Let $x \in \mathcal{E}^{\pm}(L)$, $L/\mathbb{Q}_p$ &

$$\lambda_x : \mathcal{H}_0 \to L, \quad T \mapsto \psi(T)(x)$$

where $\psi : \mathcal{H}_0 \to \mathcal{O}(\mathcal{E}^{\pm})$.

From last time, we have

<u>THM</u> Let $\omega \in \mathcal{W}$, $L'/L(\omega)$ finite.

The map $x \mapsto \lambda_x$ is a bijection

$$\mathcal{E}^{\pm}(L') \times_{\mathcal{W}} \omega \overset{1:1}{\longleftrightarrow} \text{Syst. of eigenval. appearing in } \mathrm{Symb}_p^{\pm}(\mathcal{D}_\omega(L))^{\mathbb{I}}$$

(Except when $\omega = 0$ & $\pm 1 = -1$)

§ Classical Points

<u>Def</u> Let $L/\mathbb{Q}_p$, $x \in \mathcal{E}^{\pm}(L)$ is *classical* if λ_x appears in $\mathrm{Symb}^{\pm}_{\Gamma_1(M)}(\mathcal{V}_k(L))$ for some $k, M \in \mathbb{N}$.

$\Updownarrow$

λ_x in $M^{\pm}_{k+2}(\Gamma_1(M), L)$

Fact: k is uniquely determined by x, called its weight

Warning: Not always $\kappa(x) = k$

Def Let $\lambda_x^P := \lambda_x|_{\mathcal{H}_0^P}$ $(\mathcal{H}_0 = \langle \mathcal{H}_0^P, U_p \rangle)$

We define the minimal level M_0 of x as the minimal M_0 s.t.

$$\lambda_x^P \text{ appears in } \mathrm{Symb}^I_{\Gamma_1(M_0)}(V_k(L))$$

Write $M_0 = N_0 p^{t_0}$ $(p \nmid N_0)$, so N_0 = minimal tame level & p^{t_0} = minimal wild level.

Later: If Γ has level pN $\rightsquigarrow$ $N_0 \mid N$
in def of $\mathcal{E}^{\pm}$

<u>Rmk</u> These points do not come from the classical structure used in the proof of the comparison b/w $\mathcal{E}^{\pm}$ & $\mathcal{E}$.

The "classical points" coming from this structure will now be called "very classical".

Def A point $x \in \mathcal{E}^{\pm}$ (over $L(x)/\mathbb{Q}_p$) is very classical of weight $k \in \mathbb{N}$ if

1) $\kappa(x) = k$

2) λ_x appears in $\mathrm{Symb}^I_p(V_k(L(x)))$

Clearly: VC points are classical.

Later: Not all classical pts are VC.

Now, fix $k \geq 0$, $t \geq 1$ &

$$\varepsilon : (\mathbb{Z}/p^t\mathbb{Z})^\times \to L^\times \quad (L/\mathbb{Q}_p)$$

Clearly, ε defines a pt $\mathbb{Z}_p^\times \to (\mathbb{Z}/p^t\mathbb{Z})^\times \to L^\times$ of $\mathcal{W}(L)$

So does $\omega := (z \mapsto z^k \varepsilon(z))$.

We already considered the case $\varepsilon = 1$ and most of the theory stays the same here.

<u>Fact</u> $r(\omega) = r(\varepsilon) = p^{-\nu}$ $\quad (p^\nu = \mathrm{cond}(\varepsilon))$

For $0 < r < r(\varepsilon)$, consider

$$\underbrace{\mathcal{P}_k[r](L)}_{\text{poly of deg} \leq k} \subset \underbrace{\mathcal{H}_\omega[r](L)}_{\text{analytic fcts with weight-}\omega\text{ action}}$$

<u>Lemma</u> The weight-ω action of $S_0(p)$ on $\mathcal{H}_\omega[r]$ leaves $\mathcal{P}_k[r]$ stable.

$\rightsquigarrow$ notation: $\mathcal{P}_\omega[r]$ to remember action and $\mathcal{V}_\omega[r]$ = dual

<u>Rmk</u> ε uniquely determines k.

We obtain very similar control thm's.

Prop The map
$$\rho_\omega : S_p(\mathcal{D}_\omega[r](L))^{\#} \to S_p(\mathcal{V}_\omega[r](L))^{\#}$$
is surj. & induces an $\cong$ on "slope $< k+1$" part

Prop We have an $\mathcal{H}_0$-module $\cong$
$$S_p(\mathcal{V}_\omega[r](L))^{\#} \cong S_{\Gamma_1(Np^t)}(\mathcal{V}_k(L))[\varepsilon]^{\#}$$

Rmk • ω essentially cuts out the nebentype at p.

• If $\varepsilon = \underline{1}$, we get back $\Gamma = \Gamma_1(N) \cap \Gamma_0(p)$

• The RHS has the advantage that we don't care about radius of convergence r.

THM There exists a natural $\mathcal{H}_0$-equiv. surj. map
$$S_p(\mathcal{D}_\omega(L))^{\#} \to S_{\Gamma_1(Np^t)}(\mathcal{V}_k(L))[\varepsilon]^{\#}$$
that induces an $\cong$ on "slope $< k+1$".

Prop / Def Let $L/\mathbb{Q}_p$, k & ε as above.
Every syst. of eigenvalues λ appearing in
$$S_{\Gamma_1(Np^t)}(\mathcal{V}_k(L))[\varepsilon]^{\#}$$
defines a unique point $x \in \mathcal{C}^{\pm}(L)$ s.t. $\lambda_x = \lambda$ and $\kappa(x) = (z \mapsto z^k \varepsilon(z))$.

This point is classical of wgt. k & level Np^t. We call such point "Hida classical".

Rmk $\kappa(x) \neq k$, so $HC \not\Rightarrow VC$.

Prop Let $x \in \mathcal{C}^{\pm}(L)$ such that

- $\kappa(x) = (z \mapsto z^k \varepsilon(z))$ ($\varepsilon = 1$ possible)
- $v_p(U_p(x)) < k+1$

Then, x is either VC or HC. In particular, it is classical.

We will now try to show that every classical point is VC or HC.

§ Family of Galois Rep'n Carried by the Eigencurve

Note that $\kappa : \mathcal{C}^{\pm} \to \mathcal{W}$ is a $\mathcal{C}^{\pm}$-valued point of $\mathcal{W}$, so defines a continuous map

$$\kappa : \mathbb{Z}_p^\times \to \mathcal{O}(\mathcal{C}^{\pm})$$

<u>THM</u> There exists a unique continuous pseudo-rep'n (τ, δ) of

max'l unram'd away from Np ⇝ $G_{\mathbb{Q},Np} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_{Np})$

with value in $\mathcal{O}(\mathcal{C}^{\pm})$ s.t.

1) $\tau(\mathrm{Frob}_\ell) = T_\ell$ ($\forall \ell \nmid Np$)

2) $\tau(c) = 0$ (c = $\mathbb{C}$-conj.)

3) δ factors through

$$\mathrm{Gal}(\mathbb{Q}(\zeta_{Np^\infty})/\mathbb{Q}) = \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$$

4) $\delta|_{\mathbb{Z}_p^\times} = (t \mapsto t \cdot \kappa(t))$, $\delta|_{(\mathbb{Z}/N\mathbb{Z})^\times}(a) = \langle a \rangle$.

Proof (Sketch)

Step 1 Chenevier's Interpolation Trick:

Suppose $\exists$ Zariski closed set $Z \in \mathcal{E}^{\pm}$ s.t.

1) $\forall z \in Z$, $\exists \rho_z : G_{\mathbb{Q},Np} \to GL_2(\overline{\mathbb{Q}}_p)$

2) $\mathrm{Tr}\,\rho_z(\mathrm{Frob}_\ell) = T_\ell(z)$, $\forall \ell \nmid Np$

Let $\mathcal{O}(\mathcal{E}^{\pm})^{\circ}$ = power bdd elmts in $\mathcal{O}(\mathcal{E}^{\pm})$.

Consider $ev_Z : \mathcal{O}(\mathcal{E}^{\pm})^{\circ} \to \prod_{z \in Z} \overline{\mathbb{Q}}_p$

We can show the range of ev_Z is compact & ev_Z is a homeo of its source onto its image.

Now, consider

$$\tau_Z : G_{\mathbb{Q},Np} \xrightarrow{\prod_{z \in Z} \mathrm{tr}\,\rho_z} \prod_{z \in Z} \overline{\mathbb{Q}}_p$$

Then, $\mathrm{Im}(\tau_Z) \subset \mathrm{Im}(ev_Z)$

$\rightsquigarrow$ Let $\tau := ev_Z^{-1} \circ \tau_Z$

Clearly, $\tau(\mathrm{Frob}_\ell) = T_\ell$

Step 2 Let Z = set of VC points.

It works b/c of work of Deligne and Eichler - Shimura. (ρ_z odd $\Rightarrow \tau(c) = 0$)

Step 3 Properties of δ:

We have $\delta_z = \det \rho_z = \omega_p^{k+1} \varepsilon_z(\omega_N)$, where

1) $\omega_p : G_{\mathbb{Q},N_p} \to \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) = \mathbb{Z}_p^\times$ is the cyclo char

2) $\omega_N : G_{\mathbb{Q},N_p} \to \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) = (\mathbb{Z}/N\mathbb{Z})^\times$

Hence, δ_z factors through $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ and is

$$t \overset{\in \mathbb{Z}_p^\times}{} \mapsto t \cdot \kappa(t)(z) = \omega_p(t)^{k+1}$$

$$a \overset{\in (\mathbb{Z}/N\mathbb{Z})^\times}{} \mapsto \langle a \rangle(z) = \varepsilon_z(\omega_N)$$

$\rightsquigarrow$ Let $\delta : G_{\mathbb{Q},N_p} \to \mathcal{O}(\mathcal{C}^{\pm})$ as in the statement.

Then, we have $\delta(g)(z) = \delta_z(g)$

Step 4 Check that (τ, δ) is a pseudo-repr'n & is unique. $\square$

Corollary If $L/\mathbb{Q}_p$ & $x \in \mathcal{C}^{\pm}(L)$, then $\exists!$ continuous semi-simple

$$\rho_x : G_{\mathbb{Q},N_p} \to GL_2(L)$$

s.t. 1) $\mathrm{Tr}\,\rho_x(\mathrm{Frob}_\ell) = T_\ell(x)$, $\forall \ell \nmid N_p$

2) $\mathrm{Tr}\,\rho_x(c) = 0$

3) $\det \rho_x$ factors through

$$\mathrm{Gal}(\mathbb{Q}(\mu_{N_p\infty})/\mathbb{Q}) = \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$$

s.t. $(\omega, a) \mapsto \omega \kappa_x(\omega) \cdot \langle a \rangle(x) \in L(x)^\times$

§3 Properties at $\ell \neq p$

Let $\ell \neq p$, let $\rho_\ell = \rho|_{G_{\mathbb{Q}_\ell}}$

Def The tame conductor of ρ is

$$N(\rho) = \prod_{\ell \neq p} \ell^{n_\ell(\rho_\ell)},$$

where $n_\ell(\rho_\ell) = \dim(\rho) - \dim(\rho^{I_\ell})$
$+ \text{s.w}(\text{gr}\,\rho)$

Level $\Gamma = \Gamma_1(N) \cap \Gamma_0(p)$

Prop Let f = mod. form be an $\mathcal{H}_0$-eigenf. & assume $f \neq$ exceptional Eis. series. If f has min level $N_0 p^t$, then

$$N(\rho_f) = N_0$$

Corollary If $x \in \mathcal{C}^{\pm}$, the tame conductor $N(\rho_x)$ of ρ_x divides N.

Proof Previous proposition + the function $n_\ell(\rho_x)$ is locally cst on $\mathcal{C}^{\pm}$ (See Lemma 7.4.3) $\square$

THM The only classical pts of $\mathcal{C}^{\pm}$ are VL or HL.

Def A classical point x of $\mathcal{E}^{\Sigma}$ is said to be normal if

1) λ_x is not the syst. of eigenvalues of E_2

2) λ_x is not $E_{2,\psi,\tau}$ ($\tau_{G_{\mathbb{Q}_\ell}} = \psi_{G_{\mathbb{Q}_\ell}}$, some $\ell \in \mathbb{N}$)

Note: All cusp. class. pts are normal & Eisenstein classical of wgt >2.
Even in wgt 2 if level is $\square$-free

<u>Lemma</u> Let x be classical, tame level N_0. If x is normal, $\exists$ small nbgh of x s.t. all classical pts in it have tame level N_0.

<u>Proof</u> $\exists$ new form f of tame level N_0 s.t. $\rho_f = \rho_x$

Since f is normal, one can show $n_\ell(\rho_y)$ is cst in a nbgh of x ($\ell \neq p$). $\square$

§3 Ordinary Locus

Def $\mathcal{E}^I_{ord} = \{x \mid v_p(U_p(x)) = 0\}$

Prop $\mathcal{E}^I_{ord}$ is a union of conn'd comp. of $\mathcal{E}^I$, so equidim of dim 1. The VC points are very Zar dense in $\mathcal{E}^I_{ord}$ & $\kappa : \mathcal{E}^I_{ord} \to \mathcal{W}^I$ is finite.

Proof $U_p \in \mathcal{O}(\mathcal{E}^I)$ is bdd by 1 & unit ball in $\mathcal{O}(\mathcal{E}^I)$ is cpct.

$\Rightarrow e = \lim_{n \to \infty} U_p^{n!}$ exists in $\mathcal{O}(\mathcal{E}^I)$ & takes value 1 on $\{x : |U_p(x)| = 1\}$ & 0 elsewhere

$\Rightarrow e$ is idempotent & $\mathcal{E}^I_{ord} = e(\mathcal{E}^I) = \{e = 1\}$. $\square$

Prop If $x \in \mathcal{E}^I_{ord}$, then

$$1 \to \underset{\substack{\uparrow \\ \text{HT wgt} \\ -d\kappa(x)}}{\chi_2} \to \rho_x|_{G_{\mathbb{Q}_p}} \to \underset{\substack{\uparrow \\ \text{unram'd}}}{\chi_1} \to 1$$