Syllabus
Math GR6403 Modern Geometry II
Spring 2025

Instructor: Elena Giorgi (elena.giorgi@columbia.edu)

Course Hours and Location: Monday and Wednesday, 10:10am-11:25am, in room 507 Mathematics

Overview of the Course: This course continues the series of Modern Geometry I, which this year covered the first four chapters of *"Riemannian Geometry"* by Do Carmo.

Topics of the Course: We will focus mainly on Riemannian Geometry, Lorentzian Geometry and General Relativity. A more detailed list of topics covered in the class is the following:

1. Riemannian Geometry
   - Jacobi fields (Chapter 5 Do Carmo)
   - Isometric immersions (Chapter 6 Do Carmo)
   - Complete manifolds; Hopf-Rinow and Hadamard Theorems (Chapter 7 Do Carmo)
   - Spaces of constant curvature (Chapter 8 Do Carmo)
   - Variations of energy (Chapter 9 Do Carmo)
2. Lorentzian Geometry & General Relativity
   - Causality and Global Hyperbolicity
   - Geometry of Null Hypersurfaces
   - Null structure equations
   - Trapped surfaces and Penrose incompleteness theorem
   - The Einstein equation and the Cauchy problem
   - Black holes
   - Vectorfield method and waves of black holes

Textbooks: *"Riemannian Geometry"* by Do Carmo.

Structure of the course: For the two thirds of the class we will roughly follow chapters 5, 6, 7, 8, 9 of the textbook by Do Carmo. For the second half we will mostly follow lectures notes by Aretakis (https://web.math.princeton.edu/ aretakis/columbiaGR.pdf).

In addition to the lectures, there will be about weekly problem sets for the first half of the class and biweekly problem sets for the second half of the class.

These will be listed on Coursework and should be uploaded on Gradescope. Late homeworks will not be accepted.

It is very important to do all of the problem sets to the best of your ability, as this is the most effective way to absorb the material. While you are welcome to collaborate with your peers, you must attempt all problems on your own and your submitted solutions must be written out individually.

There will be one midterm exam and one take-home final exam. The midterm will be a seventy-five minute exam which is scheduled on Wednesday March 12th during normal class hours. The final exam will be a take-home exam.

The exams will generally follow the material from the problem sets but may include some additional conceptual problems to test your understanding. The midterm will cover roughly half of the course material, while the final exam will cover all the material from the course, with slightly more emphasis on the content covered after the midterm.

Grading scheme: The course grades will be computed as follows: 20% Homework, 40% Midterm exam, 40% Final exam.

Office Hours

- The instructor will hold weekly office hours, 9-10am on Mondays and Wednesdays, in room 606 Mathematics.
- The TA will hold weekly hours in the Math Help Room in room 406 Mathematics (see the schedule here for their office hours) and you are encouraged to come to their hours with any questions or confusions you may have.

Tentative schedule of lectures: This schedule is tentative and may be modified as the course progresses.

1. January 22: Jacobi fields
2. January 27: Jacobi fields on manifolds with constant sectional curvature
3. January 29: Taylor expansion of g in local coordinates
4. February 3: Taylor expansion of det$g$, Conjugate points
5. February 5: Conjugate locus, Isometric immersions,
6. February 10: First and second fundamental forms
7. February 12: Gauss, Codazzi, Ricci equations
8. February 17: Totally geodesic, mean curvature vector
9. February 19: Complete Riemannian manifolds, Hopf-Rinow Theorem
10. February 24: Proof of Hopf-Rinow Theorem and corollaries

11. February 26: Hadamard Theorem
12. March 3: Theorem of Cartan on the determination of the metric by curvature
13. March 5: Conformal deformation of the curvature
14. March 10: Yamabe problem, Geodesics on hyperbolic space
15. March 12: MIDTERM
16. March 24: Space forms
17. March 26: Conformal maps, Theorem of Liouville
18. March 31: Variations of energy, Formula for the first variation
19. April 2: Formula for the second variation, Bonnet-Myers Theorem
20. April 7: Weinstein's theorem, Synge's theorem
21. April 9: Index forms
22. April 14: Lorentzian Geometry, Null Hypersurfaces
23. April 16: Einstein equation and the Cauchy problem
24. April 21: Trapped surfaces and Penrose Incompleteness theorem
25. April 23: Black holes
26. April 28: Wave equation on Minkowski spacetime
27. April 30: Wave equation on black holes
28. May 5: NO CLASS