The Yoneda Lemma Review

Emily Saunders

September 2020

1 The Yoneda Lemma

First we start by recalling the definiton of the Yoneda functor h^A/h_A or $Hom_{\mathcal{C}}(-,A)/Hom_{\mathcal{C}}(A,-)$ of an object $A \in \mathcal{C}$.

Definition 1.1. Given an object $A \in \mathcal{C}$ with $\mathcal{C}$ small, the contravariant Yoneda functor $h^A : \mathcal{C}^{op} \to Set$ is defined on objects by,

$$h^A(X) = Hom_{\mathcal{C}}(X, A)$$

and defined on morphisms $f : X \to X'$ by precomposition,

$$h^A(f) : Hom_{\mathcal{C}}(X', A) \to Hom_{\mathcal{C}}(X, A)$$

$$(g : X' \to A) \mapsto (g \circ f : X \to X' \to A)$$

The covariant Yoneda functor $h_A : \mathcal{C} \to Set$ is defined on objects by,

$$h_A(X) = Hom_{\mathcal{C}}(A, X)$$

and defined on morphisms $f : X' \to X$ by postcomposition,

$$h_A(f) : Hom_{\mathcal{C}}(A, X') \to Hom_{\mathcal{C}}(A, X)$$

$$(g : A \to X') \mapsto (f \circ g) : A \to X' \to X)$$

For the remainder of this lecture we will use the contravariant Yoneda functor, but all concepts have a dual notion that applies to the covariant Yoneda functor, as the covariant Yoneda functor of an object $A \in \mathcal{C}$ is simply the contravariant Yoneda functor of $A \in \mathcal{C}^{op}$.

The Yoneda functor of an object in $\mathcal{C}^{op}$ defines a functor from $\mathcal{C}$ into the functor category (known as the Yoneda embedding)

$$h^{(-)} : \mathcal{C} \to Funct(\mathcal{C}^{op}, Set)$$

We still have not specificed what this functor does on morphisms $f : A \to A'$. Recall that a morphism in a functor category is a natural transformation. The natural transformation $h^{(-)}(f)$ is defined by the components,

$$h^{(-)}(f)_X : Hom_{\mathcal{C}}(X, A) \to Hom_{\mathcal{C}}(X, A')$$

$$(g : X \to A) \mapsto (f \circ g : X \to A \to A')$$

that is, each component of the natural transformation is given by postcomposition with f. As an exercise, one can check that this is indeed a natural transformation by checking that the following diagram commutes for all $h : X \to X'$,

$$\begin{array}{ccc} Hom_{\mathcal{C}}(X', A) & \xrightarrow{h^{(-)}(f) = f \circ -} & Hom_{\mathcal{C}}(X', A') \\ \Big\downarrow{\scriptstyle h^A(h) = - \circ h} & & \Big\downarrow{\scriptstyle h^A(h) = - \circ h} \\ Hom_{\mathcal{C}}(X, A) & \xrightarrow{h^{(-)}(f) = f \circ -} & Hom_{\mathcal{C}}(X, A') \end{array}$$

STOP before going any further, make sure you understand the distinction between the Yoneda functor of a specified object $A \in \mathcal{C}$, $h^A : \mathcal{C}^{op} \to Set$, and the *Yoneda embedding* $h^{(-)} : \mathcal{C} \to Funct(\mathcal{C}^{op}, Set)$, a functor from $\mathcal{C}$ to the functor category from $\mathcal{C}^{op}$ to Set (also known as the *presheaf* category of $\mathcal{C}$) which takes every object to it's Yoneda functor. These two concepts are very important and very easy to confuse. Make sure you understand what each of these functors does to morphisms. What is a morphism in their respective codomain categories?

Now we are ready to state the Yoneda Lemma

Lemma 1 (The Yoneda Lemma)**.** For a functor $F : \mathcal{C}^{op} \to Set$,

$$Hom_{Funct(\mathcal{C}^{op}, Set)}(h^A, F) = Nat(h^A, F) \cong F(A)$$

Proof. The key point in the proof of the lemma is noticing that every natural transformation $\eta : h^A \Rightarrow F$ is entirely and uniquely determined by where the component of A sends the identity, i.e. the component of A is a function $\eta_A : h^A(A) = Hom_{\mathcal{C}}(A, A) \to F(A)$ and η is determined by $\eta_A(id_A)$

Recall that the data of a natural transformation $\eta : h^A \Rightarrow F$ is the data of components $\eta_X : h^A(X) = Hom_{\mathcal{C}}(X, A) \to F(X)$ such that the following diagram commutes for all $f : X \to X'$,

$$\begin{array}{ccc} h^A(X') = Hom_{\mathcal{C}}(X', A) & \xrightarrow{\eta_{X'}} & F(X') \\ \Big\downarrow{\scriptstyle h^A(f) = - \circ f} & & \Big\downarrow{\scriptstyle F(f)} \\ h^A(X) = Hom_{\mathcal{C}}(X, A) & \xrightarrow{\eta_X} & F(X) \end{array}$$

We want to construct a function $\varphi : F(A) \to Nat(h^A, F)$. For $u \in F(A)$ we will construct $\varphi(u)$ by declaring that $\varphi(u)_A(id_A) = u$. Now, given any $f : X \to A$, plugging this into the above diagram we get that the following must commute,

$$\begin{array}{ccc} h^A(A) = Hom_{\mathcal{C}}(A, A) & \xrightarrow[id_A \mapsto u]{\varphi(u)_A} & F(A) \\ {\scriptstyle id_A \mapsto id_A \circ f = f}\Big\downarrow{\scriptstyle h^A(f) = - \circ f} & & {\scriptstyle u \mapsto F(f)(u)}\Big\downarrow{\scriptstyle F(f)} \\ h^A(X) = Hom_{\mathcal{C}}(X, A) & \xrightarrow{\varphi(u)_X} & F(X) \end{array}$$

Thus if we want to define $\varphi(u)_X(f)$, the commutivity of the above diagram requires that it be sent to $F(f)(u)$, thus $\varphi(u)_X$ is uniquely determined to be the map,

$$\varphi(u)_X : Hom_{\mathcal{C}}(X, A) \to F(X)$$

$$(f : X \to A) \mapsto F(f)(u)$$

It still remains to show that $\varphi(u)$ satisfies the naturality condition for an arbitrary $g : X \to X'$, which we can check in the following diagram,

$$\begin{array}{ccc}
h^A(X') = Hom_{\mathcal{C}}(X', A) & \xrightarrow[f \mapsto F(f)(u)]{\varphi(u)_{X'}} & F(A) \\
{\scriptstyle f \mapsto f \circ g} \Big\downarrow {\scriptstyle h^A(g) = - \circ g} & & {\scriptstyle F(f)(u) \mapsto F(g) \circ F(f)(u)} \Big\downarrow {\scriptstyle F(g)} \\
h^A(X) = Hom_{\mathcal{C}}(X, A) & \xrightarrow[f \circ g \mapsto F(f \circ g)(u)]{\varphi(u)_X} & F(X)
\end{array}$$

We check that this diagram commutes by checking what happens to an object $f \in Hom_{\mathcal{C}}(X', A)$ along both paths. Referring to the diagram, we see that $\varphi(u)_X \circ h^A(g)(f) = F(f \circ g)(u)$ and $F(g) \circ \varphi(u)_{X'}(f) = F(g) \circ F(f)(u)$. From the fact that functors commute with composition of morphisms by definition, we know that $F(f \circ g) = F(f) \circ F(g)$, therefore we have that $F(f) \circ F(g)(u) = F(f \circ g)(u)$.

To summarize, we have now shown that for each $u \in F(A)$, there exists a *unique* natural transformation $\varphi(u)$ such that $\varphi(u)_A(id_A) = u$. Thus we have constructed a function,

$$\varphi : F(A) \to Nat(h^A, F)$$

this function is clearly injective, as $u \neq u'$ implies $\varphi(u)_A(id_A) = u \neq \varphi(u')_A(id_A) = u'$. Furthermore, this map is surjective, as any natural transformation $\eta : h^A \Rightarrow F$ is determined completely by where η_A sends id_A, thus it is equivalent to $\varphi(\eta_A(id_A))$. □

Corollary 1. The Yoneda embedding $h^{(-)} : \mathcal{C} \to Funct(\mathcal{C}^{op}, Set)$ is a full and faithful embedding of one category into another.

Proof. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is full/faithful if given any two objects $X, Y \in \mathcal{C}$, the homset function induced by F,

$$F : Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$$

is surjective/injective. Thus the statement of this corollary is equivalent to saying that for all $X, Y \in \mathcal{C}$ the homset function,

$$h^{(-)} : Hom_{\mathcal{C}}(X, Y) \xrightarrow{\sim} Nat(h^X, h^Y)$$

is an isomorphism.

First, note that $Hom_{\mathcal{C}}(X, Y) = h^Y(X)$. Now, for $f \in Hom_{\mathcal{C}}(X, Y)$, we know from above that $h^{(-)}(f)$ is given by postcomposition, so $h^{(-)}(f)_X(id_X) = f \circ id_X = f$. Therefore $h^{(-)}$ is of the form φ from our proof of the Yoneda lemma with $F = h^Y$ so we have,

$$\varphi = h^{(-)} : Hom_{\mathcal{C}}(X, Y) = h^Y(X) \xrightarrow{\sim} Nat(h^X, h^Y)$$

□

What does this mean in practice? The Yoneda lemma has many important consequences. One of the most important being the above corollary, which tells us that we can think of our category $\mathcal{C}$ as sitting inside of the larger category $Funct(\mathcal{C}^{op}, Set)$, since all information between any two objects in the image of the Yoneda embedding is preserved. This tells us that in order to specify an object in our category, it is entirely sufficient to specify how every element maps into it (or equivalently out of it by the covariant Yoneda Lemma). While this may sound like a more difficult way to define an object, notice that we have already think of many objects this way! For instance, the terminal object is the unique $* \in \mathcal{C}$ such that for every other $X \in \mathcal{C}$ there is a single unique $X \to *$. Similarly, a product $X \times Y$ is defined as the unique element such that a morphism $(f, g) : A \to X \times Y$ is uniquely determined by a pair of morphisms $f : A \to X$ and $g : A \to Y$. The Yoneda lemma tells us that, if they exists, these properties actually uniquely characterize these objects.

2 Properties of the Presheaf Category

These functor categories $Funct(\mathcal{C}^{op}, Set)$ are so important that they actually have their own name,

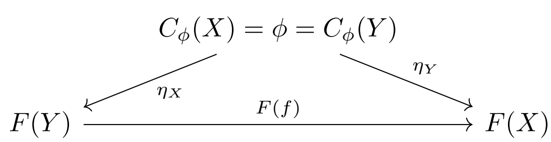
Definition 2.1. The *Presheaf category* $Presheaf(\mathcal{C})$ of a small category $\mathcal{C}$ is the category $Funct(\mathcal{C}^{op}, Set)$

Presheaf categories are nice because they allow us to naturally expand our view of our original category. Furthermore, many nice properties of Set lead to nice properties for our presheaf categories as we will see below, such as completeness and cocompleteness.

Proposition 1. All presheaf categories have an initial and terminal object

Proof. Let $C_\phi : \mathcal{C}^{op} \to Set$ be the constant functor $C_\phi(X) = \phi$ and $C_\phi(f) = id_\phi$ where $\phi \in Set$ is the empty set, the initial object in Set. We will prove that for every $F : \mathcal{C}^{op} \to Set$, there exists a unique natural transformation $\eta : C_\phi \Rightarrow F$.

For every $X \in \mathcal{C}^{op}$, define the component $\eta_X : C_\phi(X) = \phi \to F(X)$ to be the unique morphism $\phi \to F(X)$ given to us by the initial object property in Set. This property also tells is that for all $f : X \to Y$, the following diagram commutes,



which proves that η is a natural transformation. The terminal object is defined similarly as the constant function at $*$, the one element set in Set. □

Proposition 2. For any F, G in any presheaf category $Presheaf(\mathcal{C})$ the binary product $F \times G$ exists.

Proof. Again, the key insight of this proof is using the fact that all binary products in Set exist, so we can define the product on each individual element in Set, i.e. $F \times G(X) = F(X) \times G(X)$ with natural transformations to F and G given by pointwise projections $p_X^F : F(X) \times G(X) \to F(X)$ and $p_X^G : F(X) \times G(X) \to G(X)$.

We need to check that for all $H \in Presheaf(\mathcal{C})$, a natural transformation $H \Rightarrow F \times G$ is equivalent to a pair of natural transformations $\eta : H \Rightarrow F$ and $\epsilon : H \Rightarrow G$.

Given a pair of natural transformations $\eta : H \Rightarrow F$ and $\epsilon : H \Rightarrow G$ we can define a natural transformation $\eta \times \epsilon : H \Rightarrow F \times G$. We define the components to be the functions,

$$\alpha_X = \eta_X \times \epsilon_X : H(X) \to F \times G(X) = F(X) \times G(X)$$

one can easily check as an exercise that the naturality conditions on η and ϵ imply the naturality condition for α. $\square$

Notice that the proof of both of these propositions follow a similar structure. We define a special object in our presheaf category by defining it pointwise for every $X \in \mathcal{C}^{op}$ and then define the components of the necessary natural transformations pointwise as well using the desired property in Set. We can use this same style of proof to show that coproducts, equalizers, coequalizers and many more limits/colimits exist in any presheaf category. This same structure of proof can be used to show that all presheaf categories are complete and cocomplete.