots \longrightarrow \mathbb{C}_2 \xrightarrow{d_2} \mathbb{C}_1 \xrightarrow{d_1} \mathbb{C}_0 \longrightarrow \mathbb{C} \longrightarrow 0$$

where

$$\mathbb{C}_n = U\mathfrak{g} \otimes_{U\mathfrak{b}} (\mathfrak{g}/\mathfrak{b})^{\wedge n}.$$

This is by construction finitely-generated over $U\mathfrak{g}$ (since $(\mathfrak{g}/\mathfrak{b})^{\wedge n}$ is finite-dimensional), $\mathfrak{h}$-semisimple by the Cartan root decomposition ($\mathfrak{g}/\mathfrak{b} = \mathfrak{n}^-$ has $\mathfrak{h}$ acts by roots), and is locally $U\mathfrak{n}^+$-finite since we can pass $U\mathfrak{n}^+$ across the tensor $\otimes_{U\mathfrak{b}}$ to act on $(\mathfrak{g}/\mathfrak{b})^{\wedge n}$, which is finite-dimensional. Hence $\mathbb{C}_n \in \mathcal{O}$, and applying the functor $\square^{\vartheta_0}$ keeps it in $\mathcal{O}$. This checks that the terms are indeed in $\mathcal{O}$.

The claim is that Weak BGG is realized by the resolution

$$\cdots \longrightarrow {\mathbb{C}_2}^{\vartheta_0} \longrightarrow {\mathbb{C}_1}^{\vartheta_0} \longrightarrow {\mathbb{C}_0}^{\vartheta_0} \longrightarrow \mathbb{C} \longrightarrow 0.$$

Let us compute the type of these terms:

$$\begin{aligned}\operatorname{Typ} \mathbb{C}_k &= \operatorname{Typ}\left(U\mathfrak{g} \otimes_{U\mathfrak{b}} (\mathfrak{g}/\mathfrak{b})^{\wedge k}\right)\\ &= \mathrm{Wt}(\mathfrak{g}/\mathfrak{b})^{\wedge k}\\ &= \left\{\sum_{\alpha \in S \subseteq \Phi_-} \alpha\right\}_{\substack{S \subseteq \Phi_-,\\ |S|=k}},\end{aligned}$$

where in the last line we have recalled that $\mathfrak{g}/\mathfrak{b} = \mathfrak{n}^-$, so that $\mathrm{Wt}(\mathfrak{g}/\mathfrak{b}) = \mathrm{Wt}(\mathfrak{n}^-) = \Phi_-$; then it is a purely linear algebraic fact to see that, with the action of $\mathfrak{h}$ defined by Leibniz[33], the eigenvalues of the k-th wedge space are the set of sums of k distinct eigenvalues of the 1-st wedge space. In summary

$$\implies \operatorname{Typ} \mathbb{C}_k = \left\{-\sum_{\alpha \in S} \alpha\right\}_{\substack{S \subseteq \Phi_+,\\ |S|=k}}.$$

Since $\square^\vartheta$ is exact, applying this functor gives a resolution

$$\cdots \longrightarrow {\mathbb{C}_2}^{\vartheta} \longrightarrow {\mathbb{C}_1}^{\vartheta} \longrightarrow {\mathbb{C}_0}^{\vartheta} \longrightarrow \mathbb{C} \longrightarrow 0$$

[33] I.e. $T(v \wedge w) = Tv \wedge w + v \wedge Tw$.

where $\mathbb{C} = \mathbb{C}^\vartheta$ since it is a one-dimensional space. Then, by Lemma 9.7,

$$\implies \operatorname{Typ} {\mathbb{C}_k}^\vartheta = \{\psi \in \operatorname{Typ} \mathbb{C}_k : \vartheta_\psi = \vartheta\} = \{\psi \in \operatorname{Wt}(\mathfrak{g}/\mathfrak{b})^{\wedge k} : \vartheta_\psi = \vartheta\}.$$

In particular, let us take $\vartheta = \vartheta_0$ the central character of the Verma module of weight 0. Then $\operatorname{Typ} {\mathbb{C}_k}^{\vartheta_0} = \{\psi \in \operatorname{Wt}(\mathfrak{g}/\mathfrak{b})^{\wedge k} : \vartheta_\psi = \vartheta_0\}$. By the Harish-Chandra theorem (see end of Section 2),

$$\vartheta_\psi = \vartheta_0 \iff \psi = w \circ 0 \text{ for some } w.$$

Now recall that

$$w \circ 0 = w\varrho - \varrho = -\sum_{\alpha \in \Phi_+ : w^{-1}\alpha \in \Phi_-} \alpha$$

and also that[34]

$$\ell(w) = \#\{\alpha \in \Phi_+ : w(\alpha) \in \Phi_-\} = \#\{\alpha \in \Phi_+ : w^{-1}(\alpha) \in \Phi_-\} = \ell(w^{-1}).$$

Combining all this with our description of $\operatorname{Typ} {\mathbb{C}_k}^\vartheta$ and $\operatorname{Typ} \mathbb{C}_k$, we have

$$\implies \operatorname{Typ} {\mathbb{C}_k}^{\vartheta_0} = \left\{ -\sum_{\alpha \in S} \alpha : S \subseteq \Phi_+,\ |S| = k,\ -\sum_{\alpha \in S} \alpha = -\sum_{\alpha \in \Phi_+ : w^{-1}\alpha \in \Phi_-} \alpha \right\}.$$

At this point, let us cite some combinatorial lemmas.

Fact (9.8)**.**

$$\sum_{\alpha \in \Phi_+ : w^{-1}\alpha \in \Phi_-} \alpha = \sum_{\alpha \in S} \alpha$$

$$\Updownarrow$$

$$S = \{\alpha \in \Phi_+ : w^{-1}\alpha \in \Phi_-\}.$$

In fact, Exercise 7.7 of Kirillov gives much more precise answers (though we won't need this here):

Fact. For $w^{-1} = s_{i_1} \cdots s_{i_k}$ a reduced expression,

$$\{\alpha \in \Phi_+ : w^{-1}\alpha \in \Phi_-\} = \{\alpha_{i_1},\ s_{i_1}\alpha_{i_2},\ s_{i_1}s_{i_2}\alpha_{i_3},\ \cdots\ , s_{i_1} \cdots s_{i_{k-1}}\alpha_{i_k}\}.$$

9.8 then tells us that

$$\operatorname{Typ} {\mathbb{C}_k}^{\vartheta_0} = \left\{ -\sum_{\alpha \in \Phi_+ : w^{-1}\alpha \in \Phi_-} \alpha : w \in W_k \right\},$$

i.e.

$$\implies \operatorname{Typ} {\mathbb{C}_k}^{\vartheta_0} = \{w \circ 0\}_{w \in W_k}.$$

This shows that the resolution

$$\cdots \longrightarrow {\mathbb{C}_2}^{\vartheta_0} \longrightarrow {\mathbb{C}_1}^{\vartheta_0} \longrightarrow {\mathbb{C}_0}^{\vartheta_0} \longrightarrow \mathbb{C} \longrightarrow 0$$

has terms of the type claimed, as desired. ∎

[34] From the reduced simple word description we can see $\ell(w) = \ell(w^{-1})$.

5.3. **Proving Weak BGG.** Now we are finally in a position to prove Weak BGG. Let me reproduce the statement for convenience:

Theorem (Weak BGG, 9.9)**.** For $\lambda \in \Lambda_+$ and $\Pi_\lambda \in \mathsf{irRep}_{\mathrm{fd}}\,\mathfrak{g}$, there is a resolution of $\mathfrak{g}$-modules

$$0 \longrightarrow B_{|\Phi_+|} \longrightarrow \cdots \longrightarrow B_1 \longrightarrow B_0 \longrightarrow \Pi_\lambda \longrightarrow 0$$

with terms in $\mathcal{O}$

$$B_k = \left(\left(U\mathfrak{g} \otimes_{U\mathfrak{b}} (\mathfrak{g}/\mathfrak{b})^{\wedge k}\right)^{\vartheta_0} \otimes \Pi_\lambda\right)^{\vartheta_\lambda}$$

such that

$$\operatorname{Typ} B_k = \{w \circ \lambda\}_{w \in W_k}.$$

Proof. We already have the theorem proved for the base case of $\Pi_0 = \mathbb{C}$. Now we will construct a new resolution of Π_λ as follows: take the weak BGG resolution for $\mathbb{C}$ and apply to it first $\square \otimes_{\mathbb{C}} \Pi_\lambda$ then $\square^{\vartheta_\lambda}$:

$$0 \longrightarrow B_{|\Phi_+|}(\mathbb{C}) \longrightarrow \cdots \longrightarrow B_1(\mathbb{C}) \longrightarrow B_0(\mathbb{C}) \longrightarrow \mathbb{C} \longrightarrow 0$$

$$\Big\Downarrow \square \otimes_{\mathbb{C}} \Pi_\lambda$$

$$0 \longrightarrow B_{|\Phi_+|}(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda \longrightarrow \cdots \longrightarrow B_1(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda \longrightarrow B_0(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda \longrightarrow \Pi_\lambda \longrightarrow 0$$

$$\Big\Downarrow \square^{\vartheta_\lambda}$$

$$0 \longrightarrow \left(B_{|\Phi_+|}(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda\right)^{\vartheta_\lambda} \longrightarrow \cdots \longrightarrow \left(B_1(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda\right)^{\vartheta_\lambda} \longrightarrow \left(B_0(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda\right)^{\vartheta_\lambda} \longrightarrow \Pi_\lambda \longrightarrow 0,$$

where since Π_λ is a finite-dimensional vector space we know $\square \otimes_{\mathbb{C}} \Pi_\lambda$ is exact[35], so the composition of two exact functors[36] $\square^{\vartheta_\lambda}$ and $\square \otimes_{\mathbb{C}} \Pi_\lambda$ is exact. Note that by construction[37] these terms live in the category $\mathcal{O}$. We have also noted $\mathbb{C} \otimes_{\mathbb{C}} \Pi_\lambda = \Pi_\lambda$, as well as $\Pi_\lambda^{\vartheta_\lambda} = \Pi_\lambda$ since Π_λ is irreducible[38].

Having constructed a resolution, it remains to see each term has the desired type. To exhibit

$$\operatorname{Typ}\left(B_k(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda\right)^{\vartheta_\lambda} = \{w \circ \lambda\}_{w \in W_k},$$

let us first compute $\operatorname{Typ}(B_k(\mathbb{C}) \otimes \Pi_\lambda)$ by exhibiting a filtration. Recall

$$\begin{aligned}\operatorname{Typ} B_k(\mathbb{C}) = \{w \circ 0\}_{w \in W_k} \implies &\exists\, 0 = B_k(\mathbb{C})_0 \subseteq \cdots \subseteq B_k(\mathbb{C})_n = B_k(\mathbb{C}) \\ &:\ B_k(\mathbb{C})_i \big/ B_k(\mathbb{C})_{i-1} \cong M_{w_i \circ 0}\end{aligned}$$

for $w_i \in W_k$; then the filtration of $B_k(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda$ given by

$$0 = B_k(\mathbb{C})_0 \otimes_{\mathbb{C}} \Pi_\lambda \subseteq \cdots \subseteq B_k(\mathbb{C})_n \otimes_{\mathbb{C}} \Pi_\lambda = B_k(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda$$

has each quotient

$$\implies B_k(\mathbb{C})_i \otimes_{\mathbb{C}} \Pi_\lambda \big/ B_k(\mathbb{C})_{i-1} \otimes_{\mathbb{C}} \Pi_\lambda \cong B_k(\mathbb{C})_i \big/ B_k(\mathbb{C})_{i-1} \otimes_{\mathbb{C}} \Pi_\lambda \cong M_{w_i \circ 0} \otimes_{\mathbb{C}} \Pi_\lambda,$$

where we have passed the tensor product outside the quotient since $\square \otimes_{\mathbb{C}} \Pi_\lambda$ is exact (as Π_λ is a vector space).

For a similar reason[39] as in the JH case,

Fact. If

$$0 = N_0 \subseteq \cdots \subseteq N_n = M,$$

then

$$\operatorname{Typ} M = \bigsqcup_i \operatorname{Typ}(N_i/N_{i-1}).$$

[35] Recall the huge block proposition in Section 2, where we noted that the tensor product with a finite-dimensional space is exact and in particular stays in $\mathcal{O}$.

[36] Since the tensor product stayed in $\mathcal{O}$, we are allowed to use that $\square^{\vartheta_\lambda}$ is an exact functor from $\mathcal{O}$ to $\mathcal{O}$.

[37] We saw that $B_k(\mathbb{C}) \in \mathcal{O}$, so applying these functors to it keeps things in $\mathcal{O}$.

[38] Recall M^ϑ is a subrep of M.

[39] Just refine the filtration; see the earlier footnote about JH (I think it's footnote 17 though of course this changes as I edit unfortunately).

For this reason, the above filtration of $B_k(\mathbb{C}) \otimes \Pi_\lambda$ buys us

$$\operatorname{Typ}(B_k(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda) = \bigsqcup_i \operatorname{Typ} B_k(\mathbb{C})_i \otimes_{\mathbb{C}} \Pi_\lambda \big/ B_k(\mathbb{C})_{i-1} \otimes_{\mathbb{C}} \Pi_\lambda = \bigsqcup_i \operatorname{Typ}(M_{w_i \circ 0} \otimes_{\mathbb{C}} \Pi_\lambda) = \bigsqcup_i \{\mu + w_i \circ 0\}_{\mu \in \operatorname{Wt} \Pi_\lambda},$$

where we have appealed to Lemma 9.10 in the last equality. Hence

$$\implies \operatorname{Typ}\left(B_k(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda\right) = \{\mu + w \circ 0\}_{\substack{\mu \in \operatorname{Wt} \Pi_\lambda, \\ w \in W_k}}.$$

Now let us pass to $\square^{\vartheta_\lambda}$. By Lemma 9.7,

$$\operatorname{Typ}\left(B_k(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda\right)^{\vartheta_\lambda} = \{\psi \in \operatorname{Typ}(B_k(\mathbb{C}) \otimes \Pi_\lambda) : \vartheta_\psi = \vartheta_\lambda\} = \{\mu + w \circ 0 : \vartheta_{\mu + w \circ 0} = \vartheta_\lambda\}_{\substack{\mu \in \operatorname{Wt} \Pi_\lambda, \\ w \in W_k}}.$$

By Harish-Chandra again, we have

$$\vartheta_{\mu + w \circ 0} = \vartheta_\lambda \iff \mu + w \circ 0 = u \circ \lambda \text{ for some } u.$$

Now we cite another combinatorial fact about Weyl groups:

Fact (K8.22b)**.** For any $\mu \in \Lambda$, the Weyl group orbit (not under the shifted action) of μ contains exactly one element of Λ_+.

Hence, for any $\mu + w \circ 0 \in \operatorname{Typ}\left(B_k(\mathbb{C}) \otimes \Pi_\lambda\right)^{\vartheta_\lambda}$, since we know

$$\begin{aligned} \mu + w \circ 0 &= u \circ \lambda \\ \mu + w\varrho - \varrho &= u\lambda + u\varrho - \varrho \\ u^{-1}(\mu + w\varrho - u\varrho) &= \lambda, \end{aligned}$$

where $\mu + w\varrho - u\varrho \in \Lambda$ and $\lambda \in \Lambda_+$, we know that u^{-1} is the only element of W which turns $\mu + w\varrho - u\varrho$ into λ, so that there is a one-to-one correspondence between $\mu + w \circ 0$ and such u.

Moreover, since the set of weights of a finite-dimensional representation are closed under W[40], we have

$$u^{-1}\mu \in \operatorname{Wt} \Pi_\lambda \implies u^{-1}\mu \leq \lambda.$$

Since $\varrho - u^{-1}w\varrho = \sum_{\alpha \in \Phi_+ : w^{-1}u\alpha \in \Phi_-} \alpha$, we also have

$$u^{-1}w\varrho \leq \varrho.$$

Together these two give

$$u^{-1}\mu + u^{-1}w\varrho \leq \lambda + \varrho.$$

But as we saw in the last paragraph, this inequality is actually an equality, which forces

$$u^{-1}\mu = \lambda, \quad u^{-1}w\varrho = \varrho,$$

which means

$$u = w^{-1}.$$

In particular this means $\ell(u) = \ell(w) = k$, which gives

$$\implies \operatorname{Typ}\left(B_k(\mathbb{C}) \otimes_{\mathbb{C}} \Pi_\lambda\right)^{\vartheta_\lambda} = \{u \circ \lambda\}_{u \in W_k}.$$

This is precisely as claimed.

We have, at last, shown Weak BGG. ■

[40] In fact, the full statement is this: for any $V \in \mathsf{Rep}_{\mathrm{fd}}\, \mathfrak{g}$ and any $w \in W$,

$$w(\chi_V) = \chi_V,$$

where we define

$$w(e^\lambda) := e^{w(\lambda)}.$$

In particular the weights and the dimensions of their weight subspaces are invariant under W. This is Theorem K8.8. Its proof is very short and can be summarized by "reduce to $\mathfrak{sl}_2$", like many other proofs in Kirillov.

5.4. **A Corollary of Weak BGG.** As mentioned (and used!) earlier, Bott's Theorem[41] on Lie algebra cohomology can be derived as a corollary of Weak BGG;

Theorem (Bott)**.** For $\Pi \in \mathsf{irRep}_{\mathrm{fd}}\, \mathfrak{g}$,

$$\dim H^k(\mathfrak{n}^- : \Pi) = |W_k|.$$

Proof. Recall

$$H^k(\mathfrak{n}^- : \Pi) = \mathrm{Ext}^k_{U\mathfrak{n}^-}(\mathbb{C}, \Pi) = \mathrm{Tor}_k^{U\mathfrak{n}^-}(\Pi^*, \mathbb{C})^* = \mathrm{Tor}_k^{U\mathfrak{n}^-}(\mathbb{C}, \Pi^{\dagger,*})^*.$$

Let us compute the latter by resolving the second variable

$$0 \longrightarrow B_{|\Phi_+|}(\Pi^{\dagger,*}) \longrightarrow \cdots \longrightarrow B_1(\Pi^{\dagger,*}) \longrightarrow B_0(\Pi^{\dagger,*}) \longrightarrow \Pi^{\dagger,*} \longrightarrow 0;$$

then Tor is the homology of (here we suppress the Π)

$$0 \longrightarrow \mathbb{C} \otimes_{\mathfrak{n}^-} B_{|\Phi_+|} \longrightarrow \cdots \longrightarrow \mathbb{C} \otimes_{\mathfrak{n}^-} B_1 \longrightarrow \mathbb{C} \otimes_{\mathfrak{n}^-} B_0 \longrightarrow 0,$$

which is the same thing as writing

$$0 \longrightarrow B_{|\Phi_+|}\big/\mathfrak{n}^- B_{|\Phi_+|} \longrightarrow \cdots \longrightarrow B_1\big/\mathfrak{n}^- B_1 \longrightarrow B_0\big/\mathfrak{n}^- B_0 \longrightarrow 0.$$

As remarked in a footnote earlier, $\mathfrak{h}$ acts naturally on this sequence.

By Weak BGG, since $B_k \in \mathcal{O}$, we have B_k is $U\mathfrak{n}^-$-finitely-generated, so that

$$\dim B_k\big/\mathfrak{n}^- B_k < \infty.$$

In particular, $B_k/\mathfrak{n}^- B_k$ admits a weight space decomposition, where the weight vectors are precisely[42] the quotient images of the highest weight vectors of B_k, which may be obtained by looking in a filtration

$$0 = (B_k)_0 \subseteq \cdots \subseteq (B_k)_n = B_k,$$

where $(B_k)_i/(B_k)_{i-1} = M_{w_i \circ \lambda}$, and taking the highest weight vector e.g. $v^{w_i \circ \lambda} \in (B_k)_i/(B_k)_{i-1}$. Lifting this to a $\widehat{v}^{w_i \circ \lambda} \in (B_k)_i \subseteq B_k$, and then projecting down to a $\widetilde{v}^{w_i \circ \lambda} \in B_k/\mathfrak{n}^- B_k$, we obtain a nonzero weight vector (nonzero since $v^{w_i \circ \lambda}$ was taken to be highest weight and so cannot be in the image of $\mathfrak{n}^-$). Hence the weights of $B_k/\mathfrak{n}^- B_k$ are

$$\implies \mathrm{Wt}\, B_k\big/\mathfrak{n}^- B_k = \{w \circ \lambda\}_{w \in W_k},$$

and in particular

$$\implies \dim B_k\big/\mathfrak{n}^- B_k = |W_k|.$$

But the maps in the resolution $B_\bullet/\mathfrak{n}^- B_\bullet$, which commute with $\mathfrak{h}$ (see a previous footnote), must preserve weight spaces; since $w \circ \lambda \neq u \circ \lambda$ for $w \neq u$, we have these maps must actually all be zero, so that its homology is simply

$$\implies \mathrm{Tor}_k^{U\mathfrak{n}^-}(\mathbb{C}, \Pi^{\dagger,*}) = B_k(\Pi^{\dagger,*})\big/\mathfrak{n}^- B_k(\Pi^{\dagger,*}),$$

and in particular

$$\dim H^k(\mathfrak{n}^- : \Pi) = \dim \mathrm{Tor}_k^{U\mathfrak{n}^-}(\mathbb{C}, \Pi^{\dagger,*}) = \dim B_k\big/\mathfrak{n}^- B_k = |W_k|,$$

i.e.

$$\implies \dim H^k(\mathfrak{n}^- : \Pi) = |W_k|,$$

as is stated. ∎

With this we have finally cleared all of our debts; at last we have done what we set out to do. What a journey!

[41] Interestingly I can't quite find this online under this name. I thought at first BGG might be using out-of-date terminology, but Humphreys uses this name also. I'm probably just bad at using the Internet.

[42] It is clear that the process we describe below gives weight vectors, and moreover all weight vectors must arise this way since the following: for $B_k \in \mathcal{O}$, recall $\mathrm{Wt}\, B_k$ lies in a finite union of cones. Then, since $\mathfrak{n}^-$ drops the weights, $\mathfrak{n}^- B_k$ has all the weights of B_k except for the highest ones in each cone, so that $B_k/\mathfrak{n}^- B_k$ has weights precisely the highest ones.

Part II

The Functorial BGG Resolution

6. A Short Introduction to Part II

6.1. **Foreword to Part II.** In this half, we give an exposition on a "modern" categorical viewpoint towards the BGG resolution. As I understand it, the mathematics presented here is considered common knowledge in the field. This story was communicated to me by Professor Gaitsgory, Charles Fu, and Kevin Lin; I am incredibly grateful for their help and patience. In particular, I was entirely a listener. However, I could not find a reference where this story was committed to paper, and so I decided to write it down. My role is only that of a scribe.

In the interest of length, we will assume familiarity with triangulated categories, t-structures[43], and derived categories/functors. Of course, we also assume moderate familiarity with the representation theory of Lie algebras, beyond that assumed in Part I, for example to the extent of one who has read some of the book on category $\mathcal{O}$ by Humphreys [5]. It would be helpful to keep stable ∞-categories in the back of one's mind, but here we try to avoid this language as much as possible (due in no small part to the author's lack of familiarity with this language).

6.2. **A Refresher on Notations/Conventions.** In an effort to make the two parts more self-contained, we will here briefly recall some conventions/notations.

6.2.1. *Representation Theory.* We will work over $\mathbb{C}$ throughout. Unless otherwise stated, $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ will be a semisimple Lie algebra. We will write Σ for the set of simple roots; if α_i is a simple root, it will be said that $\alpha_i \in \Sigma$, and otherwise α_i denotes any indexed set of roots. In this spirit, we will denote by $I(\Sigma)$ the index set of the simple roots, i.e. $i \in I(\Sigma) \iff \alpha_i \in \Sigma$. Let ω_i be the fundamental weights. The Verma module is denoted M_λ. An irreducible representation of highest weight λ is commonly called L_λ, and sometimes to emphasize that it is finite-dimensional we may write Π_λ instead. Let $\Lambda = \mathbb{Z}\{\omega_i\}_\Sigma$ be the weight lattice, and let $Q = \mathbb{Z}\{\alpha_i\}_\Sigma$ be the root lattice; similarly denote $\Lambda_+ = \mathbb{N}\{\omega_i\}_\Sigma$ and $Q_+ = \mathbb{N}\{\alpha_i\}_\Sigma$ (our convention is that $\mathbb{N}$ contains 0). Recall the notion, for $\lambda, \mu \in \Lambda$, of

$$\mu \leq \lambda \iff \lambda - \mu \in Q_+.$$

We will write W for the Weyl group and

$$W_k \coloneqq \{w \in W : \ell(W) = k\}.$$

We will let

$$w \circ \lambda \coloneqq w(\lambda + \varrho) - \varrho$$

define the affine action of the Weyl group W on $\mathfrak{h}^*$, where recall $\varrho \coloneqq \frac{1}{2}\sum_{\Phi_+} \alpha = \sum_\Sigma \alpha$, where $\Sigma \subseteq \Phi_+$ denotes the set of simple roots. We will also write

$$\lambda \sim \mu \iff \exists\, w : \lambda = w(\mu),$$

and

$$\lambda \overset{\circ}{\sim} \mu \iff \exists\, w : \lambda = w \circ \mu.$$

[43]I'm under the impression that the 't' in 't-structures' should be in text font and not math font, because it's not a mathematical symbol.

In line with the notation of $\mathfrak{g}^\alpha$, we will also write

$$M^\mu := \{v \in M : \mathfrak{h}v = \mu(\mathfrak{h})v\}$$

for the μ-weight space of M; for example for Verma modules this would be written ${M_\lambda}^\mu$ with staggered indices.

For M a representation of $\mathfrak{g}$, we denote by M^* the linear dual equipped with the usual representation structure, and we denote by $M^\dagger$ the contragredient dual. We write $\mathrm{Hom}_\mathfrak{g}$ when perhaps we should write $\mathrm{Hom}_\mathcal{O}$; this is not a big issue until we write $\mathrm{RHom}_\mathfrak{g}$, which we consider to be right derived over $\mathcal{O}$ as opposed to all of $\mathsf{Mod}\, U\mathfrak{g}$; that is, by $\mathrm{RHom}_\mathfrak{g}$ we mean the complex whose cohomology is $\mathrm{Ext}_\mathcal{O}$.

6.2.2. *Category Theory/Homological Algebra.* When performing homological algebra, we will try to stick to cohomological indexing. A typical complex might be denoted $X^\bullet$; if the context is clear we may drop the bullet. In order to accommodate the case of complexes of $\mathfrak{g}$-modules, where we use the superscript to denote weight spaces, we will put the complex index in parentheses: namely, the k-th object of $M^\bullet$ is given by $M^{(k)}$. However, since no such conflict exists for maps, we will keep the degrees of maps unadorned; for example, a differential might be written d^k.

We write $\mathcal{F} \dashv \mathcal{G}$ for $\mathcal{F}$ being left adjoint to $\mathcal{G}$. We consider the shift functor, denoted either Σ or just $[1]$ where appropriate, in a triangulated category to be defined as an autoequivalence. We denote the category of chain complexes by $\mathsf{Ch}(\mathcal{A})$, and we denote the category of chain complexes up to homotopy by $\mathsf{K}(\mathcal{A})$.

Our convention on the category induced by a poset is that arrows $\mu \to \lambda$ correspond to $\mu < \lambda$, so that the final object is the maximal element of the poset, and the initial object is the minimal element. Whenever we write something like $\Lambda \cup \{\lambda_\infty\}$ (respectively $\Lambda \cup \{\lambda_{-\infty}\}$), we mean for the new symbol to replace/merge with the final (respectively initial) object if it exists already.

Typically, objects with a bullet in the superscript are written in cohomological indexing, and objects with a bullet in the subscript are written in homological indexing. The single major exception to this (for which we profusely apologize) is the Lie algebra chain complex (as well as its (co)homology), $C_\bullet(\mathfrak{g} : \square)$, which is taken to have cohomological indexing despite having a bullet in the subscript.

6.2.3. *Logic/Foundations.* We assume ZFC and WP (Weak Vopěnka).

7. Categorical Considerations

7.1. **Filtered Triangulated Categories.** We consider a filtered triangulated category. (For us, the shift functor in a triangulated category is defined as an autoequivalence.)

Definition. For $\mathcal{C} \in \mathsf{TriCat}$ a triangulated category and Λ a poset with a unique final object (our convention on arrows is that final means maximal), a "filtration on $\mathcal{C}$" is a functor

$$\mathcal{F} \colon \Lambda \longrightarrow \mathsf{TriCat}$$

such that all arrows go to fully faithful embeddings and the final object goes to $\mathcal{C}$.

For $\lambda \in \Lambda$, we denote

$$\mathcal{C}^{\leq\lambda} := \mathcal{F}(\lambda)$$

and

$$\mathcal{C}^{<\lambda} := \text{the smallest thick triangulated subcategory containing all } \mathcal{F}(\mu) \text{ for } \mu < \lambda.$$

We also define

$$\mathcal{C}^{=\lambda} := \mathcal{C}^{\leq\lambda}/\mathcal{C}^{<\lambda}$$

to be the Verdier quotient.

The idea is to later apply this to representation theory by considering a filtration on the derived category of $\mathcal{O}$ given by considering objects with weights less than a given weight.

We assume the reader is familiar with basic facts/constructions in triangulated categories; for example, to see more on Verdier quotients, reference Chapter 2 of [11]. In particular, Theorem 2.1.8 and Remark 2.1.10 there tell us that, in the above setting, the Verdier quotient functor

$$p_\lambda \colon \mathcal{C}^{\leq\lambda} \longrightarrow \mathcal{C}^{=\lambda}$$

has

$$\operatorname{Ker} p_\lambda = \mathcal{C}^{<\lambda}.$$

Here are some names of situations when inclusions/projections admit left adjoints:

Definition. The inclusion of a full subcategory is called "reflective" if it admits a left adjoint.
The quotient functor of a Verdier quotient is called a "colocalization" if it admits a left adjoint.

For our purposes, we will make two key assumptions on our filtered triangulated category $\mathcal{C}$:

(1) Every $i_\lambda \colon \mathcal{C}^{\leq\lambda} \longrightarrow \mathcal{C}$ is reflective.
(2) Every $p_\lambda \colon \mathcal{C}^{\leq\lambda} \longrightarrow \mathcal{C}^{=\lambda}$ is a colocalization.

We denote the left adjoint of i_λ and p_λ by $i_\lambda^\perp$ and $p_\lambda^\perp$, respectively.

Given these assumptions, one can deduce some basic facts about the functors

$$j_\lambda \colon \mathcal{C}^{<\lambda} \longrightarrow \mathcal{C}^{\leq\lambda}$$

and

$$i_{\mu\to\lambda} \colon \mathcal{C}^{\leq\mu} \longrightarrow \mathcal{C}^{\leq\lambda},$$

where the latter functor is obtained from the filtration $\mathcal{F}$ applied to the arrow $\mu \to \lambda$ (meaning $\mu < \lambda$). In particular, they both admit left adjoints.

Lemma. $i_{\mu\to\lambda} \coloneqq \mathcal{F}(\mu \to \lambda) \colon \mathcal{C}^{\leq\mu} \longrightarrow \mathcal{C}^{\leq\lambda}$ admits a left adjoint

$$i_{\mu\to\lambda}^\perp = i_\mu^\perp i_\lambda.$$

$j_\lambda \colon \mathcal{C}^{<\lambda} \longrightarrow \mathcal{C}^{\leq\lambda}$ also admits a left adjoint $j_\lambda^\perp$ which satisfies

$$j_\lambda j_\lambda^\perp = \operatorname{Cofib}(p_\lambda^\perp p_\lambda \to \operatorname{Id}_{\mathcal{C}^{\leq\lambda}}).$$

Proof. The first is easy to see. Indeed, $\operatorname{Hom}_{\mathcal{C}^{\leq\mu}}(i_\mu^\perp i_\lambda X, Y) = \operatorname{Hom}_{\mathcal{C}}(i_\lambda X, i_\mu Y) = \operatorname{Hom}_{\mathcal{C}^{\leq\lambda}}(X, i_{\mu\to\lambda} Y)$ is obvious from the fully faithfulness of the inclusion functors.

The second statement is given in Theorem 4.9.1 of [8], and the explicit form is in the proof. Indeed, $\mathcal{C}^{<\lambda}$ is thick by assumption, and the Verdier localization quotient functor and its adjoint are both triangulated functors (adjoints of triangulated functors are triangulated; see Lemma 5.3.6 of [11]). There is a subtlety that Krause deals with categories where the Verdier quotient is a localization as opposed to a colocalization, but this is easily fixed: indeed, note that in general $\mathcal{F} \dashv \mathcal{G}$ if and only if $\mathcal{F}^{\mathrm{op}} \vdash \mathcal{G}^{\mathrm{op}}$, and the opposite category of a triangulated category is still triangulated. ■

We should maybe also remark that, since $\mathcal{C}^{<\lambda}$ is a triangulated subcategory, it is baked into the definitions already that j_λ is fully faithful. Similarly by definition i_λ is fully faithful. Hence we have natural isomorphisms

$$\varepsilon \colon i_\lambda^\perp i_\lambda \xrightarrow{\sim} \operatorname{Id}_{\mathcal{C}^{\leq\lambda}}$$

and

$$\varepsilon \colon j_\lambda^\perp j_\lambda \xrightarrow{\sim} \operatorname{Id}_{\mathcal{C}^{<\lambda}}.$$

In Propositions 2.3.1 and 2.4.1 of [8], it is also shown that (again, to apply to our case, consider the opposite category) $p_\lambda^\perp$ is fully faithful, so that there is a natural isomorphism

$$\eta \colon \operatorname{Id}_{\mathcal{C}^{=\lambda}} \xrightarrow{\sim} p_\lambda p_\lambda^\perp.$$

Letting $j_{\mu\to\lambda}$ denote the inclusion of $\mathcal{C}^{\leq\mu}$ into $\mathcal{C}^{<\lambda}$ for $\mu < \lambda$, it is also easy to see the following:

Lemma. $j_{\mu\to\lambda} \colon \mathcal{C}^{\leq\mu} \longrightarrow \mathcal{C}^{<\lambda}$ admits a left adjoint given by

$$j_{\mu\to\lambda}^\perp = i_\mu^\perp i_\lambda j_\lambda.$$

Proof. This is straightforward to see. For $X \in \mathcal{C}^{<\lambda}$ and $Y \in \mathcal{C}^{\leq\mu}$, note

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{C}^{\leq\mu}}(j^{\perp}_{\mu\to\lambda}X, Y) &\cong \operatorname{Hom}_{\mathcal{C}^{\leq\mu}}(i^{\perp}_{\mu} i_\lambda j_\lambda X, Y) \\
&\cong \operatorname{Hom}_{\mathcal{C}}(i_\lambda j_\lambda X, i_\mu Y) \\
&\cong \operatorname{Hom}_{\mathcal{C}}(i_\lambda j_\lambda X, i_\lambda j_\lambda j_{\mu\to\lambda} Y) \\
&\cong \operatorname{Hom}_{\mathcal{C}^{<\lambda}}(X, j_{\mu\to\lambda} Y),
\end{aligned}$$

where the last line is by fully faithfulness of the inclusion functors. ■

To summarize, we have the following diagram:

$$\begin{tikzcd}
 & & \mathcal{C}^{=\lambda} \arrow[d, bend left, "p^{\perp}_{\lambda}"] & & \\
\mathcal{C} \arrow[rr, bend left, "i^{\perp}_{\lambda}"] \arrow[rr, phantom, "\perp"] \arrow[drr, bend left, "i^{\perp}_{\mu}"] & & \mathcal{C}^{\leq\lambda} \arrow[ll, bend left, "i_\lambda"] \arrow[u, bend left, "p_\lambda"] \arrow[u, phantom, "p_\lambda \vdash p^{\perp}_{\lambda}"] \arrow[rr, bend left, "j^{\perp}_{\lambda}"] \arrow[rr, phantom, "\perp"] \arrow[d, bend left, "i^{\perp}_{\mu} i_\lambda"] & & \mathcal{C}^{<\lambda} \arrow[ll, bend left, "j_\lambda"] \arrow[dll, bend right, "i^{\perp}_{\mu} i_\lambda j_\lambda"'] \\
 & & \mathcal{C}^{\leq\mu} \arrow[ull, bend left, "i_\mu"] \arrow[u, bend left, "i_{\mu\to\lambda}"] \arrow[u, phantom, "i_{\mu\to\lambda} \vdash i^{\perp}_{\mu} i_\lambda"] \arrow[urr, bend right, "j_{\mu\to\lambda}"'] & &
\end{tikzcd}$$

7.2. **Filtered Objects.** We may also consider the notion of an object filtered by a poset Λ.

Definition. In a category $\mathcal{C}$, a "Λ-filtered object" is a functor

$$\mathcal{X} \in \mathsf{Fun}(\Lambda, \mathcal{C}).$$

If $\mathcal{C}$ is moreover triangulated, letting Λ^0 be Λ with all arrows removed, consider the "associated graded"

$$\begin{aligned}
\mathrm{Gr}\colon \mathsf{Fun}(\Lambda, \mathcal{C}) &\longrightarrow \mathsf{Fun}(\Lambda^0, \mathcal{C}) \\
\mathcal{X} &\longmapsto \mathrm{Gr}\,\mathcal{X}
\end{aligned}$$

defined by

$$(\mathrm{Gr}\,\mathcal{X})(\lambda) = \mathrm{Fib}\Big(\mathcal{X}(\lambda) \to \lim_{\lambda\to\mu} \mathcal{X}(\mu)\Big)$$

assuming such limits exists. We might also denote this by $\mathrm{Gr}^{\lambda}\,\mathcal{X}$.

An object $X \in \mathcal{C}$ is said to have a Λ-filtration if after adding a initial/minimal element $\lambda_{-\infty}$ to Λ there exists a functor

$$\mathcal{X}\colon \Lambda \cup \{\lambda_{-\infty}\} \longrightarrow \mathcal{C}$$

such that

$$\mathcal{X}(\lambda_{-\infty}) = X.$$

The following theorem grants us the explicit form of a filtration and its associated graded in the case of an identity functor on a filtered triangulated category. We will later apply this to the setting of category $\mathcal{O}$ to obtain "approximations" of a L_λ, which will be our functorial BGG resolution.

Theorem. Under our hypotheses on $\mathcal{C}$, the functor $\mathrm{Id}_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \mathcal{C}$ admits an Λ^{op}-filtration with terms

$$\mathcal{I}\mathrm{d}(\lambda) = i_\lambda i_\lambda^\perp$$

and associated graded

$$(\mathrm{Gr}\,\mathcal{I}\mathrm{d})(\lambda) = i_\lambda p_\lambda^\perp p_\lambda i_\lambda^\perp.$$

Proof. So in the definition above we see that to endow $\mathrm{Id}_{\mathcal{C}}$ with a filtration is to define a functor $\mathcal{I}\mathrm{d}\colon \Lambda^{\mathrm{op}} \cup \{\lambda_{-\infty}\} \longrightarrow \mathsf{Fun}(\mathcal{C},\mathcal{C})$; as Λ has a unique maximal element λ_∞ by assumption, it is not necessary to union this $\{\lambda_{-\infty}\}$. We will use the same symbol λ_∞ to denote the unique minimal element of Λ^{op}. We may well define $\mathcal{I}\mathrm{d}\colon \Lambda^{\mathrm{op}} \longrightarrow \mathsf{Fun}(\mathcal{C},\mathcal{C})$ by $\mathcal{I}\mathrm{d}(\lambda) = i_\lambda i_\lambda^\perp$ and $\mathcal{I}\mathrm{d}(\lambda_\infty) = \mathrm{Id}$. Given $\lambda \overset{\mathrm{op}}{\to} \mu$ in Λ^{op}, i.e. $\mu \to \lambda$ in Λ, we should also construct an arrow $i_\lambda i_\lambda^\perp \to i_\mu i_\mu^\perp$. This is given by the map

$$\begin{aligned} \mathrm{id}_{i_\mu^\perp X} \in \mathrm{Hom}_{\mathcal{C}^{\leq\mu}}(i_\mu^\perp X, i_\mu^\perp X) &\cong \mathrm{Hom}_{\mathcal{C}}(X, i_\mu i_\mu^\perp X) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(X, i_\lambda i_{\mu\to\lambda} i_\mu^\perp X) \\ &\cong \mathrm{Hom}_{\mathcal{C}^{\leq\lambda}}(i_\lambda^\perp X, i_{\mu\to\lambda} i_\mu^\perp X) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(i_\lambda i_\lambda^\perp X, i_\mu i_\mu^\perp X). \end{aligned}$$

It is straightforward to check that the map so defined is a natural transformation of functors. Similarly, we define the arrow out of the initial object $\lambda_\infty \overset{\mathrm{op}}{\to} \lambda$ to give rise to a morphism $\mathrm{Id} \to i_\lambda i_\lambda^\perp$ given by the unit of the adjunction.

Now the claim is that $\mathrm{Gr}^\lambda \mathcal{I}\mathrm{d} = \mathrm{Fib}(i_\lambda i_\lambda^\perp \to \lim_{\lambda\leftarrow\mu} i_\mu i_\mu^\perp) = i_\lambda p_\lambda^\perp p_\lambda i_\lambda^\perp$. Recall that

$$p_\lambda^\perp p_\lambda \longrightarrow \mathrm{Id} \longrightarrow j_\lambda j_\lambda^\perp \longrightarrow \Sigma p_\lambda^\perp p_\lambda$$

is exact. As both i_λ and $i_\lambda^\perp$ are triangulated functors, it suffices to show that $i_\lambda j_\lambda j_\lambda^\perp i_\lambda^\perp = \lim_{\lambda\leftarrow\mu} i_\mu i_\mu^\perp$. As i_λ, j_λ are just fully faithful inclusion functors, it suffices to just show $(i_\lambda j_\lambda)^\perp = \lim_{\mu\to\lambda} j_{\mu\to\lambda} i_\mu^\perp$ (here we remark $i_\mu = i_\lambda j_\lambda j_{\mu\to\lambda}$). But this is almost tautological: for each $\lambda \leftarrow \mu$, we have a map $j_\lambda^\perp i_\lambda^\perp \longrightarrow j_{\mu\to\lambda} i_\mu^\perp$ given by $\mathcal{I}\mathrm{d}$, namely by taking the morphism $\mathcal{I}\mathrm{d}(\mu\to\lambda)\colon i_\lambda i_\lambda^\perp \longrightarrow i_\mu i_\mu^\perp$ and applying $j_\lambda^\perp i_\lambda^\perp$ to both sides (also using the natural isomorphism $i_\lambda^\perp i_\lambda \cong \mathrm{Id}$) to obtain $j_\lambda^\perp i_\lambda^\perp i_\lambda i_\lambda^\perp = j_\lambda^\perp i_\lambda^\perp \longrightarrow j_\lambda^\perp i_\lambda^\perp i_\mu i_\mu^\perp = j_{\mu\to\lambda} i_\mu^\perp$. Then the limit of this over all $\mu < \lambda$ is precisely object corresponding to the smallest poset element above all the μ, namely λ. ■

7.3. **t-structures and Hearts.** We did not present a brief overview of triangulated categories because there is, for the most part (as far as we are aware), no major variations in definitions across the literature. However, in the case of t-structures, there is an issue of cohomological versus homological indexing. The author tends to prefer cohomological indexing[44]. Here we briefly recall conventions.

Definition. A "t-structure" on a triangulated category $\mathcal{C}$ is the information of a pair $(\mathcal{C}^{(\leq 0)}, \mathcal{C}^{(\geq 0)})$ of full subcategories which are stable under isomorphism. This pair is required to satisfy the following:

(1) $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}^{(\leq 0)}, \mathcal{C}^{(\geq 0)}[-1]) = 0$.
(2) $\mathcal{C}^{(\leq 0)}[1] \subseteq \mathcal{C}^{(\leq 0)}$ and $\mathcal{C}^{(\geq 0)}[-1] \subseteq \mathcal{C}^{(\geq 0)}$.
(3) For any $A \in \mathcal{C}$, there exists an exact triangle $X \longrightarrow A \longrightarrow Y[-1] \longrightarrow X[1]$ such that $X \in \mathcal{C}^{(\leq 0)}$ and $Y \in \mathcal{C}^{(\geq 0)}$.

Then define the "heart" (or "core") of the t-structure to be the full subcategory

$$\mathcal{C}^\heartsuit := \mathcal{C}^{(\leq 0)} \cap \mathcal{C}^{(\geq 0)}.$$

We denote $\mathcal{C}^{(\leq n)}$ and $\mathcal{C}^{(\geq n)}$ to be full subcategories defined by

$$\mathcal{C}^{(\leq n)} := \mathcal{C}^{(\leq 0)}[-n], \quad \mathcal{C}^{(\geq n)} := \mathcal{C}^{(\geq 0)}[-n].$$

[44] I am told that algebraic geometers tend to prefer the former, whereas algebraic topologists tend to prefer the latter. I am personally not a big fan of algebraic topology (I am not a big Fan in general), so even if just as a matter of principle I will adhere to the former.

If the inclusion functors

$$\iota^{(\leq n)}\colon \mathcal{C}^{(\leq n)} \longrightarrow \mathcal{C}, \quad \iota^{(\geq n)}\colon \mathcal{C}^{(\geq n)} \longrightarrow \mathcal{C},$$

admit right and left adjoints respectively (i.e. are coreflective and reflective respectively), let their adjoints be the "truncation functors" $\tau^{(\leq n)}$ and $\tau^{(\geq n)}$:

$$\tau^{(\leq n)} \vdash \iota^{(\leq n)}, \quad \tau^{(\geq n)} \dashv \iota^{(\geq n)}.$$

As it turns out, if $\mathcal{C}$ is an *enhanced* triangulated category (i.e. arises as the homotopy category of a stable ∞-category), then automatically $\mathcal{C}^{(\geq n)}$ is stable under limits and $\mathcal{C}^{(\leq n)}$ is stable under colimits (see Corollary 1.2.1.6 of [10][45]); then, by Vopěnka's Principle (see Theorem 6.22 [1]), the inclusion functors do indeed admit the desired adjoints.

These truncation functors have many properties and related constructions which we will not delve into here. For example, it turns out the counit and unit of the adjunctions give an exact triangle

$$\tau^{(\leq 0)}X \longrightarrow X \longrightarrow \tau^{(\geq 1)}X \longrightarrow \tau^{(\leq 0)}X[1],$$

where the last map is uniquely determined. For more properties, we refer the reader to either Wikipedia or [9].

There is also a variant of t-structures for stable ∞-categories; for example, see [9] or the Wikipedia[46] article on t-structures. Here a t-structure is defined on the homotopy category and then extended to the stable ∞-category by taking the full subcategory spanned. Truncation functors are defined as before as functors on the stable versions of the pair, and the heart on this stable version of the t-structure is equivalent to the nerve of the heart of the homotopy category.

Warning: different people take different views to a t-structure. Others might define a t-structure as only one of the full subcategories in the pair we described and denote the right adjoint of its inclusion as some further information in the definition of a t-structure.

Of course, in the case of a derived category $\mathsf{D}(\mathcal{A})$ (similar definitions for $\mathsf{D}^+, \mathsf{D}^-, \mathsf{D}^0$), we define the natural t-structure to be

$$\mathsf{D}(\mathcal{A})^{(\leq 0)} := \{X : H^{>0}(X) = 0\}, \quad \mathsf{D}(\mathcal{A})^{(\geq 0)} := \{X : H^{<0}(X) = 0\}.$$

This has

$$\mathsf{D}(\mathcal{A})^{(\leq n)} := \{X : H^{>n}(X) = 0\}, \quad \mathsf{D}(\mathcal{A})^{(\geq n)} := \{X : H^{<n}(X) = 0\}$$

and

$$\mathsf{D}(\mathcal{A})^{\heartsuit} \cong \mathcal{A}.$$

The truncation functors can be described explicitly as

$$\begin{aligned}\tau^{(\leq 0)}X^\bullet &= (\cdots \longrightarrow X^{(-2)} \longrightarrow X^{(-1)} \longrightarrow \operatorname{Ker} \mathrm{d}^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots),\\ \tau^{(\geq 1)}X^\bullet &= (\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X^{(0)}/\operatorname{Ker} \mathrm{d}^0 \longrightarrow X^{(1)} \longrightarrow X^{(2)} \longrightarrow \cdots),\end{aligned}$$

whereupon it is clear we have

$$0 \longrightarrow \tau^{(\leq 0)}X^\bullet \longrightarrow X^\bullet \longrightarrow \tau^{(\geq 1)}X^\bullet \longrightarrow 0.$$

7.4. **A Dold-Kan-ish Correspondence.** We present here a correspondence between $\mathbb{N}$-filtered objects and connective complexes. We remind the reader that $\mathbb{N} = \{n \in \mathbb{Z} : n \geq 0\}$ includes zero for us.

Theorem. Let $\mathcal{C}$ be an enhanced triangulated category with a t-structure. Then

$$\Big\{\mathbb{N}\text{-filtered object } \mathcal{X} \text{ of } \mathcal{C} \text{ such that } (\operatorname{Gr}^n \mathcal{X})[-n] \in \mathcal{C}^{\heartsuit}\Big\} \overset{\text{equiv of cats}}{\cong} \Big\{\text{connective chain complexes in } \mathsf{Ch}(\mathcal{C}^{\heartsuit})\Big\}$$

where connective means $H^{>0}(C) = 0$.

[45] Important remark about referencing Lurie's HA: Lurie does everything with homological indexing as opposed to cohomological. In order to make sense of anything Lurie writes about t-structures, simply swap the symbols $\geq$ and $\leq$.

[46] Wikipedia uses cohomological indexing for the most part.

Proof. In the left to right direction, one goes from a $\mathbb{N}$-filtered object $\mathcal{X}$ to a (homologically graded) chain complex in the following way. The filtration looks like[47]

$$\mathcal{X}(0) \xrightarrow{f_0} \mathcal{X}(1) \xrightarrow{f_1} \mathcal{X}(2) \longrightarrow \cdots,$$

and the assumption is that $(\mathrm{Gr}\,\mathcal{X})(n)[-n] \in \mathcal{C}^{\heartsuit}$, namely that $\mathrm{Fib}(\mathcal{X}(n) \to \mathcal{X}(n+1))[-n] = \mathrm{Fib}(f_n)[-n] \in \mathcal{C}^{\heartsuit}$. Then the octahedral axiom for triangulated categories gives an exact triangle

$$\mathrm{Fib}(f_{n-1}) \longrightarrow \mathrm{Fib}(f_n \circ f_{n-1}) \longrightarrow \mathrm{Fib}(f_n) \longrightarrow \mathrm{Fib}(f_{n-1})[1].$$

The the corresponding complex C has terms

$$C_n \coloneqq \mathrm{Fib}(f_n)[-n] \in \mathcal{C}^{\heartsuit},$$

namely $C_n = (\mathrm{Gr}\,\mathcal{X})(n)[-n]$, with differential maps

$$\mathrm{d}_n \colon \mathrm{Fib}(f_n)[-n] \longrightarrow \mathrm{Fib}(f_{n-1})[-n+1].$$

In other words, the connective complex $C_\bullet \in \mathsf{Ch}^{(\leq 0)}(\mathcal{C}^{\heartsuit})$ is given by

$$C_n = (\mathrm{Gr}^n\,\mathcal{X})[-n]$$

with differentials

$$\mathrm{d}_n \colon (\mathrm{Gr}^n\,\mathcal{X})[-n] \longrightarrow (\mathrm{Gr}^{n-1}\,\mathcal{X})[-n+1].$$

Let this be denoted $C_\bullet(\mathcal{X})$.

To provide the backwards direction, we appeal to Theorem 1.3.3.2 in [9]. (Again, to parse Theorem 1.3.3.2 with our conventions, swap $\geq$ and $\leq$.) In particular, let us consider $\mathcal{A} = \mathcal{C}^{\heartsuit}$, and consider the identity functor $\mathrm{Id}_{\mathcal{C}^{\heartsuit}} \colon \mathcal{C}^{\heartsuit} \longrightarrow \mathcal{C}^{\heartsuit}$, which is right exact. Under the equivalence described in 1.3.3.2, there exists some (right t-exact) functor $\mathcal{F} \colon \mathsf{D}^{-}(\mathcal{C}^{\heartsuit}) \longrightarrow \mathcal{C}$ such that $\tau^{(\geq 0)} \circ \mathcal{F}|_{\mathcal{C}^{\heartsuit}} = \mathrm{Id}_{\mathcal{C}^{\heartsuit}}$. Let this $\mathcal{F}$ be the backwards direction, which shall send a connective complex $C_\bullet$ to $\mathcal{F}(C_\bullet) \in \mathcal{C}$; moreover, we equip this object of $\mathcal{C}$ with a $\mathbb{N}$-filtration by considering stupid truncations on the complex $C_\bullet$.

Now consider the functor $\mathcal{G}$ from $\mathsf{Ch}^{(\leq 0)}(\mathcal{C}^{\heartsuit})$ to $\mathcal{C}$ by sending objects of form $\mathrm{Gr}^{-\bullet}(\mathcal{X})[\bullet]$ to $X = \mathcal{X}(0)$. Clearly, when restricted to $\mathcal{C}^{\heartsuit}$, this is the identity functor. Therefore, by the equivalence described in 1.3.3.2, we must have $\mathcal{G} \cong \mathcal{F}$. In particular, one has $\mathcal{F}(C_\bullet(\mathcal{X})) \cong \mathcal{G}(C_\bullet(\mathcal{X})) \cong X$. Hence this gives equivalence in one direction.

Note well that, from the way our proof worked above, not only is there *some* backwards direction functor exhibiting the equivalence, *any* $\mathcal{F}$ satisfying the right conditions (most importantly $\tau^{(\geq 0)} \circ \mathcal{F}|_{\mathcal{C}^{\heartsuit}} = \mathrm{Id}_{\mathcal{C}^{\heartsuit}}$) will work. Also, as it turns out, for our later application, this direction will be all we need, but the other direction, that $C_\bullet(\mathcal{F}(B_\bullet)) = B_\bullet$, is also obvious from construction (recall we equip $\mathcal{F}(B_\bullet)$ with the stupid truncation). ∎

Note well that, in the case of $\mathcal{C} = \mathsf{D}^{-}(\mathcal{A})$ for some abelian category $\mathcal{A}$, we have $\mathcal{C}^{\heartsuit} \cong \mathcal{A}$ and we can take $\mathcal{F}$ to simply be the identity functor from $\mathsf{Ch}^{(\leq 0)}(\mathcal{A})$ to $\mathsf{D}^{-}(\mathcal{A})$. In fact, if we are guaranteed that the chain complexes in $\mathsf{Ch}^{(\leq 0)}(\mathcal{A})$ in consideration are bounded on the left as well, then we can consider $\mathcal{C} = \mathsf{D}^{0}(\mathcal{A})$. In the last section, we will apply this to the situation $\mathcal{C} = \mathsf{D}^{0}(\mathcal{O})$.

8. Back to Category $\mathcal{O}$

We now apply the previous categorical discussions to representation theory.

8.1. **Giving the Derived Category of $\mathcal{O}$ a Filtration.**

In our case, let $\mathcal{O}$ be the standard BGG category, and let $\mathsf{D}(\mathcal{O})$ be the derived category thereof. We write $\mathsf{D}^{?}(\mathcal{O})$, where ? is either the empty string, $+$, $-$, or 0, for the derived category, the bounded-below derived category, the bounded-above derived category, and the derived category which is bounded on both sides ('doubly-bounded'?). Where statements are true no matter which type of derived category we pick, we will put a ? in the superscript. For our poset, consider $\overline{\mathfrak{h}^*} \coloneqq \mathfrak{h}^* \cup \{\lambda_\infty\}$, endowed with a partial order via $\mu \leq \lambda \iff \lambda - \mu \in Q_+$ and by declaring λ_∞ to be the unique maximal/final element. Let us endow $\mathsf{D}^{?}(\mathcal{O})$ with a $\overline{\mathfrak{h}^*}$-filtration $\mathcal{F} \colon \overline{\mathfrak{h}^*} \longrightarrow \mathsf{TriCat}$ by defining

$$\mathsf{D}^{?}(\mathcal{O})^{\leq \lambda} \coloneqq \mathsf{D}^{?}(\mathcal{O}^{\leq \lambda})$$

[47] Another word for Fib here is Cocone, or maybe just Ne, since the two 'co's cancel out.

to be the full subcategory whose objects are complexes whose terms have weights at most λ. Note well that this notion is well-defined under quasiisomorphisms since weights spaces are preserved in a sequence of $U\mathfrak{g}$-modules. Of course, $\mathsf{D}^?(\mathcal{O})^{\leq\lambda_\infty}$ is defined to be $\mathsf{D}^?(\mathcal{O})$.

First we must show that $\mathsf{D}^?(\mathcal{O})^{\leq\lambda}$ so defined is an actual fully faithful triangulated subcategory (a full additive subcategory which is stable under isomorphisms and cones and which is fixed by shifts). But this is easy to see. Firstly $\mathcal{O}^{\leq\lambda}$ is clearly abelian, since the only thing to really check is that it is closed under finite direct sums, which is fine since $\mathrm{Wt}(M\oplus N)=\mathrm{Wt}(M)\cup\mathrm{Wt}(N)$. Secondly $\mathsf{D}^?(\mathcal{O}^{\leq\lambda})$ is a full subcategory since the property of having weight at most something is clearly closed under isomorphisms, shifts, and cones (which are defined as $\mathrm{Cofib}(M^\bullet\xrightarrow{f}N^\bullet):=M[1]^\bullet\oplus N^\bullet$ with $d_{\mathrm{Cofib}}:=\begin{pmatrix}\mathrm{d}_{M[1]} & 0\\ f[1] & \mathrm{d}_N\end{pmatrix}$).

Then ι_λ is the inclusion functor $\mathsf{D}^?(\mathcal{O}^{\leq\lambda})\longrightarrow\mathsf{D}^?(\mathcal{O})$. Note well that $\iota_{\lambda_\infty}=\mathrm{Id}_{\mathsf{D}^?(\mathcal{O})}$ is the identity on $\mathsf{D}^?(\mathcal{O})$.

Next let us describe what $\mathsf{D}^?(\mathcal{O})^{<\lambda}$ is.

Lemma. Let ? be either the empty string, $+$, $-$, or 0. The smallest thick triangulated subcategory containing $\mathsf{D}^?(\mathcal{O})^{\leq\mu}$ for every $\mu<\lambda$ is the full subcategory whose objects are given by

$$\mathsf{D}^?(\mathcal{O})^{<\lambda}=\{M^\bullet:\mathrm{Wt}\,M^{(k)}<\lambda\ \forall\,k\}.$$

In other words, $\mathsf{D}^?(\mathcal{O})^{<\lambda}=\mathsf{D}^?(\mathcal{O}^{<\lambda})$.

Proof. Note $\{M^\bullet:\mathrm{Wt}\,M^i<\lambda\ \forall\,i\}$ is thick since $M^\bullet=M_1^\bullet\oplus M_2^\bullet\implies\mathrm{Wt}\,M^\bullet=\mathrm{Wt}\,M_1^\bullet\cup\mathrm{Wt}\,M_2^\bullet$, so $M^\bullet\in\mathsf{D}^?(\mathcal{O}^{<\lambda})$ implies $\mathrm{Wt}\,M_1^\bullet\subseteq\mathrm{Wt}\,M^\bullet<\lambda$. Of course, $\mathsf{D}^?(\mathcal{O}^{<\lambda})$ is a triangulated subcategory for the same reasons as in the preceding discussion. Clearly, as λ is the smallest weight greater than all μ for which $\mu<\lambda$ (tautologically), this is the smallest such category. ■

Now that we have the $<$ and the $\leq$, we can form the Verdier quotient. Let

$$\mathsf{D}^?(\mathcal{O})^{=\lambda}:=\mathsf{D}^?(\mathcal{O}^{\leq\lambda})/\mathsf{D}^?(\mathcal{O}^{<\lambda})$$

be the Verdier quotient. As it will turn out, if $?=+$, then this is secretly just vector spaces. (One might be able to extend this for other values of ?, but to be safe we will prove only the statements presented in this paper.)

8.2. **Reflections, Colocalizations, and Pieces of the Associated Graded.** But before we show that $\mathsf{D}^+(\mathcal{O})^{=\lambda}$ is secretly vector spaces, now that we have the basic structure of a filtered triangulated category, let us check that $\mathsf{D}^0(\mathcal{O})$ indeed satisfies the assumptions we put forth earlier (inclusions are reflective, quotients are colocalizations (as it turns out, they are also localizations)). The first is that the inclusions i_λ are reflective:

Lemma. The inclusion functor

$$i_\lambda\colon\mathsf{D}^?(\mathcal{O}^{\leq\lambda})\longrightarrow\mathsf{D}^?(\mathcal{O})$$

has a left adjoint $i_\lambda^\perp$.

Proof. This follows from the weak Vopěnka's principle, as Theorem 6.22 in [1]: Assuming weak Vopěnka's principle, every full subcategory of a locally presentable category[48] $\mathcal{C}$ closed in $\mathcal{C}$ under limits is reflective in $\mathcal{C}$. Due again to Corollary 1.2.1.6 of [9], we conclude the existence of such an adjoint. ■

We should perhaps remark that, in the case of the maximal element of $\overline{\mathfrak{h}^*}$, we clearly have $\iota_{\lambda_\infty}^\perp=\mathrm{Id}_{\mathsf{D}^?(\mathcal{O})}$.

To show the second assumption that quotients are colocalizations is met, first we will show

[48] See here for the definition of a locally presentable category. We are fine because Vermas generate.

Proposition. When we take $? = +$,
$$\mathsf{D}^+(\mathcal{O})^{=\lambda} \cong \mathsf{D}^+(\mathsf{Vec}).$$
In fact, the Verdier quotient functor $p_\lambda \colon \mathsf{D}^+(\mathcal{O}^{\leq\lambda}) \longrightarrow \mathsf{D}^+(\mathcal{O})^{=\lambda} \cong \mathsf{D}^+(\mathsf{Vec})$ looks like
$$p_\lambda = \mathrm{RHom}_{\mathfrak{b}^+}(\mathbb{C}_\lambda, \square) = \mathrm{RHom}_{\mathfrak{g}}(M_\lambda, \square) = C^\bullet(\mathfrak{n}^+ : \square)^\lambda.$$

Before launching into the proof of this, let us give a few brief comments on derived functors, which we learned mostly from [4]. Given a left-exact functor $\mathcal{F} \colon \mathcal{A} \longrightarrow \mathcal{B}$, one obtains a triangulated functor $\mathrm{R}\mathcal{F} \colon \mathsf{D}^+(\mathcal{A}) \longrightarrow \mathsf{D}^+(\mathcal{B})$ by
$$\mathrm{R}\mathcal{F} = \mathrm{Lan}_{\pi^+_{\mathcal{A}}}(\pi^+_{\mathcal{B}} \circ \mathsf{K}^+\mathcal{F}) \colon \mathsf{D}^+(\mathcal{A}) \longrightarrow \mathsf{D}^+(\mathcal{B}),$$
where $\pi^+_{\mathcal{A}} \colon \mathsf{K}^+(\mathcal{A}) \longrightarrow \mathsf{D}^+(\mathcal{A})$ and $\pi^+_{\mathcal{B}} \colon \mathsf{K}^+(\mathcal{B}) \longrightarrow \mathsf{D}^+(\mathcal{B})$ are the quotient functors and $\mathsf{K}^+(\mathcal{F}) \colon \mathsf{K}^+(\mathcal{A}) \longrightarrow \mathsf{K}^+(\mathcal{B})$ is given by applying $\mathcal{F}$ on complexes term-wise. Similarly, for $\mathcal{F} \colon \mathcal{A} \longrightarrow \mathcal{B}$ a right-exact functor, one obtains a triangulated functor
$$\mathrm{L}\mathcal{F} = \mathrm{Ran}_{\pi^-_{\mathcal{A}}}(\pi^-_{\mathcal{B}} \circ \mathsf{K}^-\mathcal{F}) \colon \mathsf{D}^-(\mathcal{A}) \longrightarrow \mathsf{D}^-(\mathcal{B}).$$

In the case of hom, recall (see for example [4]) that an inner hom in $\mathsf{Ch}(\mathcal{A})$ is given by $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$, whose terms are
$$\mathrm{Hom}^{(n)}(A^\bullet, B^\bullet) = \prod_{k\in\mathbb{Z}} \mathrm{Hom}(A^{(k)}, B^{(k+n)}),$$
with differential given by
$$\mathrm{d}^n_{\mathrm{Hom}}(f) = \mathrm{d}_B \circ f - (-1)^n f \circ \mathrm{d}_A.$$
If $\mathcal{A}$ is an abelian category with enough projectives, by taking $A^\bullet$ to be a projective resolution of some desired object and restricting the above to $\mathsf{D}^+(\mathcal{A})$, this becomes the right derived hom.

We return to the proof of the proposition. We emphasize that, throughout this proof, $\mathfrak{n} = \mathfrak{n}^+$. We also remark that, in the Chevalley-Eilenberg complex for computing RHom below, when we take the λ-th weight space, we are considering $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{\wedge k}, \square)$ as a $\mathfrak{b}$-representation via the adjoint action of $\mathfrak{b}$ on $\mathfrak{n}$ and the given action on $\square$, namely utilizing the construction that $\mathfrak{g} \circlearrowright \mathrm{Hom}_{\mathbb{C}}(M, N)$ via $\rho_N(\xi) \circ \square - \square \circ \rho_1(\xi)$. Another way to think about this is $\mathrm{Hom}_{\mathbb{C}}(M, N) \cong M^* \otimes_{\mathbb{C}} N$ (if $\dim M < \infty$, which is true for our purposes), and in general $M \otimes_{\mathbb{C}} N$ carries the representation $\rho_M(\xi) \otimes 1 + 1 \otimes \rho_N(\xi)$. Indeed, by basic linear algebra,
$$\mathrm{Wt}(M \otimes_{\mathbb{C}} N) = \mathrm{Wt}(M) + \mathrm{Wt}(N),$$
$$\mathrm{Wt}(\mathrm{Hom}_{\mathbb{C}}(M, N)) = \mathrm{Wt}(N) - \mathrm{Wt}(M),$$
where the symbols above mean pairwise sums/differences.

Proof of Proposition. Let $\mathbb{C}_\lambda$ denote the one-dimensional representation of $\mathfrak{b}$ on which $\mathfrak{n} = \mathfrak{n}^+$ acts by zero and $\mathfrak{h}$ acts by λ. Consider the functor
$$\mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, \square) \colon \mathcal{O}^{\leq\lambda} \longrightarrow \mathsf{Vec}.$$
It is easy to see directly that $\mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, \square) = (\square^{\mathfrak{n}})^\lambda$, and it is also easy to see that $\mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, \square) = \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, \mathrm{Hom}_{\mathfrak{g}}(U\mathfrak{g}, \square)) = \mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}_\lambda \otimes_{\mathfrak{b}} U\mathfrak{g}, \square) = \mathrm{Hom}_{\mathfrak{g}}(M_\lambda, \square)$, so that
$$\mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, \square) = (\square^{\mathfrak{n}})^\lambda = \mathrm{Hom}_{\mathfrak{g}}(M_\lambda, \square).$$
Let us consider the right-derived version of this map and call it π_λ:
$$\pi_\lambda \coloneqq \mathrm{RHom}_{\mathfrak{b}}(\mathbb{C}_\lambda, \square) \colon \mathsf{D}^+(\mathcal{O}^{\leq\lambda}) \longrightarrow \mathsf{D}^+(\mathsf{Vec}).$$
As taking λ-weight spaces is exact, this means we are looking at
$$\mathrm{RHom}_{\mathfrak{b}}(\mathbb{C}_\lambda, \square) = (\mathrm{R}\square^{\mathfrak{n}})^\lambda = C^\bullet(\mathfrak{n} : \square)^\lambda,$$
namely taking the λ-weight space of the Lie algebra cochains of $\mathfrak{n}$ (the cochains pick up a $\mathfrak{b}$-action from the $\mathfrak{b}$-action on $\square$ and $\mathfrak{n}$, as exhibited by the inner hom complex later). In fact, we can say more explicitly what this is, once we pick the standard projective resolution $P^\bullet(\mathbb{C}_0) = U\mathfrak{n} \otimes_{\mathbb{C}} \mathfrak{n}^{\wedge -\bullet}$ of the trivial representation $\mathbb{C}_0 \in \mathsf{Rep}\,\mathfrak{n}$ given by
$$\cdots \longrightarrow U\mathfrak{n} \otimes \mathfrak{n}^{\wedge 2} \longrightarrow U\mathfrak{n} \otimes \mathfrak{n}^{\wedge 1} \longrightarrow U\mathfrak{n} \longrightarrow \mathbb{C}_0$$

with differential maps

$$\mathrm{d}_n\colon x\otimes(\xi_1\wedge\cdots\wedge\xi_n)\longmapsto\sum_i(-1)^{i+1}x\xi_i\otimes\xi_{\wedge[n]\setminus i}+\sum_{i<j}(-1)^{i+j}x\otimes\big([\xi_i,\xi_j]\wedge\xi_{\wedge[n]\setminus i,j}\big).$$

Indeed,

$$\begin{aligned}\mathrm{RHom}_{\mathfrak{b}}(\mathbb{C}_\lambda,\square)&=(\mathrm{R}\square^{\mathfrak{n}})^\lambda\\&=\mathrm{RHom}_{\mathfrak{n}}(\mathbb{C}_0,\square)^\lambda\\&=\mathrm{Hom}^\bullet_{\mathfrak{n}}(P^\bullet(\mathbb{C}_0),\square)^\lambda\\&=\mathrm{Hom}^\bullet_{\mathbb{C}}(\mathfrak{n}^{\wedge-\bullet},\square)^\lambda\\&=\prod_{k\in\mathbb{N}}\mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{\wedge k},\square^{(\bullet-k)})^\lambda.\end{aligned}$$

Just so there is no misunderstanding, $\prod_{k\in\mathbb{N}}\mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{\wedge k},\square^{(\bullet-k)})^\lambda$ is the complex whose n-th term is given by

$$\prod_{k\in\mathbb{N}}\mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{\wedge k},\square^{(n-k)})^\lambda$$

and whose differentials are those induced by that of $P^\bullet(\mathbb{C}_0)$, that of $\square$, and that of the inner hom construction recalled prior to this proof.

Firstly, note that

$$\mathsf{D}^+(\mathcal{O}^{<\lambda})\subseteq\operatorname{Ker}\pi_\lambda,$$

which is easy to see since $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{\wedge k},M^{(n-k)})$ admits a $\mathfrak{g}$-action from that of $M^{(n-k)}$ and that of $\mathfrak{n}^{\wedge k}$; if $M^\bullet\in\mathsf{D}^+(\mathcal{O}^{<\lambda})$, then in particular

$$\mathrm{Wt}\,\mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{\wedge k},\square^{(n-k)})=\mathrm{Wt}(\square^{(n-k)})-\mathrm{Wt}(\mathfrak{n}^{\wedge k})=\mathrm{Wt}(\square^{(n-k)})-\left\{\sum_{\alpha\in S}\alpha\right\}_{\substack{|S|=k\\S\subseteq\Phi_+}},$$

where we recall that

$$\mathrm{Wt}\,\mathfrak{n}^{\wedge k}=\left\{\sum_{\alpha\in S}\alpha\right\}_{\substack{|S|=k\\S\subseteq\Phi_+}};$$

as $\mathrm{Wt}\,\square^{(n-k)}<\lambda$ and clearly $\left\{\sum_{\alpha\in S}\alpha\right\}_{\substack{|S|=k\\S\subseteq\Phi_+}}\geq 0\in\mathfrak{h}^*$, it is clear that $\mathrm{Wt}\,\mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{\wedge k},\square^{(n-k)})<\lambda$ also. Hence in particular the λ-weight space is empty.

Hence, by the universal property of the Verdier quotient, π_λ factors through p_λ.

Moreover, since $\mathcal{O}^{\leq\lambda}$ (respectively $\mathcal{O}^{<\lambda}$) is generated by Vermas M_μ with $\mu\leq\lambda$ (respectively $\mu<\lambda$), we know $\mathsf{D}^+(\mathcal{O}^{\leq\lambda})$ (respectively $\mathsf{D}^+(\mathcal{O}^{<\lambda})$) is generated by shifts of Vermas whose highest weight is at most λ (respectively less than λ). Recall that right adjoints preserve limits and that left adjoints preserve colimits. Recall in general category theory that an object X is said to be "compact" if $\mathrm{Hom}(X,\square)$ preserves colimits. Also recall that compact objects in the category of R-modules are precisely the finitely-presented modules. Hence, as objects of $\mathcal{O}$ are finitely-generated, we have that in particular $\pi_\lambda=\mathrm{RHom}_{\mathfrak{g}}(M_\lambda,\square)=\mathrm{RHom}_{\mathfrak{b}}(\mathbb{C}_\lambda,\square)$ preserves colimits. Hence, as p_λ is the functor sending $\mathsf{D}^?(\mathcal{O}^{<\lambda})$ (and nothing else) to (something isomorphic to) zero, to show that $p_\lambda=\pi_\lambda$, it suffices to check that $\pi_\lambda(M_\mu)=0$ for any $\mu<\lambda$ and $\pi_\lambda(M_\lambda)\neq 0$. Indeed, the former we have already seen is true in the previous paragraph, and the latter is true since

$$\pi_\lambda(M_\lambda)=\mathrm{RHom}_{\mathfrak{g}}(M_\lambda,M_\lambda)\simeq\mathrm{Ext}^\bullet_{\mathfrak{g}}(M_\lambda,M_\lambda),$$

where at degree 0 we know $\mathrm{Ext}^0_{\mathfrak{g}}(M_\lambda,M_\lambda)=\mathrm{Hom}_{\mathfrak{g}}(M_\lambda,M_\lambda)=\mathbb{C}\neq 0$. Hence we conclude. ■

Note that, in the course of our proof above, we have seen that the Lie algebra cochain complex can be written explicitly as

$$C^\bullet(\mathfrak{n}^+:\square)^\lambda=\prod_{k\in\mathbb{N}}\mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{+,\wedge k},\square^{(\bullet-k)})^\lambda,$$

with differentials induced from those of $P^\bullet(\mathbb{C}_0)$, $\square$, and the inner hom construction.

We can also take $? = 0$:

Proposition. When we take $? = 0$,

$$\mathsf{D}^0(\mathcal{O})^{=\lambda} \cong \mathsf{D}^0(\mathsf{Vec}).$$

In fact, the Verdier quotient functor $p_\lambda\colon \mathsf{D}^0(\mathcal{O}^{\leq\lambda}) \longrightarrow \mathsf{D}^0(\mathcal{O})^{=\lambda} \cong \mathsf{D}^0(\mathsf{Vec})$ still looks like

$$p_\lambda = \mathrm{RHom}_{\mathfrak{b}^+}(\mathbb{C}_\lambda, \square) = \mathrm{RHom}_{\mathfrak{g}}(M_\lambda, \square) = C^\bullet(\mathfrak{n}^+ : \square)^\lambda.$$

Proof. In the above proof, we defined π_λ to be $\prod_{k\in\mathbb{N}} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{\wedge k}, \square^{(\bullet - k)})^\lambda$. The only thing to check is that, when restricted to $\mathsf{D}^0(\mathcal{O}^{\leq\lambda})$, π_λ so defined actually lands inside $\mathsf{D}^0(\mathsf{Vec})$. But this is obvious precisely since $\mathfrak{g}$, and therefore $\mathfrak{n}^+$, is finite-dimensional – indeed, the finite-dimensionality of $\mathfrak{n}^+$ puts a bound on the range of k in the product above, and since $\square \in \mathsf{D}^0(\mathcal{O}^{\leq\lambda})$ is assumed bounded on both sides to begin with, this also puts a bound on the range of $\bullet$. Hence π_λ does land in $\mathsf{D}^0(\mathsf{Vec})$.

Once we have that, the rest of the proof is the same as the previous proof. ■

Since in our proof above we have seen that p_λ is constructed as a right derived hom functor, one could easily expect the Verdier quotient in this case to moreover be a colocalization. Indeed, by taking $? = 0$,

Proposition. The functor $p_\lambda\colon \mathsf{D}^0(\mathcal{O}^{\leq\lambda}) \longrightarrow \mathsf{D}^0(\mathsf{Vec})$ admits a left adjoint, given by

$$p_\lambda^\perp = \square \overset{\mathrm{L}}{\otimes} M_\lambda\colon \mathsf{D}^0(\mathsf{Vec}) \longrightarrow \mathsf{D}^0(\mathcal{O}^{\leq\lambda}).$$

Since vector spaces are free, we can drop the L.

In particular, we get a functor

$$i_\lambda p_\lambda^\perp = \square \otimes_{\mathbb{C}} M_\lambda\colon \mathsf{D}^0(\mathsf{Vec}) \longrightarrow \mathsf{D}^0(\mathcal{O}).$$

Proof. This follows immediately from our previous proofs. Again, since $\mathfrak{n}^+$ is finite-dimensional, the resolution $P^\bullet(\mathbb{C}_0)$ of the trivial $\mathfrak{n}^+$-representation $\mathbb{C}_0$ is bounded, and so its (left-derived) tensor product with M_λ is also bounded, and therefore lands in $\mathsf{D}^0(\mathcal{O}^{\leq\lambda})$. Once we have that the left derived tensor product lands correctly, that it is left-adjoint to the right derived hom is immediate. (To see this, for example one can take the form $p_\lambda = \mathrm{RHom}_{\mathfrak{g}}(M_\lambda, \square)$.) ■

Perhaps we should add that one way to interpret the expression $\square \otimes_{\mathbb{C}} M_\lambda$ is to consider M_λ as a chain complex concentrated in the zeroth degree, with zero differentials everywhere.

We can almost see something of the shape $i_\lambda p_\lambda^\perp p_\lambda i_\lambda^\perp$ now; indeed, in the above proposition we see the first half is tensor with Verma. The remaining half is given in the proposition below. In the following, we also see that p_λ is not only a colocalization but also a localization (i.e. admits a right adjoint). We remind the reader that $\square^\dagger$ denotes the contragredient dual, and in particular $M_\lambda^\dagger$ denotes the (contragredient) Verma module.

Proposition. Moreover, the functor $p_\lambda\colon \mathsf{D}^0(\mathcal{O}^{\leq\lambda}) \longrightarrow \mathsf{D}^0(\mathcal{O})^{=\lambda} \cong \mathsf{D}^0(\mathsf{Vec})$ admits a right adjoint, given by

$$p_\lambda^\top = M_\lambda^\dagger \overset{\mathrm{L}}{\otimes} \square\colon \mathsf{D}^0(\mathsf{Vec}) \longrightarrow \mathsf{D}^0(\mathcal{O}^{\leq\lambda}).$$

Again, since vector spaces are free, we can drop the L.

$p_\lambda i_\lambda^\perp$ is described by the λ-weight spaces of Lie algebra chains of $\mathfrak{n}^-$, namely

$$p_\lambda i_\lambda^\perp = C_\bullet(\mathfrak{n}^- : \square)^\lambda = \mathrm{RHom}_{\mathfrak{g}}(\square, M_\lambda^\dagger)^*\colon \mathsf{D}^0(\mathcal{O}) \longrightarrow \mathsf{D}^0(\mathsf{Vec}).$$

To streamline the proof of this proposition, let us first state a lemma. (Also, in the lemma below, D^0 might be able to be replaced by any other superscript on boundedness (or lack thereof).)

Lemma.
$$C_\bullet(\mathfrak{n}^- : \Box)^\lambda = C^\bullet(\mathfrak{n}^+ : \Box)^\lambda : \mathsf{D}^0(\mathcal{O}^{\le\lambda}) \longrightarrow \mathsf{D}^0(\mathsf{Vec}).$$
It is important here that restrict to $\mathcal{O}^{\le\lambda}$; this is in general not true in all of $\mathcal{O}$.

Incidentally, the two complexes can be written explicitly: they are
$$C^n(\mathfrak{n}^+ : \Box)^\lambda = \prod_{k\in\mathbb{N}} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{+,\wedge k}, \Box^{(n-k)})^\lambda$$
and
$$C_n(\mathfrak{n}^- : \Box)^\lambda = \prod_{k\in\mathbb{N}} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{-,\wedge k}, \Box^{(n+\dim\mathfrak{n}^- - k)})^{\lambda+2\varrho}.$$
In particular, the Verdier quotient functor $p_\lambda : \mathsf{D}^0(\mathcal{O}^{\le\lambda}) \longrightarrow \mathsf{D}^0(\mathsf{Vec})$ looks like
$$p_\lambda = \mathrm{RHom}_{\mathfrak{b}^+}(\mathbb{C}_\lambda, \Box) = \mathbb{C}_\lambda^- \overset{\mathrm{L}}{\otimes}_{\mathfrak{b}^-} \Box = \mathrm{RHom}_{\mathfrak{g}}(M_\lambda, \Box) = C^\bullet(\mathfrak{n}^+ : \Box)^\lambda = C_\bullet(\mathfrak{n}^- : \Box)^\lambda;$$
the content here is in the last equality, which in general is only true on $\mathsf{D}^0(\mathcal{O}^{\le\lambda})$ as opposed to all of $\mathsf{D}^0(\mathcal{O})$.

Proof. To prove this, we recall the following fact, presented here as it was stated (Lemma A.9.1) in [12]:

Lemma (Raskin)**.** Let $\mathfrak{g}$ be any finite-dimensional Lie algebra. Then there is a canonical isomorphism of functors:
$$C_\bullet(\mathfrak{g} : \Box) \cong C^\bullet\Big(\mathfrak{g} : \Box \otimes \big(\det(\mathfrak{g})[\dim\mathfrak{g}]\big)\Big).$$

Rather than use this lemma in this form, we will appeal to the form of it as written in the fourth footnote of [12], on page 3:
$$C_\bullet(\mathfrak{n} : \Box) \cong C^\bullet(\mathfrak{n} : \Box)[\dim\mathfrak{n}] \otimes \det(\mathfrak{n}).$$

Let us apply this to the case of $\mathfrak{n}^-$. Recall $\det\mathfrak{n}^- \coloneqq \mathfrak{n}^{-,\wedge\dim\mathfrak{n}^-}$ is a one-dimensional representation of $\mathfrak{b}^-$ on which $\mathfrak{h}$ acts by the only weight $\sum_{\alpha\in\Phi_-}\alpha = -2\varrho$. Hence
$$\begin{aligned} C_\bullet(\mathfrak{n}^- : \Box)^\lambda &= \big(C^\bullet(\mathfrak{n}^- : \Box)[\dim\mathfrak{n}^-] \otimes \det(\mathfrak{n}^-)\big)^\lambda \\ &= C^\bullet(\mathfrak{n}^- : \Box)^{\lambda+2\varrho}[\dim\mathfrak{n}^-] \otimes \det(\mathfrak{n}^-)^{-2\varrho} \\ &= \prod_{k\in\mathbb{N}} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{-,\wedge k}, \Box^{(\bullet-k)})^{\lambda+2\varrho}[\dim\mathfrak{n}^-], \end{aligned}$$
where the last equality is since $\det(\mathfrak{n}^-)^{-2\varrho}$ is a one-dimensional vector space. This shows the last claim in the lemma. The second claim about $C^n(\mathfrak{n}^+ : \Box)^\lambda$ was shown earlier.

Now let us compare the two. Recall that right adjoints preserve limits and that left adjoints preserve colimits. Recall in general category theory that an object X is said to be “compact” if $\mathrm{Hom}(X, \Box)$ preserves colimits. Also recall that compact objects in the category of R-modules are precisely the finitely-presented modules. Hence, as objects of $\mathcal{O}$ are finitely-generated, we have that in particular $\mathrm{RHom}_{\mathfrak{g}}(M_\lambda, \Box) = C^\bullet(\mathfrak{n}^+ : \Box)^\lambda$ preserves colimits. That $C_\bullet(\mathfrak{n}^- : \Box)^\lambda$ preserves colimits is obvious, since it can be written as
$$C_\bullet(\mathfrak{n}^- : \Box)^\lambda = \mathbb{C}_\lambda^- \overset{\mathrm{L}}{\otimes}_{\mathfrak{b}^-} \Box.$$

Hence, as Vermas generate $\mathsf{D}^0(\mathcal{O}^{\le\lambda})$, it suffices to check that the two complexes are the same on Vermas of highest weight at most λ. Indeed, if $\mu < \lambda$, then
$$C_\bullet(\mathfrak{n}^- : M_\mu)^\lambda = \prod_{k\in\mathbb{N}} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{-,\wedge k}, M_\mu^{(\bullet+\dim\mathfrak{n}^- - k)})^{\lambda+2\varrho},$$
where each hom space has weights
$$\mathrm{Wt}\,\mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{-,\wedge k}, M_\mu^{(\bullet+\dim\mathfrak{n}^- - k)}) = \mathrm{Wt}\, M_\mu^{(\bullet+\dim\mathfrak{n}^- - k)} - \mathrm{Wt}\,\mathfrak{n}^{-,\wedge k} = \mathrm{Wt}\, M_\mu^{(\bullet+\dim\mathfrak{n}^- - k)} + \left\{\sum_{\alpha\in S}\alpha\right\}_{\substack{|S|=k \\ S\subseteq\Phi_+}};$$

clearly the highest weight this can ever attain is $\mu + 2\varrho$ (by taking $k = |\Phi_+|$), which is strictly less than $\lambda + 2\varrho$. Hence

$$C_\bullet(\mathfrak{n}^- : M_\mu)^\lambda = 0 \quad \forall\, \mu < \lambda.$$

On cochains, we can see

$$C^\bullet(\mathfrak{n}^+ : M_\mu)^\lambda = \prod_{k \in \mathbb{N}} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{+,\wedge k}, M_\mu^{(\bullet - k)})^\lambda,$$

where the hom space has weights

$$\mathrm{Wt}\,\mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{+,\wedge k}, M_\mu^{(\bullet-k)}) = \mathrm{Wt}\, M_\mu^{(\bullet-k)} - \mathrm{Wt}\,\mathfrak{n}^{+,\wedge k} = \mathrm{Wt}\, M_\mu^{(\bullet-k)} - \left\{ \sum_{\alpha \in S} \alpha \right\}_{\substack{|S|=k \\ S \subseteq \Phi_+}};$$

again clearly the highest weight this can attain is μ (by taking $k = 0$), which is strictly less than λ. Hence

$$C^\bullet(\mathfrak{n}^+ : M_\mu)^\lambda = 0 \quad \forall\, \mu < \lambda$$

also, and thus the two agree on $\mathcal{O}^{<\lambda}$.

Similarly, if $\mu = \lambda$, the above analysis shows that

$$C_\bullet(\mathfrak{n}^- : M_\lambda)^\lambda = \prod_{k \in \mathbb{N}} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{-,\wedge k}, M_\lambda^{(\bullet + \dim \mathfrak{n}^- - k)})^{\lambda + 2\varrho},$$

where each hom space has weights

$$\mathrm{Wt}\,\mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{-,\wedge k}, M_\lambda^{(\bullet+\dim \mathfrak{n}^- -k)}) = \mathrm{Wt}\, M_\lambda^{(\bullet+\dim \mathfrak{n}^- -k)} - \mathrm{Wt}\,\mathfrak{n}^{-,\wedge k} = \mathrm{Wt}\, M_\lambda^{(\bullet+\dim \mathfrak{n}^- -k)} + \left\{ \sum_{\alpha \in S} \alpha \right\}_{\substack{|S|=k \\ S \subseteq \Phi_+}};$$

clearly the only way to obtain the weight $\lambda + 2\varrho$ is by taking the highest weight of M_λ and $k = |\Phi_+|$. So $C_\bullet(\mathfrak{n}^- : M_\lambda)^\lambda$ has only one nonzero term, concentrated at $n = 0$, which is given by the tensor of the highest weight vector of M_λ and the lowest weight vector of $\mathfrak{n}^{-,\wedge \dim \mathfrak{n}^-}$:

$$C_\bullet(\mathfrak{n}^- : M_\lambda)^\lambda = \begin{cases} \mathbb{C} & \bullet = 0 \\ 0 & \text{else} \end{cases}.$$

On cochains, similarly one has

$$C^\bullet(\mathfrak{n}^+ : M_\lambda)^\lambda = \prod_{k \in \mathbb{N}} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{+,\wedge k}, M_\lambda^{(\bullet - k)})^\lambda,$$

where the hom space has weights

$$\mathrm{Wt}\,\mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{+,\wedge k}, M_\lambda^{(\bullet-k)}) = \mathrm{Wt}\, M_\lambda^{(\bullet-k)} - \mathrm{Wt}\,\mathfrak{n}^{+,\wedge k} = \mathrm{Wt}\, M_\lambda^{(\bullet-k)} - \left\{ \sum_{\alpha \in S} \alpha \right\}_{\substack{|S|=k \\ S \subseteq \Phi_+}};$$

again clearly the only way to obtain weight λ is by taking $k = 0$ and tensoring the highest weight vector of M_λ with the lowest weight vector of $\mathfrak{n}^{+,\wedge 0}$. Hence

$$C^\bullet(\mathfrak{n}^+ : M_\lambda)^\lambda = \begin{cases} \mathbb{C} & \bullet = 0 \\ 0 & \text{else} \end{cases}.$$

As the two complexes agree on all Vermas in $\mathcal{O}^{\leq \lambda}$, we conclude the lemma. ■

Now let us return to the proof of the latest proposition.

Proof of Proposition. First let us compute the right adjoint. Recall (for instance from Lemma 3.19 of [3]) that, for any M which is $\mathfrak{h}$-semisimple with finite-dimensional weight spaces (and in particular for $M \in \mathcal{O}$), one has

$$\mathrm{Hom}_{\mathfrak{g}}(M, M_\lambda^\dagger) \cong {M_{\mathfrak{n}^-}}^{\lambda,*},$$

where the $\mathfrak{n}^-$ in the subscript denotes the coinvariant space $M/\mathfrak{n}^- M$ and the last $\square^*$ denotes linear dual.

Hence let us consider

$$\square_{\mathfrak{n}^-}{}^{\lambda,*} = \mathrm{Hom}_{\mathfrak{g}}(\square, M_\lambda^\dagger)\colon \mathcal{O} \longrightarrow \mathsf{Vec},$$

and consider its right derived functor:

$$\mathrm{RHom}_{\mathfrak{g}}(\square, M_\lambda^\dagger) = R(\square_{\mathfrak{n}^-}{}^{\lambda,*}) = (L\square_{\mathfrak{n}^-})^{\lambda,*} = C_\bullet(\mathfrak{n}^- : \square)^{\lambda,*},$$

where we recall that taking λ-weight spaces are duals are exact, but the latter is contravariant, which explains the swap from R to L.

From the previous lemma, we know that on $\mathsf{D}^0(\mathcal{O}^{\leq\lambda})$ the functor p_λ looks like

$$p_\lambda = C^\bullet(\mathfrak{n}^+ : \square)^\lambda = C_\bullet(\mathfrak{n}^- : \square)^\lambda = \mathrm{RHom}_{\mathfrak{g}}(\square, M_\lambda^\dagger)^*.$$

Hence, observe, for $M \in \mathsf{D}^0(\mathcal{O}^{\leq\lambda})$ and $V^\bullet \in \mathsf{D}^0(\mathsf{Vec})$,

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{D}^0(\mathsf{Vec})}(p_\lambda M^\bullet, V^\bullet) &\cong \mathrm{Hom}_{\mathsf{D}^0(\mathsf{Vec})}(\mathrm{RHom}_{\mathfrak{g}}(M^\bullet, M_\lambda^\dagger)^*, V^\bullet) \\
&\cong \mathrm{Hom}_{\mathsf{D}^0(\mathsf{Vec})}(V^{\bullet,*}, \mathrm{RHom}_{\mathfrak{g}}(M^\bullet, M_\lambda^\dagger)) \\
&\cong \mathrm{Hom}_{\mathfrak{g}}(V^{\bullet,*} \overset{\mathrm{L}}{\otimes} M^\bullet, M_\lambda^\dagger) \\
&\cong \mathrm{Hom}_{\mathfrak{g}}(V^{\bullet,*} \otimes_{\mathbb{C}} M^\bullet, M_\lambda^\dagger) \\
&\cong \mathrm{Hom}_{\mathfrak{g}}(M^\bullet, M_\lambda^\dagger \otimes_{\mathbb{C}} V^\bullet).
\end{aligned}$$

This shows that, indeed,

$$p_\lambda^\top = M_\lambda^\dagger \otimes \square.$$

We had seen in the previous lemma that $C_\bullet(n^- : M_\mu)^\lambda = 0$ for $\mu < \lambda$. In fact, something better is true: it is always zero, unless $\mu = \lambda$, in which case it is $\mathbb{C}$ concentrated in degree 0. This reduces easily to the standard fact about Ext groups between a Verma and a dual Verma, since we had seen in the paragraph before the last paragraph that $\mathrm{RHom}_{\mathfrak{g}}(\square, M_\lambda^\dagger) = C_\bullet(\mathfrak{n}^- : \square)^{\lambda,*}$, so that

$$C_\bullet(\mathfrak{n}^- : M_\mu)^{\lambda,*} = \mathrm{RHom}_{\mathfrak{g}}(M_\mu, M_\lambda^\dagger) \simeq \mathrm{Ext}^\bullet(M_\mu, M_\lambda^\dagger).$$

That this is zero for $\mu \neq \lambda$ and $\mathbb{C}[0]$ else is standard, for example Theorem 3.20 and Proposition 4.9 in [3].

Lastly it remains to prove that $p_\lambda i_\lambda^\perp = C_\bullet(\mathfrak{n}^- : \square)^\lambda$. This is now easy. Again, since (shifts of) Vermas generate $\mathsf{D}^0(\mathcal{O})$, and since $p_\lambda i_\lambda^\perp$ and $C^\bullet(\mathfrak{n}^- : \square)^\lambda$ preserve colimits, it suffices to check they agree on Vermas. We do this by using Yoneda to check them against any $V^\bullet \in \mathsf{D}^0(\mathsf{Vec})$. Compute:

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{D}^0(\mathsf{Vec})}(p_\lambda i_\lambda^\perp M_\mu, V^\bullet) &\cong \mathrm{Hom}_{\mathsf{D}^0(\mathcal{O})}(M_\mu, i_\lambda p_\lambda^\top V^\bullet) \\
&\cong \mathrm{Hom}_{\mathsf{D}^0(\mathcal{O})}(M_\mu, M_\lambda^\dagger \otimes V^\bullet) \\
&\cong \begin{cases} V^\bullet & \mu = \lambda \\ 0 & \mu \neq \lambda \end{cases}.
\end{aligned}$$

(In the last line again we appeal to the standard facts on Ext groups between Vermas and dual Vermas.) On the other hand, since we know $C_\bullet(\mathfrak{n}^- : M_\mu)^\lambda = \mathbb{C}[0]$ if $\mu = \lambda$ and 0 else, we immediately know

$$\mathrm{Hom}_{\mathsf{D}^0(\mathsf{Vec})}(C_\bullet(\mathfrak{n}^- : M_\mu)^\lambda, V^\bullet) = \begin{cases} V^\bullet & \mu = \lambda \\ 0 & \mu \neq \lambda \end{cases}.$$

This concludes. ■

Note that in the course of the above we have seen that $C_\bullet(\mathfrak{n}^- : \square)^{\lambda,*} = \mathrm{RHom}_{\mathfrak{g}}(\square, M_\lambda^\dagger)$; if these spaces were finite-dimensional, taking double duals would grant us

$$C_\bullet(\mathfrak{n}^- : \square)^\lambda = \mathrm{RHom}_{\mathfrak{g}}(\square, M_\lambda^\dagger)^*.$$

Let us briefly justify this. We have already seen that

$$C_n(\mathfrak{n}^- : \square)^\lambda = \prod_{k\in\mathbb{N}} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{-,\wedge k}, \square^{(n+\dim \mathfrak{n}^- - k)})^{\lambda+2\varrho}.$$

As usual, if $\dim M < \infty$ (but even if $\dim N = \infty$), we have $\mathrm{Hom}_{\mathbb{C}}(M,N) \cong M^* \otimes_{\mathbb{C}} N$, so that $\mathrm{Hom}_{\mathbb{C}}(M,N) = (M^* \otimes N)^\lambda = \bigoplus_{\mu+\nu=\lambda} M^{-\mu,*} \otimes_{\mathbb{C}} N^\nu$. Hence, for $M \in \mathsf{D}^0(\mathcal{O})$, we have

$$\mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{-,\wedge k}, M^{(n+\dim \mathfrak{n}^- - k)})^{\lambda+2\varrho} = \bigoplus_{\mu+\nu=\lambda+2\varrho} \mathfrak{n}^{-,\wedge k,-\mu,*} \otimes_{\mathbb{C}} M^{(n+\dim\mathfrak{n}^- -k),\nu}.$$

Since $\mathfrak{n}^-$ is finite-dimensional, each $-\mu$-weight space of its k-th wedge is also finite-dimensional, and moreover there are only finitely many $-\mu$ for which such a weight space is nonzero. So $\mathfrak{n}^{-,\wedge k,-\mu}$ and hence $\mathfrak{n}^{-,\wedge k,-\mu,*}$ is finite-dimensional. By definition, since $M^{(n+\dim\mathfrak{n}^- -k)} \in \mathcal{O}$, its ν-weight space $M^{(n+\dim\mathfrak{n}^- -k),\nu}$ is finite-dimensional. Hence $\mathfrak{n}^{-,\wedge k,-\mu,*} \otimes_{\mathbb{C}} M^{(n+\dim\mathfrak{n}^- -k),\nu}$ is finite-dimensional. As there are only finitely many μ that appear in the weight decomposition of $\mathfrak{n}^{-,\wedge k}$, there can only by finitely many ν such that $\mu+\nu = \lambda+2\varrho$, so in particular this direct sum is finite. Hence $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{-,\wedge k}, M^{(n+\dim\mathfrak{n}^- -k)})^{\lambda+2\varrho}$ is also finite-dimensional. But then, since there can only be finitely many k for which $\mathfrak{n}^{-,\wedge k}$ is nonzero (as $\mathfrak{n}^-$ is finite-dimensional), the product $\prod_{k\in\mathbb{N}} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{-,\wedge k}, \square^{(n+\dim\mathfrak{n}^- -k)})^{\lambda+2\varrho}$ is also finite-dimensional. Hence $C_n(\mathfrak{n}^- : \square)^\lambda$ is finite-dimensional, and therefore in particular taking double duals is fine.

In the most recent proven lemma we saw that $C_\bullet(\mathfrak{n}^- : \square)^\lambda$ and $C^\bullet(\mathfrak{n}^+ : \square)^\lambda$ agree on $\mathsf{D}^0(\mathcal{O}^{<\lambda})$, but we can say something that's true in general. (This claim isn't directly necessary for any of our arguments; we include it for the sake of completeness.)

Lemma. In fact, on $\mathsf{D}^0(\mathcal{O})$,

$$C_\bullet(\mathfrak{n}^- : \square)^\lambda = C^\bullet(\mathfrak{n}^+ : \square^\dagger)^{\lambda,*}.$$

Proof. This is directly seen by $C_\bullet(\mathfrak{n}^- : \square)^\lambda = \mathrm{RHom}_{\mathfrak{g}}(\square, M_\lambda^\dagger)^* = \mathrm{RHom}_{\mathfrak{g}}(M_\lambda, \square^\dagger)^* = C^\bullet(\mathfrak{n}^+ : \square^\dagger)^{\lambda,*}$. ■

8.3. **Summary.** To summarize,

$$\begin{tikzcd}
\mathsf{D}^0(\mathcal{O}) \arrow[r, bend left, "i_\lambda^\perp"] \arrow[r, phantom, "\perp"] & \mathsf{D}^0(\mathcal{O}^{\le\lambda}) \arrow[l, bend left, "i_\lambda"] \arrow[rr, "p_\lambda = C^\bullet(\mathfrak{n}^+ : \square)^\lambda = C_\bullet(\mathfrak{n}^- : \square)^\lambda"] & & \mathsf{D}^0(\mathsf{Vec}) \arrow[ll, bend right=60, "p_\lambda^\perp = \square\otimes M_\lambda"'] \arrow[ll, bend left=60, "p_\lambda^\top = M_\lambda^\dagger \otimes \square"]
\end{tikzcd}$$

Moreover,

$$\begin{aligned} p_\lambda &= \mathrm{RHom}_{\mathfrak{b}^+}(\mathbb{C}_\lambda^+, \square) = \mathrm{RHom}_{\mathfrak{g}}(M_\lambda, \square) = C^\bullet(\mathfrak{n}^+ : \square)^\lambda \\ &= \mathbb{C}_\lambda^- \overset{\mathrm{L}}{\otimes}_{\mathfrak{b}^-} \square = \mathrm{RHom}_{\mathfrak{g}}(\square, M_\lambda^\dagger)^* = C_\bullet(\mathfrak{n}^- : \square)^\lambda, \end{aligned}$$

and we have

$$\begin{aligned} i_\lambda p_\lambda^\perp &= \square \otimes_{\mathbb{C}} M_\lambda, \\ p_\lambda i_\lambda^\perp &= C_\bullet(\mathfrak{n}^- : \square)^\lambda. \end{aligned}$$

If one really wanted to explicitly write down the (co)chain complexes, one could do so by

$$C^\bullet(\mathfrak{n}^+ : \square)^\lambda = \prod_{k\in\mathbb{N}} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{+,\wedge k}, \square^{(\bullet-k)})^\lambda$$

and[49]

$$C_\bullet(\mathfrak{n}^- : \square)^\lambda = \prod_{k\in\mathbb{N}} \mathrm{Hom}_{\mathbb{C}}(\mathfrak{n}^{-,\wedge k}, \square^{(\bullet+\dim\mathfrak{n}^- - k)})^{\lambda+2\varrho} = C^\bullet(\mathfrak{n}^- : \square)^{\lambda+2\varrho}[\dim\mathfrak{n}^-];$$

the two are related by

$$C_\bullet(\mathfrak{n}^- : \square)^\lambda = C^\bullet(\mathfrak{n}^+ : \square^\dagger)^{\lambda,*}.$$

9. Functorial BGG

Now for the punchline. Now that we have the pieces of the associated graded, we can put them together and appeal to our work from earlier.

A brief explanation of the name: this is called a *functorial* BGG resolution since it is constructed by applying a functor, namely $\mathrm{Gr}^k\,\mathcal{I}\mathrm{d}$, to appropriate irreducible modules. In some sense this forms an "approximation" of L_λ. In particular, given a map $M \longrightarrow N$, we get a functorial map from the BGG resolution of M to that of N.

A remark on the input of Kostant: in the course of the following argument, we will appeal to Kostant's theorem on cohomology in order to say what the associated graded objects of L_λ are. Although some treatments, for example [5], prove Kostant by appealing to the BGG resolution, the argument we make here is not circular – after all, Kostant's paper [7] precedes that of BGG by fifteen years.

9.1. **The Associated Graded.** Consider the identity functor

$$\mathrm{Id}_{\mathsf{D}^0(\mathcal{O})} : \mathsf{D}^0(\mathcal{O}) \longrightarrow \mathsf{D}^0(\mathcal{O}).$$

From our theorem in Section 7.2, we know this admits a $\overline{\mathfrak{h}^*}$-filtration by

$$\mathcal{I}\mathrm{d}(\mu) = i_\mu i_\mu^\perp. \tag{$*$}$$

Note well that $\mathcal{I}\mathrm{d}(\lambda_\infty) = \mathrm{Id}_{\mathsf{D}^0(\mathcal{O})}$ by construction. Moreover, its associated graded pieces are given by

$$(\mathrm{Gr}\,\mathcal{I}\mathrm{d})(\mu) = i_\mu p_\mu^\perp p_\mu i_\mu^\perp = M_\mu \otimes C_\bullet(\mathfrak{n}^- : \square)^\mu. \tag{$*$}$$

This is a functor $\mathsf{D}^0(\mathcal{O}) \longrightarrow \mathsf{D}^0(\mathcal{O})$. (Of course, as we are in a derived category, equal signs shouldn't be treated too seriously; this just means they are quasiisomorphic.)

We have seen in our earlier work that $C_\bullet(\mathfrak{n}^- : \square)^\mu = C^\bullet(\mathfrak{n}^- : \square)^{\mu+2\varrho}[\dim\mathfrak{n}^-] = C^\bullet(\mathfrak{n}^+ : \square^\dagger)^{\mu,*}$. Hence this is actually saying

$$\mathrm{Gr}^\mu\,\mathcal{I}\mathrm{d} = M_\mu \otimes_{\mathbb{C}} C_\bullet(\mathfrak{n}^- : \square)^\mu = M_\mu \otimes_{\mathbb{C}} C^\bullet(\mathfrak{n}^- : \square)^{\mu+2\varrho}[\dim\mathfrak{n}^-] = M_\mu \otimes_{\mathbb{C}} \mathrm{RHom}_{\mathfrak{g}}(\square, M_\mu^\dagger)^* = M_\mu \otimes_{\mathbb{C}} C^\bullet(\mathfrak{n}^+ : \square^\dagger)^{\mu,*}. \tag{$*$}$$

By taking cohomology[50], this is the same as[51]

$$\mathrm{Gr}^\mu\,\mathcal{I}\mathrm{d} = M_\mu \otimes_{\mathbb{C}} H_\bullet(\mathfrak{n}^- : \square)^\mu = M_\mu \otimes_{\mathbb{C}} H^\bullet(\mathfrak{n}^- : \square)^{\mu+2\varrho}[\dim\mathfrak{n}^-] = M_\mu \otimes_{\mathbb{C}} \mathrm{Ext}^\bullet_{\mathsf{D}^0(\mathcal{O})}(\square, M_\mu^\dagger)^* = M_\mu \otimes_{\mathbb{C}} H^\bullet(\mathfrak{n}^+ : \square^\dagger)^{\mu,*}. \tag{$*$}$$

Now, recall by the linkage principle that Ext groups between modules of different blocks of $\mathcal{O}$ are identically zero. Hence, when we apply $\mathrm{Gr}^\mu\,\mathcal{I}\mathrm{d}$ to a $M \in \mathcal{O}$, only the direct summand of M lying inside $\mathcal{O}^{\vartheta_\lambda}$ for $\lambda \in W \circ \mu$ is picked up. In particular, for $M \in \mathcal{O}^{\vartheta_\lambda}$,

$$(\mathrm{Gr}^\mu\,\mathcal{I}\mathrm{d})|_{\mathsf{D}^0(\mathcal{O}^{\vartheta_\lambda})} = \begin{cases} M_{w\circ\lambda} \otimes_{\mathbb{C}} \mathrm{Ext}^\bullet_{\mathsf{D}^0(\mathcal{O})}(\square, M_{w\circ\lambda}^\dagger)^* = M_{w\circ\lambda} \otimes_{\mathbb{C}} H^\bullet(\mathfrak{n}^+ : \square^\dagger)^{w\circ\lambda,*} & \mu \in W \circ \lambda \\ 0 & \text{else} \end{cases}. \tag{$*$}$$

This vanishing outside of $W \circ \lambda$ motivates us to try to change the $\overline{\mathfrak{h}^*}$-filtration to a $(W \circ \lambda)$-filtration; for the moment let us indulge in this and heuristically carry this idea out. The idea is, when applied to a

[49]Note that the sign of the differentials don't really matter since we work in the derived category, so we can just bring things like $[\dim\mathfrak{n}^-]$ into the exponent.

[50]By this we mean to say that e.g. $C_\bullet(\mathfrak{n}^- : \square)^\mu = H_\bullet(\mathfrak{n}^- : \square)^\mu$ are the same in the derived category $\mathsf{D}^0(\mathsf{Vec})$, and therefore $M_\mu \otimes_{\mathbb{C}} C_\bullet(\mathfrak{n}^- : \square)^\mu = M_\mu \otimes_{\mathbb{C}} H_\bullet(\mathfrak{n}^- : \square)^\mu$ are the same in $\mathsf{D}^0(\mathcal{O})$.

[51]Recall that when we say $\mathrm{RHom}_{\mathfrak{g}}$, we are taking $\mathrm{Hom}_{\mathfrak{g}} = \mathrm{Hom}_{\mathcal{O}}$ and deriving it on $\mathcal{O}$; hence the result is $\mathrm{Ext}_{\mathcal{O}}$ as opposed to $\mathrm{Ext}_{\mathfrak{g}}$, which are different.

$M \in \mathcal{O}^{\vartheta_\lambda}$, we obtain a $(W \circ \lambda)$-filtration as opposed to a $\overline{\mathfrak{h}^*}$-filtration. In fact, we could take this a step further and try to get a $\mathbb{N}$-filtration by

$$(\mathrm{Gr}^k \mathcal{I}\mathrm{d})|_{\mathsf{D}^0(\mathcal{O}^{\vartheta_\lambda})} = \bigoplus_{\ell(w)=k} M_{w\circ\lambda} \otimes_{\mathbb{C}} H^\bullet(\mathfrak{n}^+ : \square^\dagger)^{w\circ\lambda,*} = \bigoplus_{\ell(w)=k} M_{w\circ\lambda} \otimes_{\mathbb{C}} \mathrm{Ext}^\bullet_{\mathsf{D}^0(\mathcal{O})}(\square, M^\dagger_{w\circ\lambda})^*.$$

Let us now venture to make this happen. First let us modify the situation to obtain a $(W \circ \lambda)$-filtration. Consider the member of $\mathsf{Fun}(\mathsf{D}^0(\mathcal{O}^{\vartheta_\lambda}), \mathsf{D}^0(\mathcal{O}))$ given by

$$\mathrm{Id}|_{\mathsf{D}^0(\mathcal{O}^{\vartheta_\lambda})} \colon \mathsf{D}^0(\mathcal{O}^{\vartheta_\lambda}) \longrightarrow \mathsf{D}^0(\mathcal{O}),$$

and let us endow it with a $(W \circ \lambda)$-filtration given by[52]

$$\mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^{\vartheta_\lambda})}(w \circ \lambda) = i_{w\circ\lambda} i^\perp_{w\circ\lambda}, \tag{$*$}$$

where the maps between different terms of the filtration are just those in the original filtration. This then has associated graded $\mathrm{Gr}^{w\circ\lambda}(\mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^{\vartheta_\lambda})}) = \mathrm{Fib}\Big(i_{w\circ\lambda} i^\perp_{w\circ\lambda} \to \lim_{u\circ\lambda \to w\circ\lambda} i_{u\circ\lambda} i^\perp_{u\circ\lambda}\Big)$. However, since we know that $\mathrm{Gr}^\mu(\mathcal{I}\mathrm{d}) = \mathrm{Fib}\Big(i_\mu i^\perp_\mu \to \lim_{\nu\to\mu} i_\nu i^\perp_\nu\Big) = 0$ for $\mu \notin W \circ \lambda$, we can conclude that $i_\mu i^\perp_\mu = \lim_{\nu\to\mu} i_\nu i^\perp_\nu$ for $\mu \notin W \circ \lambda$. From this it is easy to see[53] that $\lim_{u\circ\lambda\to w\circ\lambda} i_{u\circ\lambda} i^\perp_{u\circ\lambda} = \lim_{\mu \to w\circ\lambda} i_\mu i^\perp_\mu$, so that $\mathrm{Gr}^{w\circ\lambda}(\mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^{\vartheta_\lambda})}) = \mathrm{Fib}\Big(i_{w\circ\lambda} i^\perp_{w\circ\lambda} \to \lim_{u\circ\lambda\to w\circ\lambda} i_{u\circ\lambda} i^\perp_{u\circ\lambda}\Big) = \mathrm{Fib}\Big(i_{w\circ\lambda} i^\perp_{w\circ\lambda} \to \lim_{\mu\to w\circ\lambda} i_\mu i^\perp_\mu\Big) = \mathrm{Gr}^{w\circ\lambda}\mathcal{I}\mathrm{d} = i_{w\circ\lambda} p^\perp_{w\circ\lambda} p_{w\circ\lambda} i^\perp_{w\circ\lambda}$. Hence, we can see that the filtration we have endowed $\mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^{\vartheta_\lambda})}$ with actually has associated graded

$$\mathrm{Gr}^{w\circ\lambda}(\mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^{\vartheta_\lambda})}) = \mathrm{Gr}^{w\circ\lambda}\mathcal{I}\mathrm{d} = M_{w\circ\lambda} \otimes_{\mathbb{C}} \mathrm{Ext}^\bullet_{\mathsf{D}^0(\mathcal{O})}(\square, M^\dagger_{w\circ\lambda})^* = M_{w\circ\lambda} \otimes_{\mathbb{C}} H^\bullet(\mathfrak{n}^+ : \square^\dagger)^{w\circ\lambda,*}. \tag{$*$}$$

Let us carry this even one step further and obtain a $\mathbb{N}$-filtration. Following our setup in the previous paragraph, now add the additional requirement that λ is a regular weight, i.e. $\left|\varpi^{-1}(\vartheta)\right| = |W|$ (here ϖ denotes the map $\varpi \colon \mathfrak{h}^* \longrightarrow \mathfrak{h}^* /\!\!/ W$). Since only the choice of central character matters, we may well assume λ is ϱ-dominant[54], i.e. $\lambda \in -\varrho + \Lambda_+$, by taking the maximal member of the preimage of ϑ, namely

$$\lambda = \max \varpi^{-1}(\vartheta).$$

Now endow $\mathrm{Id}|_{\mathsf{D}^0(\mathcal{O}^\vartheta)}$ with a filtration given by[55]

$$\mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^\vartheta)}(k) = \bigoplus_{\ell(w)=k} i_{w\circ\lambda} i^\perp_{w\circ\lambda}, \tag{$*$}$$

where the maps between different terms of the filtration are the same as in the original filtration if $w \circ \lambda$ and $u \circ \lambda$ are comparable and zero else. This has associated graded objects as $\mathrm{Gr}^k(\mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^\vartheta)}) = \mathrm{Fib}\Big(\bigoplus_{\ell(w)=k} i_{w\circ\lambda} i^\perp_{w\circ\lambda} \to \bigoplus_{\ell(u)=k+1} i_{u\circ\lambda} i^\perp_{u\circ\lambda}\Big)$; since we took λ to be the maximal representative of ϑ, we know $u \circ \lambda \lessdot w \circ \lambda$ happens precisely[56] when $\ell(u) = \ell(w) + 1$. Hence this is actually the same as $\mathrm{Gr}^k(\mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^\vartheta)}) = \mathrm{Fib}\Big(\bigoplus_{\ell(w)=k} i_{w\circ\lambda} i^\perp_{w\circ\lambda} \to \bigoplus_{\ell(u)=k+1} i_{u\circ\lambda} i^\perp_{u\circ\lambda}\Big) = \bigoplus_{\ell(w)=k} \mathrm{Fib}\Big(i_{w\circ\lambda} i^\perp_{w\circ\lambda} \to \lim_{u\circ\lambda\to w\circ\lambda} i_{u\circ\lambda} i^\perp_{u\circ\lambda}\Big) = \bigoplus_{\ell(w)=k} \mathrm{Gr}^{w\circ\lambda}(\mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^\vartheta)})$, or, in other words,

$$\mathrm{Gr}^k(\mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^\vartheta)}) = \bigoplus_{\ell(w)=k} \mathrm{Gr}^{w\circ\lambda}\mathcal{I}\mathrm{d} = \bigoplus_{\ell(w)=k} M_{w\circ\lambda} \otimes_{\mathbb{C}} \mathrm{Ext}^\bullet_{\mathsf{D}^0(\mathcal{O})}(\square, M^\dagger_{w\circ\lambda})^* = \bigoplus_{\ell(w)=k} M_{w\circ\lambda} \otimes_{\mathbb{C}} H^\bullet(\mathfrak{n}^+ : \square^\dagger)^{w\circ\lambda,*}. \tag{$*$}$$

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[52] We should be using a new symbol to denote this new filtration, but we write $\mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^{\vartheta_\lambda})}$ in an abuse of notation.

[53] Indeed, in this filtered limit, of course only terms $\lessdot w \circ \lambda$ matter, so by rewriting any $i_\mu i^\perp_\mu$ for $\mu \notin W \circ \lambda$ as $\lim_{\nu\to\mu} i_\nu i^\perp_\nu$, we then have ν is at least two levels lower that $w \circ \lambda$; this process effectively removes any $\mu \notin W \circ \lambda$ from the picture.

[54] Combined with the regularity assumption, which is equivalent to $\Phi^*(\lambda + \varrho) \not\ni 0$, this moreover forces $\lambda \in \Lambda_+$.

[55] Again, we should be using a new symbol to denote this new filtration, but we will use the same symbol in an abuse of notation.

[56] Also note well we assumed λ is regular.

With this in mind, we may in the future be somewhat sloppy with where we put our parentheses, e.g. simply writing $\mathrm{Gr}^k \mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^\vartheta)}$.

We want to apply this to a L_λ for $\lambda \in \Lambda_+$.

9.2. **Input from Kostant.** We now appeal to Kostant's theorem on cohomology in order to more explicitly say what $(\mathrm{Gr}^\mu \mathcal{I}\mathrm{d})(L_\lambda)$ and therefore $(\mathrm{Gr}^k \mathcal{I}\mathrm{d})(L_\lambda)$ is. Kostant's theorem was first proven in [7]; here we state this theorem as it was shown in [5], in Theorem 6.6:

Theorem (Kostant)**.** For $\lambda \in \Lambda_+$, one has the $\mathfrak{n}^+$-cohomology

$$H^k(\mathfrak{n}^+ : L_\lambda) = \bigoplus_{\ell(w)=k} \mathbb{C}_{w\circ\lambda}$$

and the $\mathfrak{n}^-$-cohomology

$$H^k(\mathfrak{n}^- : L_\lambda) = \bigoplus_{\ell(w)=k} \mathbb{C}_{-w\circ(-w_0\lambda)},$$

where w_0 is the unique longest element of W.

Let us apply $\mathrm{Gr}^\mu \mathcal{I}\mathrm{d}$ to L_λ for $\lambda \in \Lambda_+$. Recall that $L_\lambda^\dagger \cong L_\lambda$. This gives

$$(\mathrm{Gr}^\mu \mathcal{I}\mathrm{d})(L_\lambda) = M_\mu \otimes H^\bullet(\mathfrak{n}^+ : L_\lambda)^{\mu,*} = \begin{cases} M_{w\circ\lambda} \otimes {\mathbb{C}_{w\circ\lambda}}^{w\circ\lambda,*}[\ell(w)] = M_{w\circ\lambda}[\ell(w)] & \mu \in W\circ\lambda \\ 0 & \text{else} \end{cases},$$

where we note that since we are considering the linear dual of the weight space complex, though $H^\bullet(\mathfrak{n}^+ : L_\lambda)^{w\circ\lambda}$ would be concentrated in cohomological degree k, $H^\bullet(\mathfrak{n}^+ : L_\lambda)^{w\circ\lambda,*}$ is concentrated in degree $-k$.

Similarly, applying $\mathrm{Gr}^k \mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^\vartheta)}$ to L_λ (here $\vartheta = \varpi(\lambda)$), we have

$$(\mathrm{Gr}^k \mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^\vartheta)})(L_\lambda) = \bigoplus_{\ell(w)=k} M_{w\circ\lambda} \otimes H^\bullet(\mathfrak{n}^+ : L_\lambda)^{w\circ\lambda,*} = \bigoplus_{\ell(w)=k} M_{w\circ\lambda} \otimes {\mathbb{C}_{w\circ\lambda}}^{w\circ\lambda,*}[k] = \bigoplus_{\ell(w)=k} M_{w\circ\lambda}[k],$$

Note well that shifting this by $[-k]$ again will land it in $\mathsf{D}^0(\mathcal{O})^\heartsuit \cong \mathcal{O}$, i.e.

$$(\mathrm{Gr}^k \mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^\vartheta)})(L_\lambda)[-k] \in \mathsf{D}^0(\mathcal{O})^\heartsuit \cong \mathcal{O}.$$

In this footnote we perform an alternative computation[57].

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[57] In particular, the weights of $H^k(\mathfrak{n}^- : L_\lambda)$ are of form $-w\circ(-w_0\lambda)$. Recalling $\ell(w_0 w) = \ell(w_0) - \ell(w)$ and $w_0^{-1} = w_0$ and $w_0\varrho = -\varrho$, we can variable change $u \coloneqq ww_0$ to get $-w\circ(-w_0\lambda) = ww_0\lambda - w\varrho + \varrho = u\lambda - uw_0\varrho + \varrho = u\circ\lambda + 2\varrho$. Hence actually

$$H^k(\mathfrak{n}^- : L_\lambda) = \bigoplus_{\ell(w)=\dim\mathfrak{n}^- - k} \mathbb{C}_{w\circ\lambda+2\varrho}. \tag{$*$}$$

Now let us apply our associated graded to L_λ. This is

$$(\mathrm{Gr}^\mu \mathcal{I}\mathrm{d})(L_\lambda) = M_\mu \otimes H^\bullet(\mathfrak{n}^- : L_\lambda)^{\mu+2\varrho}[\dim\mathfrak{n}^-].$$

From Kostant, we know that the $(\mu+2\varrho)$-th weight space vanishes unless $\mu \in W\circ\lambda$. Hence this $\overline{\mathfrak{h}^*}$-filtration is secretly a $(W\circ\lambda)$-filtration.

We can moreover direct sum over all $\mu = w\circ\lambda$ for w of the same length k to obtain a $\mathbb{N}$-filtration:

$$(\mathrm{Gr}^k \mathcal{I}\mathrm{d})(L_\lambda) = \bigoplus_{\ell(w)=k} M_{w\circ\lambda} \otimes H^\bullet(\mathfrak{n}^- : L_\lambda)^{w\circ\lambda+2\varrho}[\dim\mathfrak{n}^-].$$

It is easy to see that $H^\bullet(\mathfrak{n}^- : L_\lambda)^{w\circ\lambda+2\varrho}[\dim\mathfrak{n}^-] = \mathbb{C}_{w\circ\lambda+2\varrho}[k]$ is a single vector space sitting in $-k$-th index. Hence

$$(\mathrm{Gr}^k \mathcal{I}\mathrm{d})(L_\lambda) = \bigoplus_{\ell(w)=k} M_{w\circ\lambda} \otimes \mathbb{C}_{w\circ\lambda+2\varrho}[k] = \bigoplus_{\ell(w)=k} M_{w\circ\lambda}[k] \tag{$*$}$$

is concentrated in index $-k$. Note well that shifting this by $[-k]$ again will land it in $\mathsf{D}^0(\mathcal{O})^\heartsuit$, i.e.

$$(\mathrm{Gr}^k \mathcal{I}\mathrm{d})(L_\lambda)[-k] \in \mathsf{D}^0(\mathcal{O})^\heartsuit \cong \mathcal{O}.$$

9.3. **Functorial BGG.** Hence we have an object $L_\lambda[0] \in \mathsf{D}^0(\mathcal{O})$ equipped with a $\mathbb{N}$-filtration $\mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^\vartheta)}(k)(L_\lambda)$ whose associated graded $(\mathrm{Gr}^k\,\mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^\vartheta)})(L_\lambda)$ is known. By the correspondence between $\mathbb{N}$-filtered objects and connective complexes, we conclude there is a connective chain complex $C^{\mathrm{BGG}}_\bullet(L_\lambda)$ whose terms are $\bigoplus_{\ell(w)=k} M_{w\circ\lambda}$. Moreover, by our work in Section 7.4, when we send it back to a $\mathbb{N}$-filtered object of $\mathsf{D}^0(\mathcal{O})$, we get the same complex $\bigoplus_{\ell(w)=\bullet} M_{w\circ\lambda}$ equipped with the stupid filtration; but this is supposed to be an equivalence, so we should have $\bigoplus_{\ell(w)=\bullet} M_{w\circ\lambda} \simeq L_\lambda[0]$ in $\mathsf{D}^0(\mathcal{O})$, which is to say the two complexes are quasiisomorphic. As $C^{\mathrm{BGG}}_\bullet(L_\lambda)$ is quasiisomorphic to a complex concentrated at degree 0, we conclude it must be exact. This is the BGG resolution.

Theorem (Functorial BGG)**.** For $\lambda \in \Lambda_+$, there is a resolution (*the* BGG resolution) of L_λ by

$$\begin{aligned} C^{\mathrm{BGG}}_\bullet(L_\lambda) &= \bigoplus_{\ell(w)=\bullet} (\mathrm{Gr}^{w\circ\lambda}\,\mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^{\vartheta_\lambda})})(L_\lambda)[-\bullet] = (\mathrm{Gr}^\bullet\,\mathcal{I}\mathrm{d}|_{\mathsf{D}^0(\mathcal{O}^{\vartheta_\lambda})})(L_\lambda)[-\bullet] \\ &= \bigoplus_{\ell(w)=\bullet} M_{w\circ\lambda} \end{aligned}$$

whose differentials are given by the octahedral axiom of triangulated categories.

10. Closing Remarks

10.1. **Uniqueness.** Though we have constructed a resolution of L_λ in the above story, we did not describe what the maps are, for they arose from the octahedral axiom, which is non-explicit. However, it is known that any two BGG resolutions (where a BGG resolution is one which terms are as prescribed by the Weyl character formula) must have the same differentials. Indeed, this story is told in Chapter 6 of [5], culminating in Section 6.8. Hence the classical and the functorial BGG resolutions we described are secretly the same.

10.2. **Further Questions.** There is a parallel of the above story with a *recollement* situation for the flag variety. By considering the flag variety as a stratified space, one can consider the categories of $\mathcal{D}$-modules on each piece; this forms a filtered triangulated category. The categorical story described above gives an associated object to a $\mathcal{D}$-module on the flag variety; by applying Beilinson-Bernstein localization, one gets back BGG.

I have not yet learned the Springer correspondence, but I wonder if one might get some analog of the BGG resolution for representations of Weyl groups by applying a similar idea to some other appropriate category of geometric objects. One of my next steps (I hope) will be to learn about Springer.

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– 李清照，《金石录后序》

"That with merely this I chronicle its beginning and its end, is also in hopes of serving as a warning to future generations of lovers of antiquity and elegance."

– Li Qingzhao, *Epilogue to Records on Metals and Stones*