Invariants of Calabi–Yau manifolds

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Surfaces

Every surface* is homeomorphic to one of these:

\[ g = 0 \]
\[ g = 1 \]
\[ g \geq 2 \]

* connected, compact, oriented
Geometric structures

Major theme in geometry

Classify manifolds which carry special geometric structures.
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- gauge fields on manifolds
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This talk is about a class of manifolds which unites these different points of view and whose classification problem is one of the most challenging and fascinating problems in modern geometry.
Calabi–Yau manifolds

A manifold is Calabi–Yau if
Calabi–Yau manifolds

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- it is complex,
Calabi–Yau manifolds

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- it is complex,
- it is Riemannian with vanishing Ricci curvature

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  \[ \text{Ricci} = 0 \quad (Einstein \ equation) \]
- it is symplectic,

so that any two of these structures determine the third one

\[ \text{metric} = \text{symplectic} \times \text{complex} \]
\[ g = \omega I \]
Calabi–Yau manifolds

Einstein’s equation is a complicated system of nonlinear PDEs and it is difficult to find nontrivial solutions.

\[
\text{Ricci}_{ij} = -\frac{1}{2} \sum_{a,b} \left( \frac{\partial^2 g_{ij}}{\partial x^a \partial x^b} + \frac{\partial^2 g_{ab}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{ib}}{\partial x^i \partial x^a} - \frac{\partial^2 g_{jb}}{\partial x^i \partial x^a} \right) g^{ab} \\
+ \frac{1}{2} \sum_{a,b,c,d} \left( \frac{1}{2} \frac{\partial g_{ac}}{\partial x^i} + \frac{\partial g_{ic}}{\partial x^a} \frac{\partial g_{jd}}{\partial x^d} - \frac{\partial g_{ic}}{\partial x^a} \frac{\partial g_{jb}}{\partial x^d} \right) g^{ab} g^{cd} \\
- \frac{1}{4} \sum_{a,b,c,d} \left( \frac{\partial g_{jc}}{\partial x^i} + \frac{\partial g_{ic}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^c} \right) \left( 2 \frac{\partial g_{bd}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^d} \right) g^{ab} g^{cd},
\]
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\]

In 1978, Yau proved a conjecture of Calabi, which gives a sufficient condition for the existence of Ricci flat metrics on complex manifolds.
Yau’s theorem allows us to produce an abundance of examples.

\[ x^4 + y^4 + z^4 + w^4 = 0 \]

complex dimension 2 (real 4)

\[ x^5 + y^5 + z^5 + w^5 + v^5 = 0 \]

complex dimension 3 (real 6)
Calabi–Yau manifolds

No classification is known in complex dimension $\geq 3$. 
Calabi–Yau manifolds

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Dimension 3 is particularly important from the viewpoint of physics (string theory, mirror symmetry) and special holonomy geometry.
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**Big Question**

Are there finitely many Calabi–Yau threefolds up to deformation?
Calabi–Yau manifolds

Standard tools of topology and complex geometry give us invariants (cohomology, Hodge numbers, Kähler cone, ...)
Calabi–Yau manifolds

Standard tools of topology and complex geometry give us invariants (cohomology, Hodge numbers, Kähler cone, . . . )

In this talk, I discuss other approaches to constructing invariants of Calabi–Yau threefolds using

- holomorphic curves
- gauge fields
Holomorphic curves

A **holomorphic curve** is a complex submanifold of complex dimension one (real dimension two, i.e. a surface), i.e. locally looks like $\mathbb{C} \subset \mathbb{C}^n$. If $X$ is a Calabi–Yau threefold, then the “expected dimension” of the space of holomorphic curves is zero.

Can we define an invariant of $X$ by counting holomorphic curves?

**Challenges**

▶ Curves don’t have to be isolated.
▶ Curves can degenerate.
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Gromov–Witten theory

Gromov 1985, Kontsevich 1995

- Replace complex structure $I$ by an almost complex structure $J$

  $$J: TX \to TX, \quad J^2 = -\text{id}$$

  which is compatible with the symplectic form: $g = \omega J$
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- Replace holomorphic curves by stable $J$-holomorphic maps.

Gromov’s compactness theorem

The space of stable $J$-holomorphic maps of given genus $g$ and homology class $\alpha \in H_2(X, \mathbb{Z})$ is compact.
Gromov–Witten theory

This leads to a sophisticated invariant which "counts" stable maps:

\[ GW_{\alpha,g}(X) \in \mathbb{Q} \]

\[ \alpha = \text{homology class}, \ g = \text{genus} \]
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The geometric meaning of these numbers is not obvious:

- **rationality**
  they are rational because maps can have symmetries

- **unboundedness**
  every curve contributes to infinitely many invariants.
Counting embedded curves

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A different approach is to consider curves as currents (measures):

\[ \Omega^2(X) \to \mathbb{R} \]

\[ \eta \mapsto \int_C \eta. \]
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A different approach is to consider curves as currents (measures):

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$$\eta \mapsto \int_C \eta.$$  

We can then try to use methods of analysis (especially geometric measure theory) to control such currents.
Counting embedded curves

By combining these techniques with recent work of Wendl on perturbations of almost complex structures, we obtain:

**Theorem** (Doan–Walpuski, 2019)
For a generic compatible almost complex $J$ there are finitely many $J$-holomorphic curves in $X$ in every homology class.
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In special situations it does not depend on $J$ (Zinger, Doan–Walpuski). In general, $N_\alpha(X, J)$ depends on $J$. 

Counting embedded curves
Gopakumar–Vafa conjecture

In 1998, based on their work in string theory, Gopakumar and Vafa speculated that Gromov–Witten invariants can be expressed in terms of numbers which have better properties.

\[
\sum_{\alpha} \sum_{g=0}^\infty \text{GW}_{\alpha, g}(X) t^{2g-2} = \sum_{\alpha} \sum_{g=0}^\infty \text{BPS}_{\alpha, g}(X) t^{2g-2} = \sum_{k=1}^\infty \frac{1}{k} \left( \frac{2}{\sin(kt/2)} \right)^2 \]
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Conjecture

The numbers \(BPS_{\alpha,g}(X)\), defined by the above formula, satisfy

- integrality: they are integers,
- finiteness: only finitely many are nonzero for every \(\alpha\).
Gopakumar–Vafa conjecture

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Their proof relies on Gromov’s compactness, which controls curves of given genus. As a result, the proof deals with $BPS_{\alpha,g}(X)$ for $\alpha$ fixed and $g$ bounded and finiteness conjecture cannot be concluded.
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This obstacle is overcome using the perspective of currents.

**Theorem** (Doan–Ionel–Walpuski, 2021)
The Gopakumar–Vafa finiteness conjecture holds.
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Theorem (Doan–Ionel–Walpuski, 2021)
The Gopakumar–Vafa finiteness conjecture holds.

A crucial part of the proof is to understand the interaction between Gromov topology, $C^\infty$ topology, and current topology on the space of curves. Allard’s regularity theory provides a bridge between them.
Summary of curve counting

- **Gromov–Witten (GW) invariant**: Computable, difficult to interpret geometrically.
- **Count of embedded curves** $N_{\alpha}(X, J)$: Geometric but not invariant, depends on $J$.
- **Gopakumar–Vafa (BPS) invariant**: Invariant, possibly geometric but not well-understood.

Question: Can we correct $N_{\alpha}(X, J)$ to get an invariant?
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A different approach: Gauge theory

The Yang–Mills equations generalize Maxwell’s equations:

\[ d_A^* F_A = 0 \iff \Delta A + \text{nonlinear} = 0 \]
\[ d^* A = 0 \]

\[ A = \sum_i A_i(x) dx_i \text{ is a connection and } F_A = dA + A \wedge A \]
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If \( X \) is a CY threefold, there is a special class of instanton solutions. The space of instantons has "expected dimension" zero.
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If \( X \) is a CY threefold, there is a special class of instanton solutions. The space of instantons has "expected dimension" zero.

In an influential 1998 paper, Donaldson and Thomas proposed to define invariants of Calabi–Yau threefolds by counting instantons.

This idea is a "complexification" of Donaldson’s revolutionary work on 4-dimensional manifolds from the 1980s.
Challenges

- Instantons are not necessarily isolated.
- Instantons can degenerate.
Gauge theory

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The original idea of defining invariants of Calabi–Yau manifolds using PDEs remains a tantalizing possibility.
Uhlenbeck’s compactness theorem says that instantons can concentrate along a codimension 4 subset $C \subset X$.

Local model:

$$|F_{A_{\epsilon}}| \sim \frac{\epsilon^2}{(\epsilon^2 + r^2)^2}$$

$r = \text{distance from } C$
Gauge theory and holomorphic curves

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Holomorphic curves prevent the count of instantons from being invariant. Recall that the count of curves is also not invariant.
Gauge theory and holomorphic curves

Donaldson and Thomas’ proposal has to be modified.

Schematically, the invariant of $X$ is count of instantons plus weighted count of curves:

$$N_\alpha(X, J) = \sum c_2(A) = \text{PD}(\alpha) \ 	ext{sign}(A, J) + \sum [C] = \alpha w(C, J)$$

The weight of a curve $C \subset X$ should jump whenever an instanton dies or is born along $C$. How to find such weights?
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New ideas

Taubes, Donaldson–Segal, Haydys–Walpuski

Good news
There are PDEs on manifolds of dimension 2, 3, 4, similar to the Yang–Mills equations, whose number of solutions jumps in a similar way: generalized Seiberg–Witten equations

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D_A \psi = 0,
\]
\[
F_A = \mu(\psi).
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Goal: Define \( w(C, J) = \) count of solutions of Seiberg–Witten equations on \( C \) depending on \( J \)
New challenges

Unexpected news

Solutions to generalized Seiberg–Witten equations can degenerate and become singular in a way that is not currently understood.
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**Theorem** (Doan–Walpuski 2018, 2019)

On every three-dimensional manifold there exist sequences \((A_n, \Psi_n)\) of solutions to generalized Seiberg–Witten equations which, as \(n \to \infty\), degenerate to a solution \((A_\infty, \Psi_\infty)\) singular along a one-dimensional subset of the manifold.
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Model of such a degeneration (Doan 2020)

\[
\epsilon \Delta u + A(x)u^\gamma + B(x, u) = 0 \quad \text{when } \epsilon \to 0.
\]
What lies ahead

connections with Algebraic Geometry
How does this story relate to invariants studied in algebraic geometry?
The MNOP Conjecture: all invariants carry the same information.

connections with Riemannian Geometry
Calabi–Yau threefolds are closely related to 7- and 8-dimensional manifolds with exceptional holonomy. Holomorphic curves and instantons have analogues in these geometries, which conjecturally can be used to categorify the invariant of Calabi–Yau threefolds.

connections with Low-Dimensional Topology
The gauge-theory side is related to Taubes' work on generalized Seiberg–Witten equations on 3- and 4-manifolds and Witten's ideas on the invariants of knots: Jones polynomial and Khovanov homology.
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