NOTES ON SOME BASICS OF LIE SUPERALGEBRAS

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The color scheme for these pseminar notes is a green value of 0.420. These truncated notes on Lie superalgebras and their representations are for the first two talks of a seminar titled "Lie Superalgebras and Categorifcation" and run by Cailan Li and Alvaro Martinez. See the seminar website at here and the syllabus here. I have decided to take the first two talks to put on my website because they are basic and general (and complete) enough to maybe be of use to people. Unfortunately the seminar quickly moved beyond the basic representation theory and onto categorification, so we never covered things like linkage. If ever one day I finish this story and complete these notes, I will (try to) be sure to update them on my website.

Any mistakes are of course my own and not the speakers'. Things called "postmortem remarks" are made by myself after the fact and can therefore be completely wrong.

Contents

1. 01/25 – Lie Superalgebra Fundamentals (Cailan Li)

1.1. **Basic definitions.** We begin with some basic definitions.

Definition 1.1.1. A "super vector space", or "vector superspace"/"superspace", is a $\mathbb{Z}_2$-graded space $V = V_0 \oplus V_1$. Given a homogeneous vector $v \in V_i$, let $|v| = i \in \mathbb{Z}_2$ denote the "parity" of the vector.

Given a superspace V, let Π be the parity-reversing functor, namely $\Pi(V)_i = V_{i+1}$ for $i \in \mathbb{Z}_2$.

Cailan and the book use $V_{\overline{0}}$ and $V_{\overline{1}}$, but I will write only V_0 and V_1 for convenience. EDIT: Cailan agrees with me.

Definition 1.1.2. A "Lie superalgebra" is a superspace $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ equipped with a $\mathbb{Z}_2$-graded bilinear operation $[\square, \square]\colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ such that for all homogeneous x, y, z we have

- (skew-supersymmetry) $[x, y] = -(-1)^{|x||y|}[y, x]$;
- (superJacobi) $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$.

In particular, e.g. the bracket of a even and an odd thing is odd. It is not hard to see that by using skew-supersymmetry one can bring superJacobi into the more symmetric form

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]].$$

Example 1.1.3. If A is an associative superalgebra, then it can be made a Lie superalgebra by setting

$$[x, y] = xy - (-1)^{|x||y|}yx.$$

Now that we have defined superspaces, what are morphisms between them?

Definition 1.1.4. A map $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ between Lie superalgebras is a homomorphism if f is even (i.e. degree 0) and

$$f([x, y]) = [f(x), f(y)].$$

Example 1.1.5. If $\mathfrak{g}$ is a Lie superalgebra, then $\operatorname{End}\mathfrak{g}$ an associative superalgebra is moreover a Lie superalgebra by the previous example. The "adjoint representation" of $\mathfrak{g}$ is then

$$\begin{aligned} \operatorname{ad}\colon \mathfrak{g} &\longrightarrow \operatorname{End}\mathfrak{g} \\ x &\longmapsto [x, \square]. \end{aligned}$$

One can check that this is a legit homomorphism because of the superJacobi identity.

Postmortem remark: I think the 'End' here refers to not strict morphisms of superspaces, since $(\operatorname{End}\mathfrak{g})_1$ should be a thing also. This latter thing means degree 1 maps surely.

Remark: because the bracket is $\mathbb{Z}_2$-graded, the restriction to the even part actually lands as $\operatorname{ad}|_{\mathfrak{g}_0}\colon \mathfrak{g}_0 \longrightarrow \operatorname{End}\mathfrak{g}_1$, i.e. $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module via the adjoint action.

Here is the main character for this seminar, $\mathfrak{gl}(m|n)$. Let $V = V_0 \oplus V_1 \cong \mathbb{C}^{m|n}$ be a super vector space, where $V_0 = \mathbb{C}^m$ and $V_1 = \mathbb{C}^n$. Then

$$\mathfrak{gl}(m|n) \coloneqq \operatorname{End}\mathbb{C}^{m|n}$$

equipped with the bracket from the previous example. Fixing a basis for $\mathbb{C}^{m|n}$, we get a natural form to write things in, namely $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The even part looks like

$$\mathfrak{gl}(m|n)_0 \ni \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

and the odd part looks like

$$\mathfrak{gl}(m|n)_1 \ni \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

Evidently, as Lie algebras,

$$\mathfrak{gl}(m|n)_0 \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n),$$

and

$$\mathfrak{gl}(m|n)_1 \cong (\mathbb{C}^m \otimes \mathbb{C}^{n*}) \oplus (\mathbb{C}^{m*} \otimes \mathbb{C}^n)$$

as $\mathfrak{gl}(m|n)_0$-modules. Note that as a set, $\mathfrak{gl}(m|n) \cong \mathfrak{gl}(m+n)$, but it is equipped with a different bracket.

Given a matrix $g \in \mathfrak{gl}(m|n)$, when written in the standard form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ earlier, define

Definition 1.1.6. The "supertrace" is

$$\mathrm{str}(g) = \mathrm{tra}(A) - \mathrm{tra}(D).$$

Here are some facts.

Fact 1.1.7. (1) $\mathrm{str}([g,h]) = 0$ for all $g, h \in \mathfrak{gl}(m|n)$;
(2) $\mathfrak{sl}(m|n) = \{g \in \mathfrak{gl}(m|n) : \mathrm{str}(g) = 0\}$ is a Lie subsuperalgebra of $\mathfrak{gl}(m|n)$;
(3) $[\mathfrak{gl}(m|n), \mathfrak{gl}(m|n)] = \mathfrak{sl}(m|n)$.

See the book for proofs.

Another familiar structure is that of bilinear forms on a vector space. There is a super version of this also.

Definition 1.1.8. A bilinear form $\langle \square; \square \rangle$ on a superspace $V = V_0 \oplus V_1$ is "supersymmetric" if

$$\langle v; w \rangle = (-1)^{|v||w|} \langle w; v \rangle.$$

It is said to be even if $\langle \text{even}, \text{odd} \rangle = 0$.

We will mostly be concerned with basic Lie superalgebras in this seminar:

Definition 1.1.9. $\mathfrak{g}$ is a "basic Lie superalgebra", if it admits a nondegenerate even supersymmetric bilinear form.

In fact, according to Cailan, "80% of the time we will be concerned with $\mathfrak{gl}(m|n)$ and the rest 20% we will be concerned with $\mathfrak{sl}(m|n)$". Such things are (almost) always basic.

Lemma 1.1.10. $\mathfrak{gl}(m|n)$ and $\mathfrak{sl}(m|n)$ (except $(m,n) = (1,1)$ and $(2,1)$ for $\mathfrak{sl}$; $\mathfrak{gl}$ is always basic) are basic Lie superalgebras.

Proof. Let $\langle x, y \rangle = \mathrm{str}(xy)$. This works. ∎

1.2. **Structural things.** Now we discuss things like Cartan, roots, and other structural things.

1.2.1. *Cartan.*

Definition 1.2.1. Let $\mathfrak{g}$ be basic. Then a "Cartan subalgebra" is a Cartan subalgebra of $\mathfrak{g}_0$, and the "Weyl group" is the Weyl group of $\mathfrak{g}_0$.

In our main case $\mathfrak{gl}(m|n)$, the even part is $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$, i.e. the diagonal matrices. Let us denote

$$I(m|n) \coloneqq \{\overline{1}, \cdots, \overline{m}, 1, \cdots, n\},$$

endowed with a total order

$$\overline{1} < \cdots < \overline{m} < 0 < 1 < \cdots < n.$$

Then the Cartan can be written

$$\mathfrak{h}(\mathfrak{gl}(m|n)) = \bigoplus_{i \in I(m|n)} \mathbb{C} E_{ii}.$$

Note that

$$\langle E_{ii}; E_{jj} \rangle = \begin{cases} 1 & \overline{1} \le i = j \le \overline{m} \\ -1 & 1 \le i = j \le n \\ 0 & i \ne j \end{cases};$$

these minus signs come up because of the supertrace.

1.2.2. *Roots.* Now that we have a notion of the Cartan, it makes sense to ask for a root decomposition.

Definition 1.2.2. Let $\mathfrak{h}$ be the Cartan of $\mathfrak{g}$. For $\alpha \in \mathfrak{h}^*$, let

$$\mathfrak{g}^\alpha = \{g \in \mathfrak{g} : [h, g] = \alpha(h)g \;\; \forall\, h \in \mathfrak{h}\}.$$

Then the "root system" for $\mathfrak{g}$ is

$$\Phi = \{\alpha \in \mathfrak{h}^* : \mathfrak{g}^\alpha \neq 0\},$$

and we can define "even/odd roots" to be

$$\Phi_0 = \{\alpha \in \Phi : \mathfrak{g}^\alpha \cap \mathfrak{g}_0 \neq 0\},$$
$$\Phi_1 = \{\alpha \in \Phi : \mathfrak{g}^\alpha \cap \mathfrak{g}_1 \neq 0\}.$$

It is not obvious to me a priori that a root should be either even or odd, or indeed either. But thankfully in the basic case we can say something stronger structurally.

Theorem 1.2.3. Let $\mathfrak{g}$ be a basic Lie superalgebra. Then

(1)
$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha;$$
(2) $\langle \Box; \Box \rangle|_{\mathfrak{h}}$ is nondegenerate and W-invariant;
(3) $\dim \mathfrak{g}^\alpha = 1$ (this relies on nondegeneracy of the previous entry)
(4) Φ_0, Φ_1 are invariant under the action of W on $\mathfrak{h}^*$. (And therefore so is Φ.)

Note that the third fact tells us that in this basic case Φ_0 and Φ_1 are thankfully disjoint.

Let's say something about roots for $\mathfrak{gl}(m|n)$. In the case of $\mathfrak{gl}(m|n)$, by definition the Cartan subalgebra is contained in the even part. This implies that the superbracket is coincides with the usual Lie bracket if the first entry is in the Cartan, i.e. the adjoint action of the Cartan on $\mathfrak{gl}(m|n)$ is the same as that of the Cartan on $\mathfrak{gl}(m+n)$. So the roots of $\mathfrak{gl}(m|n)$ are the same as the roots of $\mathfrak{gl}(m+n)$, except with the additional information of a partition of the roots into even and odd things. Let's say what this partition is. Let $\delta_i, \varepsilon_j \in \mathfrak{h}^*$ for $i \in [m]$ and $j \in [n]$ be a dual basis to $E_{\bar{i}\bar{i}}$ and E_{jj} under $\langle \Box; \Box \rangle$. Let us also denote $\varepsilon_{\bar{i}} = \delta_i$. Then the even roots are

$$\Phi_0 = \{\varepsilon_i - \varepsilon_j : i \neq j \in I(m|n) \text{ with either } i, j > 0 \text{ or } i, j < 0\},$$

and the odd roots are

$$\Phi_1 = \{\delta_i - \varepsilon_j, \varepsilon_k - \delta_l : i, l \in [m],\; j, k \in [n]\}.$$

All of this is just a complicated way of saying that the parity of a root is the sum of the parities of the two things it is a difference of.

Because $\mathfrak{h} \cong \mathfrak{h}^*$ under $h \longmapsto \langle h; \Box \rangle$, the form $\langle \Box; \Box \rangle$ induces a form[1] $(\Box; \Box)$ which is also nondegenerate on $\mathfrak{h}^*$. One can easily verify that

$$(\delta_i; \delta_j) = \delta_{ij}, \quad (\varepsilon_i; \varepsilon_j) = -\delta_{ij}, \quad (\varepsilon_k, \delta_l) = 0.$$

In the case of superalgebras, roots can exhibit some weird behavior which don't arise in the usual cases. In particular they can have 'zero length', a phenomenon called isotropy.

Definition 1.2.4. A root $\alpha \in \Phi$ is "isotropic" if $(\alpha; \alpha) = 0$. Let $\overline{\Phi}_1$ be the set of isotropic odd roots.

It bears saying that isotropic automatically implies odd because even roots are actual roots of the Lie algebra $\mathfrak{g}_0$, and the Killing form is positive-definite on the $\mathbb{Q}$-span of Φ, so that in particular $(\alpha; \alpha) > 0$. (Maybe this argument needs a little checking; what is the relationship between $(\Box; \Box)$ and the Killing?)

Example 1.2.5. In the case $\mathfrak{gl}(1|1)$, consider the (only) odd root $\delta_1 - \varepsilon_1$. Compute $(\delta_1 - \varepsilon_1; \delta_1 - \varepsilon_1) = (\delta_1; \delta_1) + (\varepsilon_1; \varepsilon_1) = 1 - 1 = 0$. So the odd root has zero length...

[1]Postmortem remark, mostly for myself: So I guess this would be the dagger rather than the star in the way I learned Lie algebras. The key is that the star doesn't really make general sense in the context of superalgebras since the denominator $(\alpha; \alpha)$ might be zero.

Note that this example easily generalizes to show that all odd roots of $\mathfrak{gl}(m|n)$ are isotropic. So the moral here is that drawing the root picture for Lie superalgebras in general is rather dangerous because of these 'invisible' roots. So instead let us draw them as roots of $\mathfrak{gl}(m+n)$ instead.

1.2.3. *Positive/simple roots.* Now we discuss positive roots.

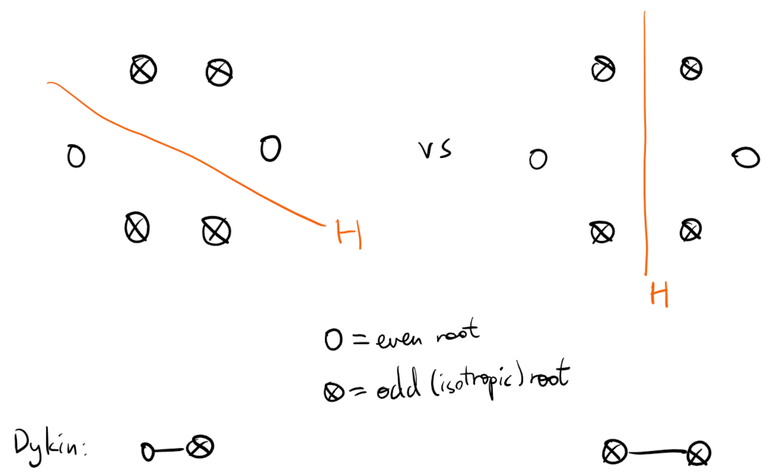
Definition 1.2.6. For $\mathfrak{g} = \mathfrak{gl}(m|n)$ basic, let H be a hyperplane in the picture for $\mathfrak{gl}(m+n)$ not containing any roots and K be the Killing form for $\mathfrak{gl}(m+n)$. Then define

$$\Phi^+(H) = \{\alpha \in \Phi : K(H, \alpha) > 0\}.$$

Let $\Sigma(H)$ be the set of simple roots of $\Phi^+(H)$, i.e. a 'fundamental system'. (Simple here still means not expressible as a positive linear combination of positive roots.)

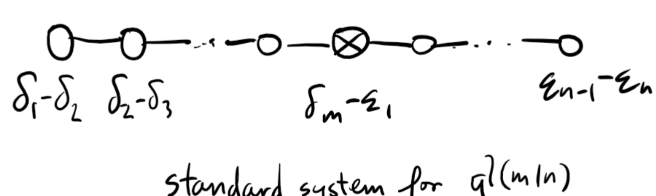
Let me say that Cailan and likely the book use $\Pi(H)$ rather than $\Sigma(H)$. As you might have suspected from the notational choice above, the choice of H actually matters here; different choices may not be conjugate to each other under the Weyl group.

Example 1.2.7. Take $\mathfrak{gl}(2|1)$. There are two odd roots, which are both isotropic, and one even root. Here's a picture:

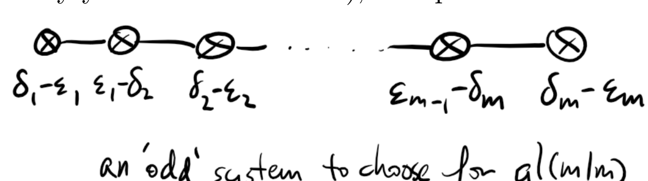


The simple roots, as usual, are the 'closest' ones to the hyperplane H. Note that on the left there is one even simple root and one odd simple root, whereas on the right both simple roots are odd. Hence the Weyl group, being $S_2 \times S_1$ and keeping Φ_i within itself, cannot bring one to the other.

Because of this poor behavior, people tend to stick to a prescribed standard for the simple roots. For $\mathfrak{gl}(m|n)$, this "standard system" is simply consecutive differences of 'diagonal entries'. In terms of Dynkin diagrams this looks like



You can look at nonstandard systems too. If $n = m$, you can have a fundamental system consisting of entirely odd roots (not sure why you would want that); as a picture this is



Now that we have a notion of positive roots, we can say what $\mathfrak{n}^+$ is. Given a choice of hyperplane H, define the nilpotent Lie algebras as follows.

Definition 1.2.8. Given a choice of H, define the "nilpotent" and "Borel" subalgebras as

$$\mathfrak{n}^+(H) = \bigoplus_{\alpha \in \Phi^+(H)} \mathfrak{g}^\alpha,$$
$$\mathfrak{n}^-(H) = \bigoplus_{\alpha \in \Phi^-(H)} \mathfrak{g}^\alpha,$$
$$\mathfrak{b}(H) = \mathfrak{h} \oplus \mathfrak{n}^+(H).$$

Warning: in this definition, it is *not* the case that the Borel is the *maximal* solvable subalgebra.

1.2.4. *Odd reflections.* Now let us discuss 'odd reflections'. Postmortem remark: As I understand it, this is supposed to atone for the failure of the Weyl group in Example 1.2.7.

Lemma 1.2.9 (Serganova)**.** Let $\mathfrak{g}$ be basic, Σ be a fundamental system for Φ^+, α an isotropic odd root. Then

$$\Phi^+_\alpha := \{-\alpha\} \cup \Phi^+ \setminus \{\alpha\},$$

i.e. Φ^+ except replace α with $-\alpha$, is also a set of positive roots with fundamental system given by

$$\Sigma_\alpha = \{-\alpha\} \cup \{\beta \in \Sigma : (\beta;\alpha) = 0,\ \beta \neq \alpha\} \cup \{\beta + \alpha : \beta \in \Sigma,\ (\beta;\alpha) \neq 0\},$$

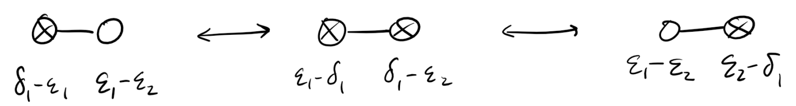
i.e. roughly leave β as is if $(\beta;\alpha) = 0$ and add α to it otherwise.

We can call this procedure r_α, in a satire of the s_α reflection for usual Lie algebras. This r_α is a map of sets, sending

$$r_\alpha \colon \alpha \longmapsto -\alpha,$$
$$\beta \longmapsto \begin{cases} \beta & (\beta;\alpha) = 0 \\ \beta + \alpha & (\beta;\alpha) \neq 0 \end{cases}.$$

Warning: unlike the s_α, this does *not* extend to a linear map.

Example 1.2.10. Consider $\mathfrak{gl}(1|2)$. If we think of this set-theoretically/pictorially as $\mathfrak{gl}(3)$, as we know, there are essentially three choices of hyperplanes in the root picture. Two of these were drawn in Example 1.2.7. The Dynkin pictures for these three choices are



Note well that if we start at the left and pick $\alpha = \delta_1 - \varepsilon_1$, then $\beta = \varepsilon_1 - \varepsilon_2$ has $(\varepsilon_1 - \varepsilon_2; \delta_1 - \varepsilon_1) = -(\varepsilon_1;\varepsilon_1) + (\varepsilon_2;\varepsilon_1) = 1 \neq 0$, so that $r_{\delta_1 - \varepsilon_1}$ tells us to add α to β which brings us to the middle picture. If we start at the middle and pick $\alpha = \delta_1 - \varepsilon_2$, then $(\varepsilon_1 - \delta_1; \delta_1 - \varepsilon_2) = -1 \neq 0$ again and $r_{\delta_1 - \varepsilon_2}$ brings us to the right.

In some sense the Weyl group action can be thought of as 'even reflections'. Then there is a theorem, also due to Serganova:

Theorem 1.2.11. The odd reflections r_α, as defined above, together with the Weyl action, is transitive on the set of fundamental systems.

So once you add in these r_α's you actually hit every possible choice of H.

I am giving the talk this week and will thus be unable to live-TeX. So I am dead-TeXing instead. For references see also Serganova's lecture notes and Brundan's survey.

Here's a brief remark that might come up later: the super-analog of ϱ, the sum of fundamental weights, is given by

$$\varrho = \frac{1}{2}\sum_{\alpha\in\Phi_0^+}\alpha - \frac{1}{2}\sum_{\beta\in\Phi_1^+}\beta.$$

Note that this depends on a choice of polarization/$\mathfrak{b}$. Also recall the constructions of Φ_α^+ and Σ_α from the odd reflections last time. In that construction, we might also define (the book calls this '$\mathfrak{b}^\alpha$' instead)

$$\mathfrak{b}_\alpha := \mathfrak{h} \oplus \bigoplus_{\beta\in\Phi_\alpha^+}\mathfrak{g}^\beta.$$

Let me also remind myself that the definition of the coroot to α here is such that $\langle h_\alpha; \mathfrak{h}\rangle = \alpha(\mathfrak{h})$. Think of $\mathfrak{gl}(m|n)$ and the supertrace.

2.1. **Highest weight theory.**

2.1.1. *$\mathcal{U}\mathfrak{g}$ and PBW.* As in the usual case, there is a notion of a universal enveloping algebra, denoted also $\mathcal{U}\mathfrak{g}$. It is defined identically by adding 'super' in front of everything, and is constructed through the usual means (tensor algebra modded out by stuff).

Definition 2.1.1. For $\mathfrak{g}$ a Lie superalgebra, its "universal enveloping (super)algebra" is an associative unital superalgebra equipped with a morphism of Lie superalgebras $\mathfrak{g} \longrightarrow \mathcal{U}\mathfrak{g}$ through which every other morphism of Lie superalgebras to an associative superalgebra factors.

In particular, as usual, representations of $\mathfrak{g}$ are precisely modules over $\mathcal{U}\mathfrak{g}$. As usual there is also the PBW[2] theorem.

Theorem 2.1.2 (PBW)**.** If $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$ are bases for $\mathfrak{g}_0$ and $\mathfrak{g}_1$ respectively, then the set

$$\left\{x_1^{a_1}\cdots x_m^{a_m}y_1^{b_1}\cdots y_n^{b_n} : a_i \in \mathbb{N},\ b_i \in \{0,1\}\right\}$$

is a basis of $\mathcal{U}\mathfrak{g}$.

(Of course $b_i \in \{0,1\}$ since the product of two odd things is even.) Then we can filter $\mathcal{U}\mathfrak{g}$ by setting $\mathcal{U}^{\le k}\mathfrak{g}$ to be the span of the above basis elements for which $\sum a_i + \sum b_j \le k$, and the graded superalgebra with respect to this grading is $\mathrm{Gr}\,\mathcal{U}\mathfrak{g} = \mathrm{S}\mathfrak{g}_0 \otimes \bigwedge\mathfrak{g}_1$.

2.1.2. *Solvable things.* Solvable Lie superalgebras are defined identically to the usual case:

Definition 2.1.3. A Lie superalgebra is "solvable" if $\mathfrak{g}^{(n)} = 0$ eventually, where $\mathfrak{g}^{(n)} := [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]$ and $\mathfrak{g}^{(0)} := \mathfrak{g}$.

Suppose $\mathfrak{g}$ is finite-dimensional solvable such that

$$[\mathfrak{g}_1, \mathfrak{g}_1] \subseteq [\mathfrak{g}_0, \mathfrak{g}_0].$$

Given[3] $\lambda \in \left(\mathfrak{g}_0/[\mathfrak{g}_0,\mathfrak{g}_0]\right)^*$, define $\mathbb{C}_\lambda = \mathbb{C}v_\lambda$ by setting

$$\mathfrak{g}_0 v_\lambda = \lambda(\mathfrak{g}_0)v_\lambda,$$
$$\mathfrak{g}_1 v_\lambda = 0.$$

We should be thinking of $\mathfrak{b}$ here of course, where $\mathfrak{b}_1 = \mathfrak{n}_1^+$ (since $\mathfrak{n}_0^+$ is in $[\mathfrak{b}_0, \mathfrak{b}_0]$, it also acts by zero). As it turns out such modules exhaust all finite-dimensional simples. Remark: representations of $\mathfrak{g}$ are supposed to be supermodules over superalgebras, and so it should come with a $\mathbb{Z}_2$-grading also. This

[2] Peanut Butter Welly

[3] Of course $\lambda \in \left(\mathfrak{g}_0/[\mathfrak{g}_0,\mathfrak{g}_0]\right)^*$ is equivalent to $\lambda \in \mathfrak{g}_0^*$ such that $\lambda([\mathfrak{g}_0, \mathfrak{g}_0]) = 0$.

above construction doesn't seem to specify whether this one-dimensional space lives in even or odd degree, but I'm pretty sure both are valid. So I presume that the following lemma means up to degree shift. The first page of this paper (which I did not read) makes me suspect that people typically think of this in degree zero.

Lemma 2.1.4. If $\mathfrak{g}$ is finite-dimensional solvable with $[\mathfrak{g}_1,\mathfrak{g}_1] \subseteq [\mathfrak{g}_0,\mathfrak{g}_0]$, then every finite-dimensional simple $\mathfrak{g}$-module is one-dimensional and exhausted by the $\mathbb{C}_\lambda$ construction above, for $\lambda \in \left(\mathfrak{g}_0/[\mathfrak{g}_0,\mathfrak{g}_0]\right)^*$.

Proof. Recall Lie's theorem[4]. Let us first show that any finite-dimensional simple $\mathfrak{g}$-module V is one-dimensional and in fact a quotient of $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathbb{C}v_\lambda$.

Indeed, note $\mathfrak{g}_0{}^{(n)} \subseteq \mathfrak{g}^{(n)}$ and $\mathfrak{g}$ solvable $\implies$ $\mathfrak{g}_0$ also solvable. Applying Lie's to $V|_{\mathfrak{g}_0}$, we get that there exists $\lambda \in \left(\mathfrak{g}_0/[\mathfrak{g}_0,\mathfrak{g}_0]\right)^*$ and $v_\lambda \in V$ such that $\mathfrak{g}_0 v_\lambda = \lambda(\mathfrak{g}_0)v_\lambda$, so that there is an inclusion of $\mathfrak{g}_0$-modules

$$\mathbb{C}v_\lambda \hookrightarrow V.$$

Under Frobenius $\operatorname{Hom}_{\mathfrak{g}}(\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathbb{C}v_\lambda, V) \cong \operatorname{Hom}_{\mathfrak{g}_0}(\mathbb{C}v_\lambda, V|_{\mathfrak{g}_0})$ this becomes some surjection (V being simple) of $\mathfrak{g}$-modules

$$\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathbb{C}v_\lambda \twoheadrightarrow V.$$

It remains to show that $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathbb{C}v_\lambda$ has a Jordan-Holder series whose quotients are all one-dimensional, for as V is simple, this would force V to also be one-dimensional.

So now let us show $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathbb{C}v_\lambda$ has a Jordan-Holder series whose quotients are all one-dimensional.

We do so explicitly. Indeed, note that $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathbb{C}v_\lambda \cong \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{g}_0} \mathbb{C}v_\lambda$ where $\mathcal{U}\mathfrak{g}$ acts by multiplication on the left factor; this in turn is isomorphism to $\bigwedge \mathfrak{g}_1 \otimes_{\mathbb{C}} \mathbb{C}v_\lambda$ as a $\mathfrak{g}_0$-module, where $\mathfrak{g}_0$ acts on $\mathfrak{g}_1$ by adjoint and therefore on $\bigwedge \mathfrak{g}_1 \otimes \mathbb{C}v_\lambda$ via Leibniz[5]. Hence apply Lie to the adjoint action $\mathfrak{g}_0 \circlearrowright \mathfrak{g}_1$, which gives a filtration $\mathfrak{g}_1 = Y_1 \supset \cdots \supset Y_n \supset 0$ of $\mathfrak{g}_1$; taking basis vectors $y_1, \cdots, y_n$ according to this filtration[6] such that $xy_i = \lambda_i(x)y_i \mod V_{i+1}$, we know that

$$\mathfrak{g}_0 \cdot y_i \subseteq \bigoplus_{j=i}^{n} \mathbb{C}y_i, \qquad [\mathfrak{g}_0,\mathfrak{g}_0] \cdot y_i \subseteq \bigoplus_{j=i+1}^{n} \mathbb{C}y_j,$$

where the action is via adjoint. Then consider an order on the obvious basis of $\bigwedge \mathfrak{g}_1$ given inductively by $\mathcal{B}_1 = \{1, y_1\}$ and $\mathcal{B}_k = \{\mathcal{B}_{k-1}, \mathcal{B}_{k-1}y_k\}$. This looks like a binary tree[7]; note that having higher numbers move you to the right, and up to reordering attaching a new number will also never move to you the left. Of course this gives a basis also of $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathbb{C}v_\lambda$; from left to right let us label this basis $v_1, \cdots, v_{2^n}$ and thus obtain a filtration

$$\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathbb{C}v_\lambda = W_1 \supset \cdots \supset W_{2^n} \supset 0,$$

where $W_i = \bigoplus_{j=i}^{2^n} \mathbb{C}v_j$. From staring at footnote 5 and knowing that $\mathfrak{g}_0$ hitting y_i will only move the index (nonstrictly) up, it is evident that this is a filtration of $\mathfrak{g}_0$-modules with one-dimensional quotients. But it is moreover a filtration of $\mathfrak{g}_1$-modules (and therefore $\mathfrak{g}$-modules); to see this[8], note

$$y_r y_{i_1} \cdots y_{i_k} v_\lambda = \begin{cases} -y_{i_1} y_r y_{i_2} \cdots y_{i_k} v_\lambda + [y_r, y_{i_1}] y_{i_2} \cdots y_{i_k} v_\lambda & r \neq i_1, \\ \frac{1}{2}[y_r, y_{i_1}] y_{i_2} \cdots y_{i_k} v_\lambda & r = i_1 \end{cases},$$

[4] Lie says that if $V \in \mathsf{Mod}^{\mathrm{fd}}\, \mathfrak{g}$ for $\mathfrak{g}$ solvable, then $\exists\, V = V_0 \supset \cdots \supset V_n = 0$ such that $\mathfrak{g}V_i \subseteq V_i$ and $\dim V_i/V_{i+1} = 1$. A corollary of this, sometimes also called Lie's theorem, is that for the same setup, $\exists\, 0 \neq v \in V$ such that $xv = \lambda(x)v \ \ \forall\, x \in \mathfrak{g}$ for some $\lambda \in \left(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]\right)^*$.

[5] This is since $xy_{i_1} \cdots y_{i_k} v_\lambda = [x, y_{i_1}] y_{i_2} \cdots y_{i_k} v_\lambda + y_{i_1}[x, y_{i_2}] y_{i_3} \cdots y_{i_k} v_\lambda + \cdot + y_{i_1} \cdots [x, y_{i_k}] v_\lambda + y_{i_1} \cdots y_{i_k} x v_\lambda$, where $xv_\lambda = \lambda(x)v_\lambda$.

[6] so that Y_i is spanned by $y_i, \cdots, y_n$

[7] Start from $\emptyset$ at level n; from level n the left branch does nothing and the right branch adds n from the left. As an example, for $n = 4$ this gives $\emptyset$, 1, 2, 12, 3, 13, 23, 123, 4, 14, 24, 124, 34, 134, 234, 1234. 'Moving left/right' will refer to this order.

[8] this is sort of an inductive argument

where note the cases can be combined. Of course if $r < i_1$ then we don't even need to do anything. If $r > i_1$ then we want to pass y_r to the right; this can be done at the cost of adding a term with the bracket, but $[y_r, y_{i_1}] \in [\mathfrak{g}_1, \mathfrak{g}_1] \subseteq [\mathfrak{g}_0, \mathfrak{g}_0]$ acts on y's by strictly raising the subscript, which[9] moves you to the right. If $r = i_1$, then we go down to $k-1$ elements of $\mathfrak{g}_1$ in front of v_λ, which might move us to the left; but $[y_r, y_{i_1}] \in [\mathfrak{g}_1, \mathfrak{g}_1] \subseteq [\mathfrak{g}_0, \mathfrak{g}_0]$ acting on $y_{i_2} \cdots y_{i_k} v_\lambda$ will move it back to the far right since $[\mathfrak{g}_0, \mathfrak{g}_0]$ strictly increases the subscript. This is the construction.

Now that we know V is a quotient of $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathbb{C}v_\lambda$, it is automatic that $\mathfrak{g}_0$ acts on it as we claimed, and moreover $\mathfrak{g}_1$ must act by zero since it changes parity and V is one-dimensional. ∎

Remark: note that if we started out the proof with a nonsimple V, we would get that some submodule of V (namely the image of the map from $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathbb{C}v_\lambda$) has 1-dimensional JH quotients and in particular must have a 1-dimensional submodule.

2.1.3. *Weights.* Let $\mathfrak{g}$ be basic; think $\mathfrak{gl}(m|n)$. Pick a positive system Φ^+, to which is then attached the information of $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Then the setting in the last subsubsection applies for the Borel $\mathfrak{b}$ since $\mathfrak{b}_1 = \mathfrak{n}_1^+$ and so $[\mathfrak{b}_1, \mathfrak{b}_1] = [\mathfrak{n}_1^+, \mathfrak{n}_1^+] \subseteq \mathfrak{n}_0^+ = [\mathfrak{h}, \mathfrak{n}_0^+] \subseteq [\mathfrak{b}_0, \mathfrak{b}_0]$. Hence $\mathbb{C}_\lambda$ is a representation of $\mathfrak{b}$ for $\lambda \in \mathfrak{h}^* = (\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}])^*$. Recall the notion of a highest weight module (HWM).

Proposition 2.1.5. For $\mathfrak{g}$ basic and $\mathfrak{b}$ a choice of Borel, any finite-dimensional simple is a $\mathfrak{b}$-HWM.

Proof. For any L a finite-dimensional simple $\mathfrak{g}$-module we know $L|_{\mathfrak{b}}$ contains some $\mathbb{C}_\lambda = \mathbb{C}v_\lambda$ by the remark at the end of the last section, so that by PBW L is spanned by the action of $\mathfrak{n}^-$ on v_λ, i.e. $L = \mathrm{Span}_{U\mathfrak{n}^-}^L v_\lambda$. Letting[10] $L^\mu := \{v \in L : \mathfrak{h}v = \mu(\mathfrak{h})v\}$ be the μ-weight space, this gives via brackets a weight space decomposition for L into $L = \bigoplus_{\mu \in \mathfrak{h}^*} L^\mu$, where as usual the weights appearing have $\lambda - \mu \in \mathbb{N}\Phi^+$. As usual, since this v_λ is killed by $\mathfrak{n}^+$, it is called a "highest weight vector with respect to $\mathfrak{b}$" or "$\mathfrak{b}$-highest weight vector". L cannot have another highest weight vector (of necessarily smaller weight) for it would then generate a submodule. ∎

Modules generated by a single such highest weight vector are called "highest weight modules", which as in the usual case are automatically indecomposable[11] and admit unique maximal submodules and unique simple quotients[12]. Also as usual, in a simple module there is *the* (up to scaling) highest weight vector[13]. We shorten 'highest weight module with respect to $\mathfrak{b}$' to '$\mathfrak{b}$-HWM', or simply 'HWM'. A 'highest weight module with respect to $\mathfrak{b}$ of highest weight λ' is shortened to '$\mathfrak{b}$-HWM(λ)'. As two simples of the same highest weight are necessarily isomorphic[14], *the* (finite-dimensional) simple of highest weight λ is denoted L_λ.

In the usual case as well as the super case, by picking different polarizations one gets different 'highest weights', but in the usual case all such highest weights are related to each other via the Weyl group. This is no longer true for the super case, as we will see in an example later. But we can still say what the extremal weights are. For $\alpha \in \Phi^+$ isotropic odd, let h_α be the coroot to α and pick $e_\alpha \in \mathfrak{g}^\alpha$ and $f_\alpha \in \mathfrak{g}^{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha$. The following is a straightforward calculation.

Lemma 2.1.6. For L (not necessarily finite-dimensional) a simple $\mathfrak{g}$-module, $v^\lambda \in L$ a $\mathfrak{b}$-HWV(λ), and α an isotropic odd simple root,

- $\lambda(h_\alpha) = 0 \implies L$ is a $\mathfrak{b}$-HWM(λ), and v^λ is a $\mathfrak{b}_\alpha$-HWV;
- $\lambda(h_\alpha) \neq 0 \implies L$ is a $\mathfrak{b}_\alpha$-HWM($\lambda - \alpha$), and $f_\alpha v^\lambda$ is a $\mathfrak{b}_\alpha$-HWV.

[9] modulo an induction argument for reordering things since we now have one less term; similarly in the next case

[10] reserving subscripts for highest weight labels

[11] any proper submodule cannot hit the λ-space without breaking properness, so the sum of all proper submodules in particular fails to hit the λ-space

[12] the unique maximal being the sum of all proper submodules

[13] a HWV of smaller weight would generate a submodule

[14] either because HWMs have unique simple quotient or because clearly the HWVs must go to each other which determines the map

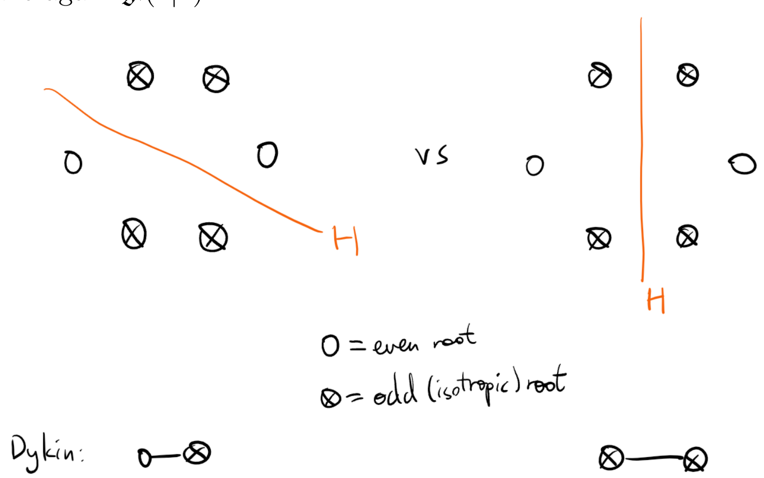
Proof. Note[15][16] the following three things:

$$\begin{aligned} e_\alpha f_\alpha v^\lambda &= \lambda(h_\alpha) v^\lambda, \\ e_\beta f_\alpha v^\lambda &= 0 \qquad\qquad \forall\, \beta \in \Phi^+ \setminus \alpha, \\ f_\alpha^2 v^\lambda &= 0. \end{aligned}$$

If $\lambda(h_\alpha) = 0$, then $f_\alpha v^\lambda = 0$ since else it would be a HWV of smaller weight. Then the 'upper nilpotent' of $\mathfrak{b}_\alpha$, namely $\mathfrak{n}_\alpha^+$ which is $\mathfrak{n}^+$ except with e_α replaced with $e_{-\alpha} = f_\alpha$, kills v^λ, so that v^λ is a $\mathfrak{b}_\alpha$-HWV.

If $\lambda(h_\alpha) \neq 0$, then $e_\alpha f_\alpha v^\lambda = \lambda(h_\alpha) v^\lambda \neq 0 \implies f_\alpha v^\lambda \neq 0$ is of weight $\lambda - \alpha$, and $f_\alpha(f_\alpha v^\lambda) = e_\beta(f_\alpha v^\lambda) = 0$ implies $f_\alpha v$ is a $\mathfrak{b}_\alpha$-HWV. ■

Example 2.1.7. Take again $\mathfrak{gl}(2|1)$.



Let us begin with the polarization on the left. Consider the isotropic odd simple root $\alpha_2 = \delta_2 - \varepsilon_1$, for which the coroot is the matrix $h_{\alpha_2} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}$, and consider the weight $\lambda = \delta_1 - \varepsilon_1 = \alpha_3$. Then $\lambda(h_{\alpha_2}) = -1 \neq 0$, so that with respect to $\mathfrak{b}_{\alpha_2}$, i.e. the polarization on the right, the new extremal weight is $\lambda - \alpha_2 = \delta_1 - \delta_2$, as one might have naively guessed. The problem is λ was odd, and this new thing $\delta_1 - \delta_2$ is even, so the two can't be related by the Weyl group.

I'd like to emphasize that if you had to blame someone for all the strange business going on, it would have to be the even/odd distinction, which makes it so that $f_\alpha^2 = 0$ for odd α. We'll see later this is really the key to a lot of the unusual behavior we see.

2.1.4. *Representations.* We have called the finite-dimensional simple of highest weight λ by the name L_λ, but what type of λ's fit the bill here? The answer turns out to be very similar to the usual case. We'll only do $\mathfrak{gl}(m|n)$.

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{h}$ be the standard Cartan, and Φ^+ be the standard choice of polarization giving rise to $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. The even subalgebra admits a compatible triangular decomposition:

$$\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h} \oplus \mathfrak{n}_0^+.$$

In addition to the even/odd grading for $\mathfrak{g}$, it also admits a $\mathbb{Z}$-grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1},$$

[15]The first is obvious, the second is since $e_\beta f_\alpha v^\lambda \in L^{\lambda+\beta-\alpha}$ and $\beta - \alpha$ is a positive linear combination of positive roots which is either not a root at all or itself in Φ^+, which brings the weight $\lambda + \beta - \alpha$ outside the range allowed by the highest weight λ, and the third is since by an earlier theorem (Theorem 1.18.(10) in the book) $2\alpha \notin \Phi$ so that $f_\alpha^2 = \frac{1}{2}[f_\alpha, f_\alpha] \in \mathfrak{g}^{2\alpha} = 0$.

[16]Theorem 1.18.(10) says that

$$k\alpha \in \Phi \text{ for } k \neq \pm 1 \iff \alpha \text{ is nonisotropic odd},$$

in which case $k = \pm 2$.

where $\mathfrak{g}_{-1} \oplus \mathfrak{g}_{+1} = \mathfrak{g}_1$ and the two meanings of $\mathfrak{g}_0$ agree. Here $\mathfrak{g}_{+1}$ are e.g. things looking like[17] $\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ & & 0 \end{pmatrix}$ and $\mathfrak{g}_{-1}$ are things looking like $\begin{pmatrix} 0 & 0 & \\ 0 & 0 & \\ * & * & 0 \end{pmatrix}$. Note that they are abelian.

For $\lambda \in \mathfrak{h}^*$, let $L_\lambda(\mathfrak{g}_0)$ denote the simple $\mathfrak{g}_0$-module of highest weight λ. As $\mathfrak{g}_0 = \mathfrak{gl}_m \oplus \mathfrak{gl}_n$, this is really $L_\mu(\mathfrak{gl}_m) \otimes L_\nu(\mathfrak{gl}_n)$ for some μ, ν. We can extend this to a module over $\mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ by making $\mathfrak{g}_{+1}$ act by zero. Inducting this to all of $\mathfrak{g}$ gives the so-called Kac module:

Definition 2.1.8. The "Kac module" is defined by

$$K_\lambda := \operatorname{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_{+1}}^{\mathfrak{g}} L_\lambda(\mathfrak{g}_0) \overset{\mathsf{Vec}}{\cong} \bigwedge \mathfrak{g}_{-1} \otimes L_\lambda(\mathfrak{g}_0).$$

As you might guess from the definition, this is reminiscent of Verma modules, except that because the definition of the superbracket is different, this Kac module is finite-dimensional (for 'integral' λ). Like Vermas, this will project to simples.

Proposition 2.1.9. Up to scalars, there is a unique $K_\lambda \twoheadrightarrow L_\lambda$, and

$$\dim L_\lambda < \infty \quad \Longleftrightarrow \quad \dim L_\lambda(\mathfrak{g}_0) < \infty \quad \Longleftrightarrow \quad \dim K_\lambda < \infty.$$

Proof. As $\operatorname{Hom}_{\mathfrak{g}}(\operatorname{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_{+1}}^{\mathfrak{g}} L_\lambda(\mathfrak{g}_0), L_\lambda) \cong \operatorname{Hom}_{\mathfrak{g}_0 \oplus \mathfrak{g}_{+1}}(L_\lambda(\mathfrak{g}_0), L_\lambda|_{\mathfrak{g}_0 \oplus \mathfrak{g}_{+1}})$, the obvious embedding $L_\lambda(\mathfrak{g}_0) \hookrightarrow L_\lambda|_{\mathfrak{g}_0 \oplus \mathfrak{g}_{+1}}$ gives rise to a map $K_\lambda \twoheadrightarrow L_\lambda$. For the second part, L_λ being finite-dimensional implies its $\mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$-submodule $L_\lambda(\mathfrak{g}_0)$ is finite-dimensional; $L_\lambda(\mathfrak{g}_0)$ being finite-dimensional implies $K_\lambda = \bigwedge \mathfrak{g}_{-1} \otimes L_\lambda(\mathfrak{g}_0)$ is finite-dimensional; and K_λ being finite-dimensional implies its quotient L_λ is finite-dimensional. ■

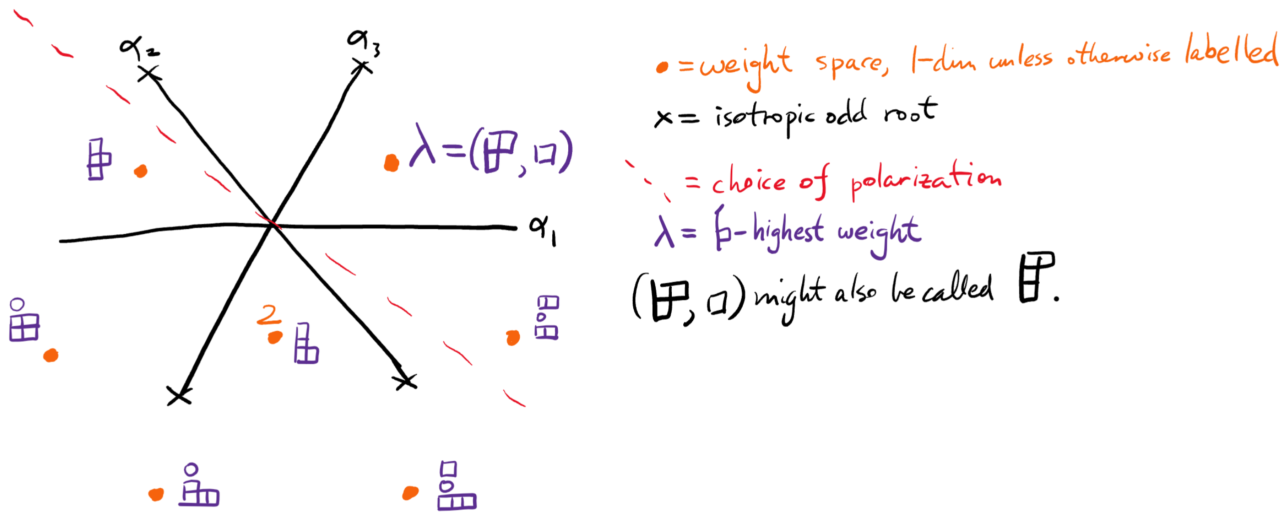
Hence $L_\lambda \leftrightsquigarrow L_\lambda(\mathfrak{g}_0)$, which in turn are $L_\mu(\mathfrak{gl}_m) \otimes L_\nu(\mathfrak{gl}_n)$. This shows that

Proposition 2.1.10. The finite-dimensional $\mathfrak{g}$-simples are L_λ for $\lambda = (\mu, \nu)$ a pair of partitions 'up to scaling' (meaning that we only require $\lambda_i \geq \lambda_{i+1}$, not that $\lambda_i \geq 0$, so that $|\lambda|$ may be negative) such that $\ell(\mu) \leq m,\ \ell(\nu) \leq n$. More precisely, the actual weight is $\lambda = \mu_1\delta_1 + \cdots + \mu_m\delta_m + \nu_1\varepsilon_1 + \cdots + \nu_n\varepsilon_n$.

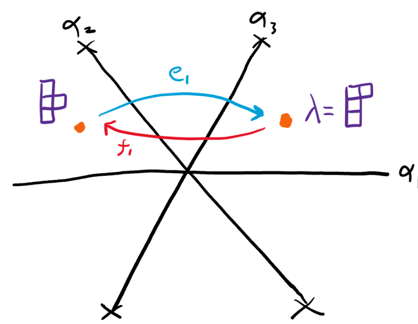
Let's see an example of this.

Example 2.1.11. Take $\mathfrak{gl}(2|1)$ with the standard polarization, and consider the highest weight $\lambda = \left(\ydiagram{2,1}, \ydiagram{1}\right)$, which we might write as $\ydiagram{2,1,1}$ in an abuse of notation. Let me present the weight diagram for K_λ immediately.

[17] in other words $\mathfrak{g}_{+1}$ is spanned by E_{ij} with $i, j \in I(m|n)$ such that $i < 0 < j$, and $\mathfrak{g}_{-1}$ is similarly except with $i > 0 > j$

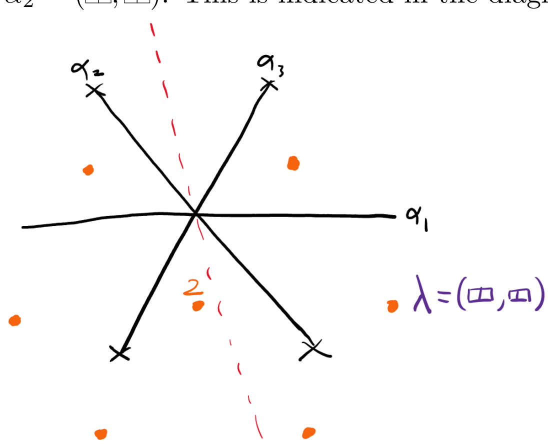


Let me explain. We start with $L_\lambda(\mathfrak{g}_0) = L_{\ydiagram{2,1}}(\mathfrak{gl}_2) \otimes L_{\square}(\mathfrak{gl}_1)$ a 2-dimensional module over $\mathfrak{gl}_2 \oplus \mathfrak{gl}_1$; sitting inside the A_2 root picture, this looks like



Then to get K_λ we induct this from $\mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ to all of $\mathfrak{g}$, i.e. we let $\mathfrak{g}_{-1} = \mathbb{C}\{f_2, f_3\}$ act freely. But due to being odd, both these lowering operators square to zero. So what we get is finite-dimensional, and in particular it is $\bigwedge \mathfrak{g}_{-1}$ which is 4-dimensional. This gives the picture we started with, which is 8-dimensional.

The claim is that this K_λ is already simple. We will see this by checking that the highest weights of L_λ with respect to different polarizations actually are the same as the weights around the perimeter of K_λ. Indeed, let us odd-reflect across α_2. Note $\lambda(h_{\alpha_2}) = \ydiagram{2,1}\left(\begin{pmatrix} & & \\ & 1 & \\ & & 1 \end{pmatrix}\right) = 2 \neq 0$, so that the new highest weight should be $\lambda - \alpha_2 = (\ydiagram{2}, \ydiagram{2})$. This is indicated in the diagram below.



If we then reflect across the odd simple root α_3 now we would find the new highest weight is the bottom right dude; I won't TeX this out. But anyways this shows $K_\lambda = L_\lambda$. (If you want you can check manually that no one-dimensional subspace in the center of the weight diagram forms a subrepresentation.)

The above proposition, 2.1.10, makes it very easy to see what λ's are possible with respect to the standard choice for $\mathfrak{b}$. But what about some different choice?

Proposition 2.1.12. For any choice of $\mathfrak{b}$ and Σ, $L_\lambda(\mathfrak{b})$ is finite-dimensional if and only if for any[19] $\beta \in \Sigma_0$ and any Σ' obtained from Σ via odd reflections such that $\beta \in \Sigma'$ or $\beta/2 \in \Sigma'$, we have

$$\frac{2(\lambda;\beta)}{(\beta;\beta)} \in \mathbb{N}.$$

It also turns out that if L_λ is finite-dimensional then K_λ is the unique maximal finite-dimensional quotient of Δ_λ the Verma, which we now discuss.

There is also the actual Verma module.

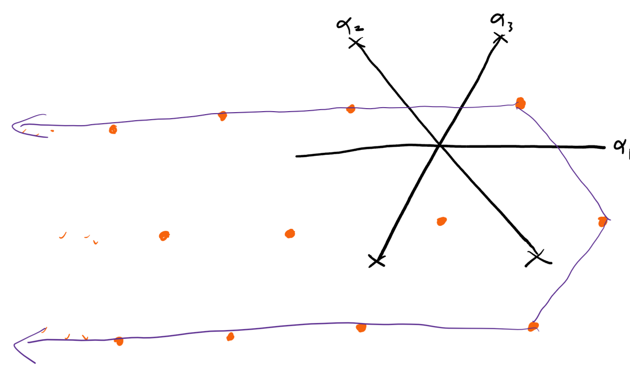
Definition 2.1.13. With respect to a polarization, the Verma associated to a $\lambda \in \mathfrak{h}^*$ is

$$\Delta_\lambda \coloneqq \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}v_\lambda,$$

where as usual $\mathfrak{n}^+$ acts trivially on v_λ.

Being a HWM, Δ_λ admits a unique simple quotient which is then necessarily L_λ. However, in the super case Vermas may look rather strange.

Example 2.1.14. Take again $\mathfrak{gl}(2|1)$ and again $\lambda = \ydiagram{2,1}$. Then the weights of the Verma $\Delta_{\ydiagram{2,1}}$ are, up to multiplicity (which I was too lazy to compute),



As usual the strange 'finite behavior' of the Verma here is ultimately due to the odd roots having $f_\alpha^2 = 0$; in fact, we can dial this strangeness to the max and have finite-dimensional Vermas.

Example 2.1.15. Take $\mathfrak{gl}(1|1)$. If we start at any λ in the A_1 picture, since $f^2 = 0$, the Verma will have only two dimensions, in weights λ and $\lambda - \alpha$. So the Verma is finite-dimensional. This λ could be possibly super far from the center. The unique simple quotient L_λ will be 2-dimensional 'most of the time', the exception being at weights for which $|\lambda| = |\mu| + |\nu| = 0$. (This is since $[e,f] = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$, so that $|\lambda| = 0$ allows both e and f to act by zero.)

As we can see from the examples above, finite-dimensional modules no longer exhibit weights symmetric about the center – they can be shifted from the origin.

A comment about the grading of representations: in this subsubsection I've been a little sloppy about this, but it's pretty clear the discussion holds up to parity switch. We can for example take the convention that the HWV be even.

We could let $\mathcal{O}$ be the category of finitely-generated $\mathfrak{h}$-semisimple $\mathfrak{g}$-modules which are locally $\mathfrak{n}^+$-finite. Note the dependence on $\mathfrak{b}$. We could instead think about the category $\mathcal{O}_{\text{super}}$ of *super*modules, but it doesn't really make a difference. Indeed, if you define $\operatorname{par}\colon \mathbb{C} \longrightarrow \mathbb{Z}/2$ by $\operatorname{par}(a+bi) = \lfloor a \rfloor$ and then $\operatorname{par}\colon \mathfrak{h}^* \longrightarrow \mathbb{Z}/2$ by $\operatorname{par}(\lambda) \coloneqq \operatorname{par}(|\lambda_+|) \coloneqq \operatorname{par}((\lambda; \varepsilon_1 + \cdots + \varepsilon_n))$, then this latter function has the property that $\mathrm{par}(\lambda+\varepsilon_i) = \mathrm{par}(\lambda)+1$ and $\mathrm{par}(\lambda+\delta_i) = \mathrm{par}(\lambda)$. Then any supermodule in super-category $\mathcal{O}$ decomposes as

$$M = M_+ \oplus M_- = \bigoplus_\lambda M^\lambda{}_{\mathrm{par}(\lambda)} \oplus \bigoplus_\lambda M^\lambda{}_{\mathrm{par}(\lambda)+1}.$$

Recall[20] the parity switching functor, Π. Let $\mathcal{O}_+ \subseteq \mathcal{O}_{\mathrm{super}}$ be the full subcategory of things such that $M = M_+$; then

$$\mathcal{O}_{\mathrm{super}} = \mathcal{O}_+ \oplus \Pi\mathcal{O}_+.$$

For this reason the grading on our representations sort of don't really matter (provided there's no parity conflict).

2.2. **Characters.** As a generalization of the usual character, we may define the 'supercharacter'. Recall the superdimension is $\mathrm{sdim}\, V = \dim V_0 - \dim V_1$.

Definition 2.2.1. The "supercharacter" of a $\mathfrak{g}$-module M is

$$\chi_M \coloneqq \sum_\lambda \mathrm{sdim}\, M^\lambda e^\lambda.$$

Something called 'atypicality' will also show up later; for now let me define what it means to not be atypical. I would have personally called this following notion 'ϱ-typical', but whatever.

Definition 2.2.2. A weight λ is "typical" if $(\lambda+\varrho;\alpha)\neq 0$ for any isotropic odd α.

Turns out we can compute the supercharacters of finite-dimensional simples of typical weight. Note that in the purely even case the following is just the usual Weyl character formula.

Theorem 2.2.3. Take some choice of $\mathfrak{b}$. For $\lambda-\varrho$ typical with L_λ finite-dimensional,

$$\chi_{L_\lambda} = \frac{\prod_{\alpha\in\Phi_1^+}(e^{\alpha/2}-e^{-\alpha/2})}{\prod_{\alpha\in\Phi_0^+}(e^{\alpha/2}-e^{-\alpha/2})}\sum_{w\in W}(-1)^w e^{w(\lambda+\varrho)}.$$

Another way to write this is

$$\chi_{L_\lambda} = \sum_{w\in W}(-1)^w \chi_{\Delta_{w\circ\lambda}},$$

where as usual $w\circ\lambda \coloneqq w(\lambda+\varrho)-\varrho$.

In fact, K_λ has a unique simple submodule, and for typical $\lambda-\varrho$,

$$K_\lambda = L_\lambda.$$

Proof. Let's just prove this for standard $\mathfrak{b}$. It's easy[21] to see that

$$\chi_{\mathcal{U}\mathfrak{n}^-} = \frac{\prod_{\alpha\in\Phi_1^+}(1-e^{-\alpha})}{\prod_{\alpha\in\Phi_0^+}(1-e^{-\alpha})},$$

so that

$$\chi_{\Delta_\lambda} = e^\lambda\frac{\prod_{\alpha\in\Phi_1^+}(1-e^{-\alpha})}{\prod_{\alpha\in\Phi_0^+}(1-e^{-\alpha})} = e^{\lambda+\varrho}\frac{\prod_{\alpha\in\Phi_1^+}(e^{\alpha/2}-e^{-\alpha/2})}{\prod_{\alpha\in\Phi_0^+}(e^{\alpha/2}-e^{-\alpha/2})}.$$

As $K_\lambda = \bigwedge\mathfrak{g}_{-1}\otimes L_\lambda(\mathfrak{g}_0)$, we have

$$\chi_{K_\lambda} = \prod_{\alpha\in\Phi_1^+}(1-e^{-\alpha})\chi_{L_\lambda(\mathfrak{g}_0)} = e^{\varrho_1}\prod_{\alpha\in\Phi_1^+}(e^{\alpha/2}-e^{-\alpha/2})\chi_{L_\lambda(\mathfrak{g}_0)} = \frac{e^{\varrho_1}\prod_{\alpha\in\Phi_1^+}(e^{\alpha/2}-e^{-\alpha/2})}{\prod_{\alpha\in\Phi_0^+}(e^{\alpha/2}-e^{-\alpha/2})}\sum_{w\in W}(-1)^w e^{w(\lambda+\varrho_0)},$$

so that it suffices to show $K_\lambda = L_\lambda$.

[20] recall this changes the action by $x\cdot v = (-1)^x xv$

[21] numerator has minus e to the blah since odd roots switch parity and we are looking at superdimension, denominator has minus since geometric series

First note that any $\mathfrak{g}$-submodule of K_λ must contain the same $\mathfrak{g}_0$-submodule, namely[22] $\det \mathfrak{g}_{-1} \otimes L_\lambda(\mathfrak{g}_0)$. Two distinct simple submodules of K_λ should intersect trivially; hence K_λ has a unique simple submodule, say L_μ.

Next observe that $\lambda' = \lambda - \sum_{\alpha\in\Phi_1^+} \alpha$ is the highest weight of L_μ with respect to $\mathfrak{b}_{-1} := \mathfrak{b}_0 \oplus \mathfrak{g}_{-1}$ the 'anti-distinguished' Borel; this is because λ' is the $\mathfrak{b}_0$-highest weight in $\det \mathfrak{g}_{-1} \otimes L_\lambda(\mathfrak{g}_0)$ and any more applications of $\mathfrak{g}_{-1}$ will kill this module. Hence

$$L_\mu = L_{\lambda - \sum_{\alpha\in\Phi_1^+} \alpha}(\mathfrak{b}_{-1}).$$

Now consider $L_\lambda = L_\lambda(\mathfrak{b})$; if we try to find out what the highest weight of this would be for $\mathfrak{b}_{-1}$, by applying the odd reflections repeatedly and using that $\lambda - \varrho$ is typical, i.e. $\lambda(h_\alpha) \neq 0$, we find that the $\mathfrak{b}_{-1}$-highest weight is also $\lambda - \sum_{\alpha\in\Phi_1^+} \alpha$. Hence $L_\mu = L_\lambda$, i.e. the unique simple submodule of K_λ is equal to the unique simple quotient, so that $K_\lambda = L_\lambda$. ■

Remark about the proof: this proof is taken from Serganova's notes since I don't know enough about Zuckerman/Bernstein functors to parse the proof in the book. However Serganova's stipulation is that λ is typical; but I think this doesn't allow the last step of the proof to go through, since in the standard polarization for $\mathfrak{gl}(2|1)$ the weight $\lambda = \delta_2$ has $\lambda(h_{\alpha_3}) = 0$ and yet $\lambda + \varrho = \delta_1$ so that $(\lambda+\varrho)(h_{\alpha_3}) = 1 \neq 0$. You might object that δ_2 is not a valid weight for a finite-dimensional simple, but I think this can be modified to work in general.

2.3. **Central characters.** Denote

Definition 2.3.1. $Z\mathfrak{g} = Z(\mathfrak{g})_0 \oplus Z(\mathfrak{g})_1$ is the "center" of $\mathcal{U}\mathfrak{g}$, where

$$Z(\mathfrak{g})_i = \Big\{ z \in \mathcal{U}(\mathfrak{g})_i : zx = (-1)^{i\cdot|x|} xz \quad \forall\, x \in \mathfrak{g} \text{ pure} \Big\}.$$

It turns out that secretly $Z\mathfrak{g} = Z(\mathfrak{g})_0$.

Let $\mathrm{S}\mathfrak{g}$ denote the symmetric superalgebra in an abuse of notation. Both this and $\mathcal{U}\mathfrak{g}$ admit a $\mathfrak{g}$-action via adjoint and Leibniz[23]. Note $\mathrm{Gr}\,\mathcal{U}\mathfrak{g} \cong \mathrm{S}\mathfrak{g}$ as vector spaces as usual. Note also that $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}} = Z\mathfrak{g}$. In fact,

Proposition 2.3.2. The supersymmetrization map $\mathrm{S}\mathfrak{g} \longrightarrow \mathcal{U}\mathfrak{g}$ is an isomorphism of $\mathfrak{g}$-modules and induces a linear isomorphism $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} Z\mathfrak{g}$.

This is evidently proved in the same way as usual[24].

For the sake of readability, in the following I will adhere to the book's choice of notation. From PBW, let

$$\phi\colon \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{U}\mathfrak{h}$$

denote the map killing any term with factors from outside $\mathfrak{h}$ in it[25]. Let its restriction to $Z\mathfrak{g}$ also be denoted

$$\phi\colon Z\mathfrak{g} \longrightarrow \mathcal{U}\mathfrak{h} = \mathrm{S}\mathfrak{h}.$$

Writing a random $z \in Z\mathfrak{g}$ as $z = h_z + \mathfrak{n}^- \mathcal{U}\mathfrak{g} + \mathcal{U}\mathfrak{g}\mathfrak{n}^+$, we know that $[\mathfrak{h}, z] = 0 = [\mathfrak{h}, \mathfrak{n}^-]\mathcal{U}\mathfrak{g} + \mathfrak{n}^-[\mathfrak{h}, \mathcal{U}\mathfrak{g}] + [\mathfrak{h}, \mathcal{U}\mathfrak{g}]\mathfrak{n}^+ + \mathcal{U}\mathfrak{g}[\mathfrak{h}, \mathfrak{n}^+] \subseteq \mathfrak{n}^- \mathcal{U}\mathfrak{g} + \mathcal{U}\mathfrak{g}\mathfrak{n}^+$, where we note $[\mathfrak{h}, \mathfrak{n}^\pm] \subseteq \mathfrak{n}^\pm$. So in order for this thing to be zero, surely there must be some cancellation going on, i.e. every term in $z - h_z$ must have at least one thing from $\mathfrak{n}^\pm$ on each end, i.e.

$$z = h_z + \mathfrak{n}^- \mathcal{U}(\mathfrak{g})\mathfrak{n}^+.$$

Note that $\phi(z) = h_z$ then. By consider the bracket of an odd guy with this, you can convince yourself that z must in fact be even.

[22] being the bottom level of the 'waterfall'

[23] Important: I'm pretty sure that in the super case Leibniz says $[x, yz] = [x, y]z + (-1)^{xy} y[x, z]$.

[24] i sure aint checking it

[25] more precisely this is the projection associated to $\mathcal{U}\mathfrak{g} = \mathcal{U}\mathfrak{h} \oplus (\mathfrak{n}^- \mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{n}^+)$

Recall what $P(\lambda)$ means for $P \in \mathrm{S}\mathfrak{h}$. Define

$$\begin{aligned} \vartheta_\lambda \colon Z\mathfrak{g} &\longrightarrow \mathbb{C} \\ z &\longmapsto \phi(z)(\lambda). \end{aligned}$$

Such a morphism is called a "central character", as usual.

Lemma 2.3.3. For $\mathfrak{g}$ basic, any $z \in Z\mathfrak{g}$ acts by $\vartheta_\lambda(z)$ on any HWM(λ).

Proof. Let the module in question be called M. As M^λ is 1-dimensional and z commutes with any h, we have $hzv^\lambda = zhv^\lambda = \lambda(h)zv^\lambda$, so that zv^λ must be some multiple of v^λ. Any other vector is arrived at by applying $\mathfrak{n}^-$ to v^λ, but z commutes with $\mathfrak{n}^-$, so z acts by the same scalar on that too. This scalar is given by $zv^\lambda = (h_z + \sum_i f_i x_i e_i)v^\lambda = \lambda(h_z)v^\lambda = \vartheta_\lambda(z)v^\lambda$. ■